Kāshī's Lunar Measurements

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Abstract: Jamshīd Ghiyāth al-Dīn al-Kāshī (d.22 June 1429 AD) measured the fundamental parameters of Ptolemy's lunar model (the radius of the epicycle, and the mean motions in longitude and anomaly) from his observations of the lunar eclipses of 2 June 1406, 26 November 1406, and 22 May 1407 at Kāshān. He presents his data and his process of computations in the prolegomenon to the *Khāqānī zīj* (1413-1414 AD). His data make up the third of four surviving full accounts of lunar measurements carried out during the late medieval Islamic period. Kāshī's error in the time of the maximum phase of the second eclipse is only ~ –8 minutes, an achievement that bears witness to his skill in making tolerably precise astronomical observations and also shows that the simple water and sand clocks available to him were relatively accurate. His input data include theoretical values for the longitude of the Sun and of the lunar ascending node, which he derives from the *Īlkhānī zīj* (Maragha, ca. 1270 AD), based, in the solar theory, upon Ibn Yūnus' (1009 AD) *Ḥākimī zīj*. Kāshī computes a value of ~ 5;17 for the epicycle radius; this does not represent an improvement over Ptolemy's 5;15, but is more precise than other values measured in medieval Islamic astronomy. He uses Muḥyī al-Dīn al-Maghribī's (d.1283 AD) last value for the mean lunar longitudinal motion (measured from the latter's observations at Maragha between 1262 and 1275 AD) and Hipparchus' value for the mean motion of the Moon in anomaly in order to compute the mean lunar motions in longitude and in anomaly respectively in the time intervals between his triple eclipses. As a result, his final values for the motional parameters of the Moon remain very close to those of his two predecessors.

Keywords: Ptolemy, Medieval Islamic Astronomy, Ghiyāth al-Dīn Jamshīd al-Kāshī, Lunar Model, Eclipse, Epicycle, Mean Motion

1. Introduction

This paper aims to present a reasonably complete description and evaluation of the measurement of the fundamental parameters of the Moon in Ptolemy's model,

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carried out by the Persian mathematician and astronomer Jamshīd Ghiyāth al-Dīn al-Kāshī (d. 22 June 1429 AD). Kāshī presented his lunar measurements in the prolegomenon to his *Zīj-i Khāqānī dar takmīl-i Zīj-i Īlkhānī* (Zīj dedicated to the Great Khan, in completion of the Īlkhānī zīj),[1] completed in 816 H/1413-1414 AD, and later dedicated to Ulugh Beg (d. 1449 AD).[2]

As its title and introductory remarks make clear, except in the case of the Moon, Kāshī based his *zīj* on the radices and parameter values used in Naṣīr al-Dīn al-Ṭūsī's (1201-1274 AD) *Īlkhānī zīj*, compiled at the Maragha observatory about 1270 AD, which was, in turn, based upon Ibn Yūnus' (d. 1009 AD) *Ḥākimī zīj* in the case of the solar and lunar parameter values (see Section 3.2 and the end of Section 3.4). Kāshī was also well acquainted with Muḥyī al-Dīn al-Maghribī (d. June 1283 AD), the outstanding observational astronomer at the Maragha observatory, and with his *Adwār al-anwār mada 'l-duhūr wa-'l-akwār* (Everlasting cycles of lights), the last *zīj* Muḥyī al-Dīn composed on the basis of his independent observations performed at Maragha between 1262 and 1275 AD, which he explained in full detail in his *Talkhīṣ al-majisṭī* (*Compendium of the* Almagest).[3] Kāshī calls Muḥyī al-Dīn a «sage/wise man» (*ḥakīm*) and explicitly mentions the *Adwār*, which he calls *Zīj al-kabīr* (*Great zīj*) — an alternative title for the *Adwār* which can also be found in other sources prior to Kāshī's time.[4] For example, in *Khāqānī zīj* II.2.1, he mentions some corrections of (scribal) errors in the sine ta-

1. For bio-bibliographical information on Kāshī, see A.P. Youschkevitch's and B.A. Rosenfeld's article in *DSB*, Vol. 7, pp. 255-262; P.G. Schmidl's entry in *BEA*, pp. 1161-1164; J. Vernet's short entry in *EI*$_2$, Vol. 4, pp. 702-703; and the references mentioned therein. The contents of Kāshī's *zīj* were listed and surveyed in Kennedy 1956, pp. 164-166, 1998a, 1998b. Kennedy also studied the various parts of this *zīj*: see the papers collected in Kennedy 1983, pp. 122-124 (double-argument tables for planetary longitudes, 164-169 (parallax theory), 522-525 (an interpolation scheme) and in Kennedy 1998c, Traces VII (spherical astronomy), VIII (equation of time), XVIII (calculation of the ascendant). For Kāshī's writings on astronomical instrumentation, see Kennedy 1960 (on an equatorium) and the papers in Kennedy 1983c, pp. 394-404 (observational instruments), 440-480 (equatoria). Kennedy also prepared a translation of Kāshī's *zīj*, which has not been published (I owe this information to Dr. Benno van Dalen).

2. Since the base meridian of Kāshī's *zīj* is Shiraz, it seems that he primarily intends to offer his work to Iskandar b. ʿUmar Shaykh Mīrzā I (1384-1415 AD). Iskandar, ruler of central Iran from 1409 AD, who had a strong interest in knowledge and culture and was Kāshī's patron (Kennedy 1998a, p. 2).

3. See Mozaffari 2018a; 2018-2019.

4. E.g., in Kamālī's *Ashrafī zīj* (compiled at the turn of the 14th century), F: ff. 231v, 232r, 233r, G: f. 248v.

ble in the *Īlkhānī zīj* and in Muḥyī al-Dīn's *zīj*,[5] and in III.2.7, he refers to Muḥyī al-Dīn's method for the calculation of the slant component of the latitude of Mercury.[6] More notably, Kāshī deploys Muḥyī al-Dīn's value for the mean daily lunar motion in longitude as a provisional value for the computation of the mean lunar longitudinal motions in the periods between his trio of lunar eclipses (see Section 3.3), and, inevitably, the final value he derives for this parameter is very close to Muḥyī al-Dīn's (cf. Section 3.4).

Kāshī's lunar measurements constitute the most original part of his *zīj*, at least as far as observational astronomy is concerned, in which he is keen to show his skill as an observational astronomer and for this reason, he included these measurements in its introduction. Other innovations and improvements he presents in this work, as summarized in a long detailed list in the prologue (prior to its last section devoted to the lunar measurements), are related to computational procedures and methods of calculation. The most interesting instance of the latter is his novel approach to the computation of planetary latitudes on the basis of Ptolemy's models in *Almagest* XIII.[7]

According to *Almagest* IV and V, the quantification of Ptolemy's lunar model needs the observation of a trio of lunar eclipses (as close to each other in time as possible) in order to derive the radius of the epicycle. One such observation would also suffice for the derivation of the mean motions of the Moon in longitude and in latitude. The derivation of the eccentricity needs an observation of the Moon near quadrature under some special conditions. Kāshī's lunar measurements represent the third of four full accounts of lunar measurements that have come down to us from the medieval Islamic period: the first two are by Abū al-Rayḥān al-Bīrūnī (the lunar eclipses of 1003-1004, Ghazna) and Muḥyī al-Dīn al-Maghribī (the lunar eclipses of 7 March 1262, 7 April 1270, and 24 January 1274), and the

5. The sine table in the Maragha *zījes* was in fact taken from Ibn Yūnus. It is worth noting that the reference to Muḥyī al-Dīn in II.2.1 can only be found in MS. Q1 (f.24r) of Kāshī's *zīj*; in MS. S (f.28v), the passage in question is written in the right margin, in which the reference to Muḥyī al-Dīn's *zīj* has been blacked out; and in the MSS of the final edition (IO: f. 32r, Q2: f. 24v, C: p. 46), there is no mention of Muḥyī al-Dīn in that place. The different versions of the *Khāqānī zīj* will be briefly discussed below (see Section 2).

6. Kāshī, *Zīj*, IO: f. ff. 103v-104r, P: —, Q1: f. 88r, Q2: f. 50v, S: f. 75v, C: pp. 179-180. The passage in question is found in al-Maghribī's *Adwār* II.5.3, CB: ff. 17v-18r; M: f. 18v. Wābkanawī later revised it in his *Muḥaqqaq zīj* III.5.4: T: ff. 54v-55r; P: ff. 83r-v; Y: ff. 99r-100r.

7. See Van Brummelen 2006, for a detailed study.

fourth is due to Taqī al-Dīn Muḥammad b. Maʿrūf (the lunar eclipses of 1576-1577 observed at Istanbul, Cairo, and Thessalonica).[8]

In the study of the medieval astronomical corpus, it is intriguing to try to establish the degree of precision astronomers attained when making direct observations. To do so, we must examine the extent to which their observational data were accurate (within the intrinsic constraints of their naked-eye empirical instruments) in tracking the motions of the celestial objects. This information is available in the few observational records preserved from them or in other relevant materials (e.g., star tables), which can serve as evidence in this respect. Or, in the absence of any surviving observational data, we must establish to what extent the unprecedented values they adopted for the various structural and motional parameters of the Ptolemaic solar, lunar, and planetary models were precise. The times of the maximum phases of the triple lunar eclipses Kāshī observed in his native city, Kāshān, in central Iran, are the only evidence found in his works that can help us judge his practical skill as an astronomical observer. The paper is organized as follows. In Section 2, an English translation of Kāshī's passage on his lunar measurements is presented. Section 3 is devoted to commentary: his input data (both observational and theoretical) are analysed in 3.1 (times) and 3.2 (solar longitudes); in 3.3, his derivation of the lunar epicycle radius is discussed and commented upon; and, finally, in 3.4, his measurement of the mean lunar motions in longitude and in anomaly is examined. Finally, Section 4 presents my conclusions.

2. Text

The translation of the section on the lunar measurements from the prolegomenon of the *Khāqānī zīj* is presented below on the basis of seven manuscripts identified by the sigla IO, Q1, Q2, P, C, L, and S,[9] whose bibliographical information and copying dates (if available) are given in the list of references at the end of the paper. (For the edition of the original Persian text, see the Appendix.) MS. S is, maybe,

8. See Mozaffari 2014; Mozaffari and Steele 2015; Mozaffari 2018a, esp. pp. 599-600, 607. Of the lunar measurements in other communities in the medieval period, the ones made by Levi ben Gershon (1288-1344) are worth mentioning; for his eclipse observations, see Goldstein 1979, and for his non-Ptolemaic lunar models, see Goldstein 1972; 1974a; 1974b.

9. Kāshī, *Zīj*, IO: ff. 4r-6v, P: pp. 24-28, Q1: ff. 2r-3v, Q2: ff. 3v-4v, S: ff. 4r-6v, C: pp. 6-9, L: pp. 389-393.

an autograph. In MS. L, there are only two fragments from Kāshī's *zīj* in nine pages, while the other six manuscripts contain the complete (or at least a large part of the) contents of this work.

MSS. IO, Q2, P, and C make up one family, while MSS. S and L present another version. The late Prof. E.S. Kennedy already made a comparison between MSS. IO and S, and from «a slight indication from the simplified tables of latitudes of Mercury», he tentatively concluded that MS. S «may be the earlier version».[10] The differences between the two groups of manuscripts are also evident in other places, for example, the table of the first lunar equation: in the final version (MSS. IO, Q2, P, and C), the first thirty entries are given to seconds of arc and the rest to arc-minutes, but in the first edition (MS. S), all entries are given to arc-minutes. The two editions differ clearly on four occasions in the section on the lunar measurements, which we have marked by underlining the relevant sentences in our edition/translation. One of them is where Kāshī refers, for the third time, to Euclid's *Elements*: the correct reference is Euclid II.6, which can be found in the first group of MSS, while MSS. S and L erroneously refer to III.6. In some cases, however, two MSS belonging to the two different editions resemble each other. A prominent example is found in the first table containing the solar, lunar, and nodal positions: MSS. L and P coincide in using the Arabic names for the written numbers assigned to the triple lunar eclipses (while the other MSS all have the Persian equivalents) as well as in committing a scribal error in writing بعد («distance»), instead of نقل («shift»). MS. Q1 is almost identical to the final version, but in certain places, it also resembles the earlier edition (notably, in mistakenly referring to Euclid III.6). Perhaps it represents an intermediate stage in the evolution of this *zīj* from the early version to the final edition. The considerations given above are correct as far as the introductory remarks on the lunar measurements are concerned, but a thorough analysis is necessary in order to identify, record, and classify all the differences between the two families of extant manuscripts of Kāshī's *zīj*.

MSS. IO and Q2 are closely related to each other, since the marginal comments and glosses in both are identical and arranged similarly (in diagonal lines, above or below a sentence, etc.). Thus, one of them might have served as a prototype for the other, or they might have a common origin. These comments can also be found in MS. P. A few of these additions are inserted into the main text in MSS. IO and

10. Kennedy 1998a, p. 3.

Q2, but all of them were placed inside the text in MS. P, so that there is no way of recognizing that they do not belong to the original.

The other fragment preserved in MS. L is also related to the Moon: it refers to the procedure of the calculation of the longitude of the Moon, according to Ptolemy's method.[11] This was later included in *Khāqānī zīj* III.2.3.[12] Its title (*Burhān bar taqwīm-i qamar*, «The proof for [the procedure of the calculation of] the longitude of the Moon) differs slightly from the one in both versions of the *zīj* (*Dar taqwīm-i qamar*, «On the longitude of the Moon»). It ends with the statement «I will [continue to] write on these problems».

We emphasize only the major differences (all underlined in our edition/translation) which clearly distinguish the two families of MSS from each other. Our additions to the original text are only for the sake of clarity and are placed within square brackets. Some useful marginal and commentary notes inserted into the text in the various manuscripts consulted for the present study (mostly, from the first group; especially, MSS. IO and Q2) are given inside curly brackets. Most of the selected comments are highlighted in order to clarify the terminology applied to the text and to indicate the arcs and angles standing for the differences in the epicyclic anomaly and equation between the triple lunar eclipses. Other marginal glosses are added to explain the simple geometrical basics and computational procedures, which we do not need to include in our edition/translation.

Remark on the correction of the mean motions (*awsāṭ*) of the Moon from the observations of lunar eclipses:

[I] We observed three lunar eclipses at the city of Kāshān, and derived the mean motions of the Moon from them in the same manner as Ptolemy did, except that he assumed that the centre of the [lunar] epicycle at the [times of] lunar eclipses is in the plane of the ecliptic, and [therefore] took the ecliptic point diametrically opposite (*naẓīr*) to the longitude (*taqwīm*) of the Sun at the mid-eclipse as the longitude of the Moon [at that time]. [Instead,] we took the intersection of the equator of the inclined (*māʾil*) [sphere/orb of the Moon] and the great circle that passes through the centre of the umbra (*ẓill*) and is perpendicular to the plane of [the equator of] the inclined [sphere/orb] of the Moon as the longitude (*mawḍiʿ*) of the Moon, because the «mid-

11. Kāshī, *Zīj*, L: pp. 394-397.

12. Kāshī, *Zīj*, IO: ff. 95v-97r, P: —, Q1: ff. 83r-v, Q2: ff. 44v-45v, S: ff. 70v-71v, C: pp. 168-170.

eclipse» means the moment when the Moon is located at that point.[13] Ptolemy demonstrates this in proposition 2 in chapter 6 of Book VI of the *Almagest*, but for the sake of convenience, he did not take it into account [in his lunar measurements].[14]

[II] The **first eclipse** took place on the night of the thirtieth of the old (*qadīm*) *Shahrīwar-māh* [i.e., the month of *Shahrīwar*, the sixth month] in the year 775 Yazdigird. There had passed 3;14,30 absolute{, i.e., true,} hours (*sāʿāt-i muṭlaqa*{, *yaʿnī*, *ḥaqīqiyya*})[15] or 2;56,29 equated hours (*sāʿāt-i muʿaddala*){, i.e., mean hours adjusted by the equation of time (*yaʿnī*, *wasaṭiyya muʿaddala bi-taʿdīl al-ayyām*)}[16]—that is, in terms of the assumed radix which will be mentioned in the third Book (*maqāla*) [of this work][17]—from midnight to the mid-eclipse.

The **second eclipse** took place in the night of the twenty-seventh of the old *Isfandārmadh-māh* [i.e., the month of *Isfand*, the twelfth month] in the mentioned year. The time passed from midnight to the mid-eclipse: 1;13,5 absolute hours or 0;48,46 equated hours.

The middle of the **third eclipse** took place on the night of the eighteenth of the old *Shahrīwar-māh* in the year 776 Yazdigird. The time passed from midnight to the mid-eclipse was absolutely 4;18,30 hours or as equated: 3;58,46 hours.

[III] We computed the longitude of the Sun and the mean longitude (*wasaṭ*) of the [lunar] ascending node at the middle of the triple lunar eclipses as follows:

[Table 1: The solar and lunar positions for the times of the mid-eclipse phases of Kāshī's trio of the lunar eclipses.]

	[1]	[2]	[3]	[4]	[5]
First	78;55,10,41°	274;32,46°	6;32, 3°	+0;1,29, 2°	258;56,39,43°
Second	252;13,53,38	283;54,51	3;51,15	+0;0,52,40	72;14,46,18
Third	68;14,18,43	293;17,40	1;31,59	−0;0,20,58	248;13,57,45

13. A marginal note inserted in the text in MSS. P and C reads: {The middle of the eclipse is always closer to the node than the point at which the true conjunction [or opposition] takes place.}

14. MSS. S and L differ slightly here: Ptolemy mentions this in chapter 6 of Book VI of the *Almagest*, but did not take it into account.

15. A marginal comment in MSS. IO, Q2, and P.

16. A marginal addition in MSS. IO, Q2, and P.

17. This parenthetical phrase is found only in MSS. S, L, and Q1.

[1] The solar longitude at the mid-eclipse [$\lambda\odot$]

[2] The mean longitude of the node [i.e., the longitude of the ascending node as counted in the direction of decreasing longitude: 360° — λ☊]

[3] The solar distance from the [closest] node

[4] The equation of shift (*taʿdīl-i naql*) [*s*]

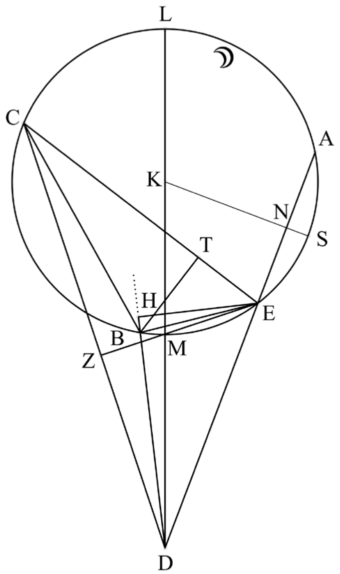
[5] The longitude of the Moon in the inclined sphere/orb at the mid-eclipse [λ☽]

[IV] The lunar motion {in longitude}[18] from the middle of the first eclipse to the middle of the second is equal to 173;18,6,35°, and the period between the two eclipses is equal to 176 days plus 21;58,35 absolute{, i.e., true}[19] hours or 21;52,17 equated{, i.e., mean}[20] hours. In this interval of time, {i.e., the period in terms of the equated hours,}[21] the mean [lunar] motion in longitude is 171;3,13,26°, and the [mean lunar] motion in anomaly is 151;20,36,10°. It is clear that the [difference in the epicyclic] equation corresponding to this arc [i.e., the mean lunar epicyclic motion just mentioned] is additive and is equal to 2;14,53,9°{, because the mean motion in longitude is less than the true longitudinal motion}.[22] Also, the lunar motion {in longitude}[23] from the middle of the second eclipse till the middle of the third is equal to 175;59,11,27°, and the period between the two eclipses is 177 days plus 3;5,25 absolute hours or 3;10,0 equated hours. In this time span, the mean motion in longitude is equal to 173;57,39,20°, and the [mean] motion in anomaly is 154;13,33,34°. It is clear that the [difference in the epicyclic] equation corresponding to this arc [i.e., the mentioned mean lunar anomalistic motion] is additive and is equal to 2;1,32,7°.

[V] Then, we combine figures 3, 4, and 5 from chapter 5 of Book IV of the *Almagest* into a single figure here [Figure 1]. We take the circle *ABC* to be the orb of the [lunar] epicycle, and the points *A*, *B*, [and] *C* as being the positions of the Moon at the middle of the trio of lunar eclipses in sequence. The three intercepted arcs between these points are known, because arc *AB* is the [mean] anomalistic motion between the first and second eclipses, and its [corresponding difference in the epicyclic] equation is additive, and arc *BC* is the [mean] epicyclic motion between the second and third eclipses, and its [corresponding difference in the epicyclic] equation is also additive. Thus, there remains arc *CA* = 54;25,50,16°. The [difference in the epicyclic] equation corresponding to this arc is subtractive and is equal to 4;16,25,16°{, which is the

18. A marginal comment in MSS. IO, Q2, and C.
19. A marginal note in MSS. IO, Q2, and P.
20. A marginal gloss in MSS. IO, Q2, and P.
21. A marginal comment in MSS. IO, Q2, P, and C.
22. A marginal gloss in MSS. IO, Q2, P, and C.
23. A marginal note in MSS. IO, Q2, and P.

sum of the two mentioned [differences in the epicyclic] equations [i.e., the positive differences in the epicyclic equations corresponding to arcs *AB* and *BC*]}.[24] It is no longer unclear whether the apogee [of the epicycle] does not lie on the two arcs *AB* and *BC*, because both are less than half a cycle [i.e., the arc of a semicircle = 180°] and their [corresponding differences in the epicyclic] equations are additive. Thus, it should be located on arc *CA*. Let *D* be the centre of the ecliptic. We join the lines *DA*, *DB*, and *DC*. *DA* intersects with the epicycle at the point *E*. We join the lines *BC*, *EC*, and *EB*. We drop the two perpendiculars *EZ* and *EH* from the point *E* [, respectively,] to the lines *DC* and *DB*, and the perpendicular *BT* from the point *B* to the line *CE*.



[FIGURE 1: The lunar epicyclic positions in Kāshī's triple lunar eclipses as drawn by him in the *Khāqānī zīj*.]

[VI] The sine of the angle *ADC* {—which is the sum of the two mentioned differences in the [epicyclic] equation—}[25] is 4;28,16,32, which is the length (*miqdār*, «size») of the line *EZ* in terms of units (*ajzā'*, pl. of *juz'*, «part») of which [the length of] the line *DE* is 60. Since the angle *AEC* taken as a central angle is equal to 54;25,50,16° and taken as an inscribed angle is equal to 27;12,55,8° and the angle *ADC* is equal to 4;16,25,16°, the angle *ECD* remains, according to Euclid I.32: 22;56,29,52°; its sine:

24. A marginal comment in MSS. IO, Q2, and C.
25. A marginal note in MSS. IO, Q2, P, and C.

23;23,15,21, which is the length of *EZ* in terms of units of which *EC* is 60. Thus, the length of *EC* is equal to 11;28,15,3 in terms of the units of which *DE* is 60. Also, the sine of the angle *ADB* {—which is the angle of the difference in the [epicyclic] equation between the first and second eclipses—}[26] is 2;21,12,56, which is the length of *EH* in terms of the units of which *DE* is 60. Since the angle *AEB* taken as a central angle is equal to 208;39,23,50° and taken as an inscribed angle is 104;19,41,55° and the angle *ADB* is 2;14,53,9°, the angle *EBD* remains equal to 102;4,48,46°; and its supplement, i.e., the angle *EBH*, is equal to 77;55,11,14°; its sine: 58;40,16,48, which is the length of *EH* in terms of units of which *EB* is 60. Thus, *EB* is equal to 2;24,24,48 in terms of the units of which *DE* is taken as 60. Also, the angle *CEB* {—which [i.e., the corresponding arc *BC*] is the arc of the [mean motion in] anomaly between the second and third eclipses—}[27] taken as a central angle is equal to 154;13,33,34° and taken as an inscribed angle is equal to 77;6,46,47°; its sine: 58,29,19,22, and its cosine: 13;22,54,14. These are[, respectively,] the lengths of the lines *BT* and *ET* in terms of the units of which *EB* is 60. Hence, in terms of the units of which *EB* is 2;24,24,48—that is, in terms of the units of which *DE* is 60—the length of *BT* is 2;20,46,33, the length of *ET* is 0;32,12,30, and the length of *EC* is 11;28,15,3. Thus, the length of *CT* remains equal to 10; 56,2,33; its square: 1,59; 33,11,45,42,30,9. The square of *BT* is 5;30,17,50,6,54,9. The sum of the two squares is 2,5;3,29,35,49,24,18, which is the square of *BC*; its root: 11;10,58,36, which is the length of *BC* in terms of the units of which *DE* is 60. In terms of the same units, the length of *EB* is 2;24,24,48. But the chord *BC* is 116,58,38,44 in terms of units of which the half-diameter of the epicycle is 60. Thus, in terms of the same units, *DE* is 627;37,13,55. In terms of the same units, the chord *BE* is 25;10,36,46. Thus, the arc *EB* is 24;13,19,42°, and the arc *AEB* is 151;20,36,10°. Hence, the arc *AE* is 127;7,16,28°; its chord, i.e., [the length of] the line *AE*, is 107;26,55,18. This is less than the diameter [of the epicycle, taken as 120]. As a result, the epicycle centre must be outside the segment *AE*. Let us take it to be the point *K*. We draw a line from the point *D*, such that it passes through the point *K* and cuts through the epicycle at [the points] *L* and *M*, both of which are [the epicyclic apsides: respectively, the point of] the greatest distance [i.e., apogee] and [the point of] the least distance [i.e., perigee]. Then, the rectangle contained by *ED*, which is 627;37,13,55, and the whole *AD*, which is 735;4,9,13, is equal to 2,8,9,4;32,22,14,49,15,55. This is equal to the rectangle *MD*, *LD*, as is known from Euclid III.35 [read: 36]. Then, if we add to it the square on *MK*, which is 1;0,0, according to Euclid II.6,[28] this would result in the square on *DK*, which is

26. A marginal addition in MSS. IO and Q2.
27. A marginal gloss in MSS. IO, Q2, and C.
28. MSS. S, L, and Q1: III.6.

equal to 2,9,9,4;32,22,14,49,15,55; its root: 681;52,6,18, which is the length of *DK* in terms of units of which *KM* is 60. Therefore, the length of *KM*, the half-diameter of the epicycle, is 5;16,46,36 in terms of units of which *DK* is 60.

[VII] Then, we draw the perpendicular *KNS* from the point *K* to *AE*, and join *AK*. Thus, according to Euclid III.3, *AN* is equal to *EN*. Then, if we add half of *AE*, which is 53;43,27,39, to *DE*, which is 627;37,13,55, this results in 681;20,41,34, which is the length of *DN* in terms of the units of which *DK* is 681;52,6,18—that is, in terms of the units of which *KM* is 60. Thus, *DN* is 59;57,14,10 in terms of the units of which *DK* is taken as 60. This is the sine of the angle *SKM*; its arc: 87;45,17,2°. This is [the size of] the arc *SEM*; its supplement is 92;14,42,58°, which is [the size of] the arc *LS*. The sum of the two arcs *SE*—which is half of [the arc] *ASE*—and *EB*, as just mentioned, is equal to 87;46,57,56. Thus, the arc *LEB*, [which is] the distance of the Moon from the true [epicyclic] apogee (*dhurwa-i marʿī*), is 180;1,40,54°, which is the equated [epicyclic] anomaly (*khāṣṣa-i muʿaddala*) [of the Moon] at the midpoint of the duration of the second eclipse. The angle *NDK*, which is the complement of the angle *NKD*, is 2;14,42,58°. And the angle *ADB* is 2;14,53,9°. Thus, the angle *KDB* remains equal to 0;0,10,11°. This is the partial (or small, *juzʾī*) [epicyclic] equation, by which the mean longitude of the Moon is less than its true longitude [at the time]. We subtract it from the longitude of the Moon in its inclined sphere/orb at the middle of the second eclipse, which is equal to 72;14,46,18°; there remains: 72;14,36,7°. The result is the mean longitude of the Moon at the middle of the second eclipse. The mean longitude of the Sun at the time is 252;52,28,57°. The double elongation is 358;44,14,20°. The equation of anomaly {for this *centrum* [= double elongation]}:[29] 0;11,6,45°. The equated anomaly: 180;1,40,54°. Thus, the mean anomaly would be 180;12,47,39°.

[VIII] The mean longitude of the Moon at the midpoint of the second lunar eclipse Ptolemy observed at Alexandria—as mentioned in the *Almagest*—is 29;30°; the equated anomaly: 64;38°; the double elongation: 0;3,38°; the equation of anomaly: 0;32,40°; the mean anomaly: 64;5,20°. That had occurred 497 years, 363 days, and 13 absolute hours at the longitude of Alexandria, which is 11;23,36 hours at the longitude of Kāshān, before the Yazdigird era.

[IX] Thus, the interval of time between the two observations is 1272 years, 354 days, and 0;36,41 absolute [hours] or 0;46,23 equated [hours]. In the period between the two observations, the mean motion in longitude [of the Moon] is, after 17006

29. An addition in MSS. IO and Q2.

complete cycles (*dawr*), 42;44,36,7°, and the [mean] motion in its anomaly is, after 16862 complete cycles, 116;7,27,39. So, we divided each of the mentioned [mean] motions, in longitude and in anomaly, plus the complete cycles, into the time span between the two observations in terms of days and fractions of a day, so that the [mean] daily motions were obtained. Multiplying them [by the number of days], we obtained the [mean] motions in days, months, and years [Table 2]. The period from the middle of [our] second eclipse to the midday of the first day of the year of 781 Yazdigird is five years, nine days, and 11;11,14 hours at the [base] meridian of the *zīj*, for the longitude of Kāshān, which is 11;3,14 hours at the [base] meridian of the *zīj* for the longitude of Shīrāz. We determined the [lunar] mean motion in longitude and in anomaly in this period and added them to the corresponding quantities at the middle of the second eclipse, so that the results are the mean longitude and [mean] anomaly [of the Moon] at noon on the first day of the year 781 [Yazdigird] at the [base] meridian of the *zīj* for the longitude of 88° [Shīrāz]. From the mean longitude of the Sun and the mean longitude of the Moon, we obtained the double elongation and its daily, monthly, and annual motions.

[Table 2: Kāshī's mean lunar daily, monthly, and annual motions.]

	Mean motion in anomaly							
	signs	degrees	minutes	seconds	thirds	fourths	fifths	sixths
In one day	0	13	3	53	56	30	37	20
In one month	1	1	56	58	15	18	40	0
In one year	2	28	43	8	46	17	6	40
In 100 years	7	21	54	37	8	31	6	40
In 600 years	10	11	27	42	51	6	40	0
At the beginning of 781 [Y]	0	27	24	39	48	56	18	12

	Mean motion in longitude								
	signs	degrees	minutes	seconds	thirds	fourths	fifths	sixths	sevenths
In one day	0	13	10	35	1	52	47	50	50
In one month	1	5	17	30	56	23	55	25	0
In one year	4	9	23	6	26	11	4	14	10
In 100 years	11	8	30	43	38	27	3	36	40

In 600 years	7	21	4	21	50	42	21	40	0
At the beginning of 781 [Y]	4	3	49	31	8	45	23	55	34 [54?]

[X] This is the result of the observations of the lunar eclipses we have mentioned. If my lifetime provides us with an opportunity and the government of the world's king helps us, we will observe the remaining planets, on the basis of which we will compose a *zīj*. Here we bring forward what we are able to do for the moment.

3. Commentary

3.1. The times of the lunar eclipses

Kāshī's dates and times of the mid-eclipse phases of the trio of lunar eclipses, as given in paragraph **[II]**, are summarized in Table 3.[30] The first column indicates the number assigned to each eclipse, and the second column contains the dates Kāshī gives in the Yazdigird era and their equivalents in the Julian calendar and Julian Day Number (JDN). As can be easily understood from the date given for the second eclipse, Kāshī's dates are in accordance with the reform of the Yazdigird calendar carried out after 1007 AD, according to which the five epagomenal days were moved from the end of the eight month (*Ābān*) to that of the last month (*Isfandārmadh*).[31] Therefore, the term *qadīm* in the text has nothing to do with the «early» Persian calendar.[32] Kāshī also gives the civil date for eclipses nos. 1 and 2, but the astronomical date (from noon to noon) for eclipse no. 3 (the civil date of eclipse no. 3 is 19-6-776 Y).[33] The third and fourth columns include our author's values for the times of the maximum phases respectively in apparent and mean local times. The differences between them are related to the equation of time, which, as noted in paragraph **[I]**, was explained in Kāshī's *zīj* III.1.1.[34] Finally, the last

30. Nos. 08220, 08221, and 08222 in 5MCLE.

31. See B. van Dalen's entry *Ta'rīkh* in *EI*2, Vol. 10, esp. pp. 262-263.

32. Kennedy (1998a, p. 5) gives the date of the second eclipse as 1 November (?) 1406. It seems that he had the month of December in mind, and that he mistakenly took the date to refer to the early Yazdigird calendar.

33. Kennedy (1998a, p. 5) gives the date of the third eclipse as 21 May 1407.

34. Kāshī, *Zīj*, IO: ff. 77r-78r, P: —, Q1: ff. 72r-v, Q2: ff. 32v-33r, S: ff. 60v-61r, C: pp. 142-144. Kāshī's table of the equation of time (see below, note 53) has already been investigated at length in a

column gives the errors in the time. Table 4 presents a detailed account of the situations of the triple lunar eclipses at Kāshān, including the times of the occurrence of their various phases in Mean Local Time (MLT) and the horizontal coordinates of the Moon at each (altitude/azimuth, *h*/*Az*) together with the times of moonrise and moonset and the eclipse magnitudes. The errors listed in the last column in Table 3 are in fact the differences between Kāshī's mean times and the times of the mid-eclipse phases in Table 4. Unlike Bīrūnī, Muḥyī al-Dīn, or Taqī al-Dīn, Kāshī does not provide us with any magnitude estimations for his only partial eclipse (no. 1).

Table 3: Kāshī's values for the times of the mid-eclipse phase of the triple lunar eclipses of 1406-1407.

Nos.	Date	Apparent time	Mean time	Error
1	Night of 30-6-775 Y 2 June 1406 JDN 2234752	$3;14,30^h$	$2;56,29^h$	$-1;10^h$
2	Night of 27-12-775 Y 26 November 1406 JDN 2234929	1;13, 5	0;48,46	–0; 8
3	Night of 18-6-776 Y 22 May 1407 JDN 2235106	4;18,30	3;58,46	–0;41

Table 4: The circumstances of the three lunar eclipses of 1406-1407.

Nos.	Moonrise	S. Partial Phase	S. Total Phase	Mid-Eclipse	E. Total Phase	E. Partial Phase	Moonset	Mag
1	18:44 (-1^d)	2:31 +21;44°/ 38;59°	—— ——	4: 6 + 7;38°/55;27°	—— ——	5:42 – 8;54°/68;25°	4:52	0.77

1988 paper by Kennedy, reprinted in Kennedy 1998c, Article VIII.

2	16:33	23:17 (−1ᵈ)	0:20	0:57	1:34	2:37	7:11	1.31
	(−1ᵈ)	+75;50°/329;16°	+76;2°/30;36°	+70;58°/54;29°	+64;27°/68;23°	+52;14°/82;20°		
3	18:37	2:44	3:51	4:40	5:28	6:35	4:51	1.49
	(−1ᵈ)	+20;10°/ 42;40°	+9;59°/54;13°	+ 1;48°/61;23°	− 6;36°/67;40°	−19;23°/75;42°		

Notes:

– S. = Start, E. = End.

– In each row, the first line gives the times (all in MLT), and the second (shaded) line the horizontal coordinates *h*/*Az*.

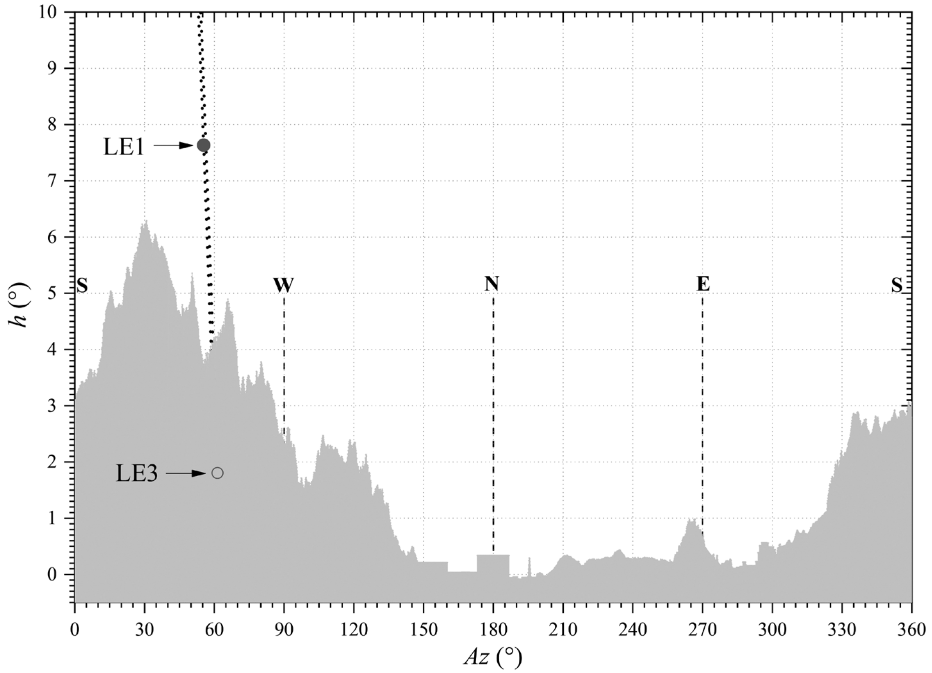


FIGURE 2: Horizon profile as seen from Kāshān, displaying the altitude limitations due to the geographic bearings. The dots depict the lunar apparent motions with respect to the local horizon for eclipses nos. 1 and 3 during one hour before moonset. The two points at which the maximum phases occurred are magnified and are marked with arrows pointing to the right.

As for times, Kashī only reports the times of the mid-eclipse phases, without providing us with any details as to how he measured and/or computed them. In the introduction to Book V of the *Khāqānī zīj*, he briefly describes a simple clepsydra (*fankān* or *bankām*) in the shape of a bowl or container with a tiny hole at its apex, which could be used as either an inflow or an outflow water-clock, and in V.I.1, he mentions an hourglass (*shīsha-i sā'at*).[35]

Figure 2 shows the horizon profile as seen from Kāshān: the region highlighted in light grey at the bottom of the graph shows the altitude limitations, as seen from the city, due to its peripheral geographical bearings. A mountainous area blocks a great deal of the view to the southwest, with a maximum of ~ 6.3° at an azimuth of ~ 30°, which is caused by Siyāh kūh (Mt. Siyāh, «Black»), with a height of ~ 3000 m above sea level, located about 18 km from the city.

The midpoint of a partial eclipse is difficult to determine directly from observation (maybe it could be detected with the aid of an auxiliary optical device, like a camera obscura). Naturally, for a total eclipse (especially, if it shows a perceptible duration) it would be wholly impossible to estimate its maximum point. For both types of eclipse, one can derive the moment of the mid-eclipse from the measurement of the times of the beginning and end of the partial and/or the total phase. There was apparently no difficulty in this regard in Kāshī's second eclipse, since the Moon was far above the horizon at Kāshān during its occurrence (see Table 4), and for this reason Kāshī's time is likely to be exceptionally precise in this case. The maximum phase in eclipses nos. 1 and 3 occurred at times when the Moon was located very near the local horizon. The lunar nocturnal motions during these two eclipses in the altitudinal region below 10° are displayed in Figure 2: note that the two paths are inextricably entwined. Neither eclipse was observable from Kāshān at its end (cf. Table 4). The maximum phase of the first eclipse, a partial one, could be observed at an altitude of ~ **7.5**° above the mountainous area in the southwest; but the abovementioned difficulty cannot in any way account for Kāshī's egregious error of more than one hour for its time. For the third eclipse, a total one, it is not known how he was able to measure the time of its maximum phase. Such a serious problem is not encountered in the other lunar measurements surviving from the late medieval Islamic period, as the Moon was above the horizon from

35. Kāshī, *Zīj*, IO: ff. 183r-184r, P: —, Q1: ff. 155r-156r, Q2: ff. 80v-81r, S. ff. 140v-141r, C: pp. 321-323. Both terms used by Kāshī for the water-clock are Arabicized forms of the Persian term *pangān*. See Mozaffari 2018, pp. 620-628 on these names and for a short history of clepsydras in Islamic astronomy. Four other observational instruments and measurement devices are also mentioned in the introduction to Book V: the parallactic instrument, the portable quadrant, the mural quadrant, and line and plumb.

the beginning until the end of all the sets of triple lunar eclipses observed by Bīrūnī,[36] Muḥyī al-Dīn,[37] and Taqī al-Dīn.[38]

Nevertheless, it should be borne in mind that Kāshī had no choice: from *Almagest* IV.6 and IV.11, it is known that for the purpose of measuring the lunar epicycle radius, one should select a trio of lunar eclipses that are close to each other in time, so as to minimize the effect of the probable errors in the lunar mean motion in longitude and in anomaly. The almost excessive rigour Kāshī sought in order to achieve a precise derivation suggests that he bore this condition in mind when he planned for his lunar measurements. From 1400 AD, when he was a very young man, until the time he finished his *zīj* (ca. 1413-1414 AD), only three other lunar eclipses were observable at Kāshān, at least, from the beginning to the maximum phase: those occurred on 3 August 1403, 21 March 1410, and 2 September 1411 (all three were total and could actually be observed from the beginning to the end). So, of the six lunar eclipses observable from Kāshān during the time interval in question, the triple eclipses of 1406-1407 were the closest to each other in time and therefore met the essential requirement laid down by Ptolemy.[39]

As we have seen, Kāshī must inevitably have computed the times of the maximum phases of the first and third eclipses with the aid of a priori known theoretical data, and so his observation reports of these eclipses seem to have gone through a process of analysis, rather than being representative of the results of pure empirical work. As we shall see below, it is practically certain that not only the *Īlkhānī*

36. Bīrūnī observed his first two lunar eclipses at Jurjān (Gurgān, northern Iran) and the third at Jurjāniyya (now in Turkmenistan). During the eclipses the altitude of the Moon was never less than 5°; its lowest altitudes were at the beginning of the first eclipse (~ **5 1/4**°) and at the end of the third (~ **7**°).

37. All of Muḥyī al-Dīn's three lunar eclipses were fully observable at Maragha from the beginning till the end of the partial phase, as the altitude of the Moon was never below 5°; its lowest altitude was at the beginning of the partial phase of the first eclipse, ~ **5;11**° at an azimuth of ~ **271;4**° (see the horizon profile as seen from the Maragha observatory in Mozaffari 2018a, Figure 3 on p. 620), which took place about 23 minutes after sunset at 18:5 MLT.

38. During all of Taqī al-Dīn's triple lunar eclipses the Moon was above the local horizon, at altitudes of not less than 18°, at Istanbul and Cairo; note that, as Taqī al-Dīn remarks, the third eclipse was observed by someone else in Cairo, due to the cloudy weather in Istanbul, and the data (we are explicitly told, the altitude of Aldebaran, α Tau: ~ **71;14**° at the mid-eclipse phase at Cairo) were then transmitted to him (see Mozaffari and Steele 2015, p. 356).

39. Bīrūnī and Taqī al-Dīn also worked with triple lunar eclipses occurring within two years, but Muḥyī al-Dīn gave preference to a trio taking place between 1262 and 1274, which he believed to have observed «with extreme accuracy».

zīj, but also Muḥyī al-Dīn's *Adwār al-anwār*, were among Kāshī's main sources in the derivation of the theoretical data he needed in order to accomplish his lunar measurements. A computation of the times of the true oppositions for Kāshī's trio of the lunar eclipses on the basis of the parameter values adopted in the two *zījes* mentioned, adjusted for the meridian of Kāshān ($L = 86°$ from the Fortunate Isles), results in the values listed in the tabulation below:

Nos.	*Īlkhānī zīj*	Error	*Adwār*	Error
1	4: 4	− 2^{m}	3:50	−1;10^{h}
2	1:42	+45	1:29	−0; 8
3	5: 2	+22	4:54	−0;41

It is obvious that for the first and third eclipses both sets of theoretical times are significantly more precise than Kāshī's, but the opposite is true for the second one.

A possible way to derive the times of the maximum phases of eclipses nos. 1 and 3 was to measure the times of the beginning of the partial phases of these two eclipses, and then to add to them half of their durations, as derived from a specific reliable *zīj*. This procedure could be safe and have no undesirable consequences if both the observational and theoretical input were sufficiently precise. If Kāshī's observations of the times of the first contact in eclipses nos. 1 and 3 were as accurate as the time he gives for the maximum phase of the second eclipse, he could achieve tolerably accurate values for the maximum points of the first and third eclipses, because the theories established in the Maragha astronomical tradition provided him with sufficiently precise values for the durations of the two eclipses in question, as we will now explain. According to the *Īlkhānī zīj* II.8, the practitioner should enter the lunar eclipse table with the true lunar daily motion in longitude (v) and latitude (β) in order to derive the magnitude and the half-duration of the partial phase and of the totality.[40] For the first and third eclipses, the *Īlkhānī zīj* gives $v = 11;57°$, as computed from the differences in the lunar longitude at noon between the two consecutive days around the eclipses. As we will see later,

40. Ṭūsī, *Īlkhānī zīj*, C: pp. 47-49, P: ff. 17r-v, M: ff. 30r-v, T: ff. 23r-v, B: ff. 26r-v, F: ff. 22v-23r, L: ff. 27v-28r, Fl: ff. 25v-26r, O: ff. 23r-v.

Kāshī's values for the lunar longitude (λ) and the longitude of the ascending node (λ☊) — already presented, respectively, in Cols. **[5]** and **[2]** in [Table 1] — were in fact computed on the basis of the *Īlkhānī zīj*, from which we have $\beta = +0;34°$ for the first eclipse and $\beta = -0;8°$ for the third one; by interpolating these quantities in the lunar eclipse table, we reach the following theoretical results:[41]

Nos.	Half-duration	Error[42]	Magnitude	Error
1	1;25h	~ −11^m	7.7 ≈ 8 digits	~ −1 digit[43]
3	1;49	~ − 7	Total	~ −1 digit

The above errors in the times of the maximum phases of the first and the third eclipse compare favourably with Kāshī's error in the corresponding time of the second eclipse. Therefore, it seems unlikely that he applied this procedure to derive the maximum points of his first and third eclipses.

Another possibility is that he adjusted the theoretical values computed for the times of the maximum phases of eclipses nos. 1 and 3 by taking into account the difference he found between the observed and theoretical values for the time of the maximum phase of the second eclipse. Support for this hypothesis comes from the fact that all differences between his times and those derived from the Maragha theories (as given earlier) amount to about one hour.

41. The relevant entries in the table read as follows:

v →	*11;48°*		*12; 0°*	
$\lvert\beta\rvert$ ↓	Mag	Half-duration	Mag	Half-duration
8′	Total	1;49^h	Total	1;49^h
34′	7;35	1;23	7;47	1;25

Ṭūsī, *Īlkhānī zīj*, C: pp. 88-89, P: ff. 29v-30r, M: ff. 54r-v, B: ff. 45v-46r, F: ff. 38v-39r, L: ff. 48r-v. The tables in the other three manuscripts consulted (T: Suppl. P: ff. 22v-24r, Fl: ff. 44v-46r, O: ff. 45v-48r) are similarly distorted, as all give mag = 7;5 and half-duration = 1;23^h for $\lvert\beta\rvert = 34'$ and either $v = 11;48°$ or $12;0°$.

42. From Table 4, the duration of the first eclipse was 3;11 hours, and the third one lasted 3;51 hours.

43. Note that a modern magnitude of 0.77 (cf. Table 4) corresponds to about 9 digits.

3.2. Solar theory

All the theoretical values Kāshī put forward in paragraph **[III]**, i.e., the longitudes of the Sun and of the lunar ascending node at the maximum points of the triple lunar eclipses ([Table 1], Cols. **[1]** and **[2]**), are in excellent agreement with the ones computed on the basis of the *Īlkhānī zīj* as adapted to the terrestrial longitude $L = 86^\circ$ (Kāshān). All basic parameters of the solar theory in the *Īlkhānī zīj*, except for the longitude of the solar apogee, were, in turn, taken from Ibn Yūnus' (d. 1009 AD) *Ḥākimī zīj*, very likely via Muḥyī al-Dīn al-Maghribī's *ʿUmdat al-ḥāsib wa ghunyat al-ṭālib* (*Mainstay of the astronomer, sufficient for the student*), the first work that the latter composed after joining the Maragha team.[44]

As laid down in Section 6[45] and the solar tables of the *Ḥākimī zīj*, Ibn Yūnus' solar theory consists of the following parameters on the basis of the ancient eccentric model: (1) the eccentricity $e = 2;6,10^p$ (the radius of the geocentric orbit, the deferent, is taken as $R = 60^p$) from the maximum equation of centre $q_{max} = 2;0,30^\circ$;[46] (2) the motion in longitude $\omega = 0;59,8,19,44,10,31,13,58...^\circ$, as derived from the solar mean motion in one Persian year 359;45,40,3,44°,[47] which corresponds to (3) the length of the tropical year $T_y = 365;14,32...$ days; (4) the longitude of the solar apogee $\lambda_A = 86;10^\circ$ for 372 Y, whose beginning is equivalent to 16-3-1003 AD (JDN 2087478); and (5) the mean longitude at mean noon in Cairo ($L = 65^\circ$) on 1-1-980 AD, as derived from the tables, $\bar{\lambda} = 285;25^\circ$.

In the *Īlkhānī zīj*, we find the tabular value for λ_A equal to 86;24,21° for the beginning of 601 Y (17-1-1232 AD = JDN 2171046), to which 2;0,30° should be added (due to the always-additive, but not displaced, table of the solar equation of centre); so, $\lambda_A = 88;24,51^\circ$.[48] It is obvious that the difference of 2;14,51° between the values of λ_A in Ibn Yūnus' *zīj* and the *Īlkhānī zīj* does not fit any of the

44. See Mozaffari 2018-2019, pp. 154-156, 206.

45. Ibn Yūnus, *Zīj*, L: p. 120; Caussin 1804, p. 215-217.

46. Table of the solar equation of centre: Ibn Yūnus, *Zīj*, L: pp. 173-174.

47. Solar mean motion tables: Ibn Yūnus, *Zīj*, L: pp. 137-138, 155-156.

48. Ṭūsī, *Īlkhānī zīj*, the tables of solar mean eccentric and apogeal motions: C: pp. 56-59, P: ff. 20v-21v, M: ff. 33v-35v, T: ff. 26r-27v, Q: ff. 32r-33r, B: ff. 29v-31r, F: ff. 26r-27r, L: ff. 31r-32v, Fl: ff. 28r-29v, O: ff. 29r-30v, Ca: ff. 22v-24r (in T, the yearly motions are for the period 703-803 Y/1333-1433 AD, while in the others, for 601-701 Y/1232-1331 AD); the table of the solar equation of centre: C: pp. 60-65, P: ff. 21v-23r, M: ff. 36r-38v, T: ff. 28r-30v, Q: f. 33v (incomplete), B: ff. 31v-34r, F: ff. 27v-29r, L: ff. 33r-35v, Fl: ff. 30r-32v, O: ff. 31r-33v, Ca: ff. 24v-27r.

well-known values for the rate of precession in use at the time (i.e., ψ = 1° in 66, 70, or 70 $^{1}/_{4}$ Persian years).[49] Moreover, the abovementioned radix value of λ_A in the *Īlkhānī zīj* is slightly different from the epoch value of λ_A = 88;20,47° in Muḥyī al-Dīn al-Maghribī's two Maragha works, i.e., *Talkhīṣ* and *Adwār*, which, as the latter explains in detail in *Talkhīṣ* IV.4-5, was drawn from the value 88;50,43° measured for 16 December 1264 on the basis of his four solar observations in 1264-1265.[50] Accordingly, the value of λ_A in the *Īlkhānī zīj* must have been the result of solar observations and measurements made at Maragha, independent of Muḥyī al-Dīn's. In his *Tuḥfa* and *Ikhtiyārāt*, Quṭb al-Dīn al-Shīrāzī (d. 1311 AD) states that the solar eccentricity is 2;5,51, according to the modern astronomical observers (it is Ibn al-Aʿlam's value),[51] and that the longitude of its apogee is equal to 87(9?);6,51° at the beginning of 650 Y, according to the *new observations* (*raṣad al-jadīd*).[52] I can see no reason to doubt Shīrāzī's remark, even though it is imprecise. It is worth noting that Ibn Yūnus' updated value 89;51,13° (= 86;10° + (631-372)/70$^{1}/_{4}$) was more accurate than both the values used in the *Īlkhānī zīj* and the one measured by Muḥyī al-Dīn at the time (modern: **89;46**° for 17-1-1232 AD).

Kāshī worked with this revised solar theory, without realizing that in principle it belonged to Ibn Yūnus. Thus, from λ_A = 88;24,51° for the beginning of 601 Y in the *Īlkhānī zīj* and adopting ψ = 1°/70^{y}, he computes λ_A = 90;0° for the year 712 Y, as mentioned in the heading of his table for the equation of time.[53] He also states in III.2.2 that «in the new observation (*raṣad-i jadīd*), they found e = 2;6,9»,[54] from which he computes q_{max} = 2;0,29,***19***° [...,21] for the mean anomaly of 92;0°;[55] in the table of the solar distance from the Earth, he gives the maximum distance as

49. See Mozaffari 2017, pp. 6-7.

50. al-Maghribī, *Talkhīṣ*, ff. 57v-61v. Muḥyī al-Dīn's solar tables on the basis of his new theory worked out in Maragha can be found in *Talkhīṣ*, ff. 64r-v and *Adwār*, CB: ff. 73v-74r, 80v-81r, M: ff. 75v-76r, 82v-83r. For a presentation of Muḥyī al-Dīn's solar observations and measurements at Maragha and their evaluation, see Saliba 1985; Mozaffari 2018a, esp. pp. 598, 606; 2018b, pp. 193, 195-197, 204, 206-207, 229, 235.

51. See Mozaffari 2013, Part 1, p. 326, 330.

52. Shīrāzī, *Tuḥfa*, f. 38v; *Ikhtiyārāt*, f. 50v. The tabular value in the *Īlkhānī zīj* for this year is 87;6,21°, and so λ_A = 89;6,51°.

53. Kāshī, *Zīj*, IO: ff. 126v-127r, P: p. 106, Q1: ff. 103v-104r, Q2: —, S: ff. 91v-92r, C: pp. 215-216.

54. Kāshī, *Zīj*, IO: f. 95v, P: —, Q1: f. 82v, Q2: f. 44v, S: f. 70v, C: p. 167.

55. Kāshī, *Zīj*, IO: ff. 130v-131v, P: pp. 113-115, Q1: ff. 107v-108v, Q2: —, S: ff. 95r-96r, C: pp. 223-225.

62;6,9,0.[56] By «new observation», he is undoubtedly referring to work carried out at the Maragha observatory; it can be seen here how, without considering the historical chain of the development of the Islamic *zīj* literature, even an outstanding figure as Kāshī was liable to make anachronistic mistakes as well as alterations in the fundamental parameter values. He gives the values 170;1,9,58° and 90;59,8,34°, respectively, for the solar mean anomaly and longitude of apogee at the epoch, that is, the beginning of 781 Y (4-12-1411 AD, JDN 2236763), which are in full agreement with the *Īlkhānī zīj*.[57] The only component of Kāshī's solar theory that departs from Ibn Yūnus and the *Īlkhānī zīj* is the solar parallax: in the table of the horizontal parallax of the Sun in V.1.2, he gives π_{max} = 0;2,21° for the least Sun-Earth distance and π_{max} = 0;2,11° for its greatest distance;[58] both values appear to have been derived from the mean solar distance of 1523;2,5$^{E.r.}$, which he had already calculated in his *Sullam al-samā'* (*The stairway to the heavens*, 1407 AD).[59]

It is curious to see whether and how Ibn Yūnus' solar theory could have been in use for over four centuries and still provide tolerably accurate result. As we have discussed in detail elsewhere, a solar theory constructed on the basis of the eccentric model can potentially trace the solar motion within the degree of precision attainable in observational instruments of the pre-telescopic era, on condition that its parameters are determined with acceptable precision. This is principally because of the simplicity of the Earth's motion, in comparison with that of the planets, which means that the theoretical deviation between the eccentric motion in a circular orbit and the Keplerian motion in an elliptical orbit is only small. Ibn Yūnus' value for T_y is only +9 seconds in error, which causes the error in ω to be only -2.57339×10^{-7}°; thus the error in the mean longitude accumulates to 1′ after the passage of 177.3 years, presumably because, unlike the Mumtaḥan team (first half of the ninth century) and al-Battānī (d.929 AD), Ibn Yūnus did not use Ptolemy's faulty equinox times with the errors of more than +1 day. For this reason, his solar theory would be expected to present a stable behaviour even over long periods. In addition, he measured tolerably accurate values for the solar orbital elements: his value for e is in error by about $+4\times10^{-4}$ (if R = 1) and his

56. Kāshī, *Zīj*, IO: f. 157r, P: —, Q1: f. 132r, Q2: —, S: f. 117, C: p. 284.
57. Kāshī, *Zīj*, IO: f. 127v, P: p. 107, Q1: f. 105r, Q2: —, S: f. 92v, C: p. 217.
58. Kāshī, *Zīj*, IO: f. 185r, P: —, Q1: f. 157r, Q2: f. 82r, S: f. 142r, C: p. 325.
59. Kāshī, *Sullam al-samā'*, N: f. 8v, D: f. 12v, V: f. 5r; cf. Kennedy 1998, p. 39.

value for λ_A is only +0.4° off. Furthermore, Ibn Yūnus' value for for the beginning of 980 AD is in error by about +1′.[60]

We have computed, by Benno van Dalen's very useful software Historical Horoscopes, solar ephemerides on the basis of Ibn Yūnus' theory for 13,000 days after 980.0 and 1280.0 AD and on the basis of its revised version adopted in the *Īlkhānī zīj* for 13,000 days after 1280.0 and 1400 AD The longitudinal errors $d\lambda$ are plotted against time as well as against the modern longitudes λ', respectively, in Figures 3 and 4, and their statistics are given in Table 5. They do not exceed ±6′ even after the passing of some 400 years. In Figure 5, we have plotted the errors in the solar declinations $d\delta$ (= the errors in the noon altitude of the Sun) that are pertinent to the errors $d\lambda$ in the *Īlkhānī* solar theory for time intervals of 13,000 days after 1280.0 and 1400.0 AD: they are below ±2.5′. It is very difficult to detect such small errors with naked-eye devices.

TABLE 5: The statistics for the errors in Ibn Yūnus's solar theory and its revised version in the *Īlkhānī zīj*, which was used by Kāshī.

	Epoch	Mean (′)	σ (′)	*MAE* (′)	Min (′)	Max (′)
Ibn Yūnus	980.0	+1.2	2.1	2.1	−2.8	+4.7
Ibn Yūnus	1280.0	−0.3	2.7	2.5	−4.5	+4.7
Īlkhānī	1280.0	−0.3	3.5	3.1	−4.9	+5.9
Kāshī	1400.0	−0.8	4.0	3.7	−6.2	+6.0

In the prolegomenon to Rukn al-Dīn Āmulī's *Zīj-i Jāmiʿ Bū-saʿīdī* (written ca. 1438), there is a list of the corrections to the radices adopted in the *Īlkhānī zīj*. We are told that they were suggested by Quṭb al-Dīn al-Shīrāzī in the case of all planets, except for the Sun and Mercury (including a correction of +30′ in the mean lunar longitude), and by a team of Maragha astronomers in the case of the Sun, after Ṭūsī's death. Although, as discussed elsewhere,[61] a part of the story given in Rukn al-Dīn's account is fanciful, it is interesting that the corrective quantities in this anecdotal history appear to have a basis in reality: for the Sun, the correction was to decrease the radix of its mean eccentric anomaly by 0;3°. We are told that

60. Mozaffari 2018b, esp. pp. 212, 220, 226, 235.

61. Āmulī, *Zīj*, T1: ff. 1v-2r, P1: f. 1v, P2: f. 103r. See Mozaffari and Zotti 2013, pp. 57-59 (the text has been edited on the basis of MSS T1 and P1 on p. 146).

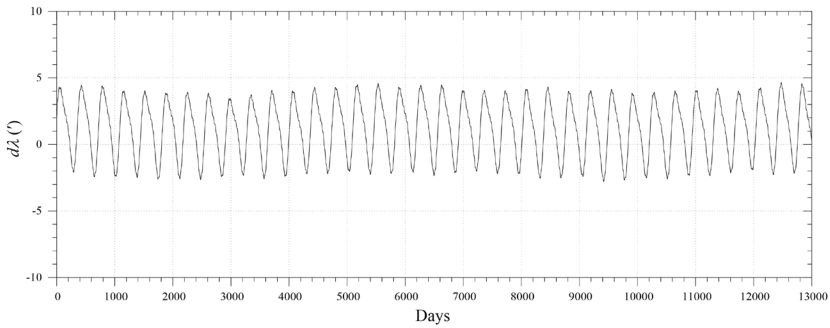
the reason was to keep the Persian solar calendar (beginning with the instant of the vernal equinox) in order, so that a reduction of $3'$ was to compensate for a persistent error of about half an ecliptic sign in the horoscope of the year (equivalent to an error of ~ 1 hour in the derivation of the time of the beginning of the year), and thus to prevent it from drifting further away from the time of the vernal equinox. As can be seen in Figures 4(a)-4(c), the correction is in clear agreement with the errors in Ibn Yūnus' solar theory used in the *Īlkhānī zīj*, reaching ~ $3'$–$4'$ at the times of the vernal equinox. We do not know who was responsible for the correction in the *Īlkhānī* solar theory, but its examination and the discovery of an error of a few minutes of arc in the solar longitude and/or a few hours in the Sun's arrival at such critical points/times as the Spring equinox was only possible with the aid of a large-size instrument, like the mural copper quadrant of the Maragha observatory (radius ~ 266 cm and graduated to subdivisions of $1'$).[62] Kāshī was apparently unaware of these corrections, since none of them were applied in his *zīj* (for the correction of the Moon's mean longitude, see the remark at the end of Section 3.4).

3.3. The computation of the size of the lunar epicycle

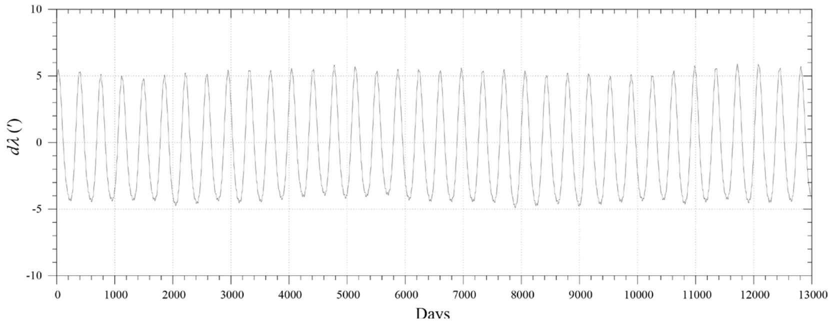
The input data for deriving the radius of the lunar epicycle are the longitude of the Moon, $\lambda_☽$, at the middle of each eclipse and its mean motions in longitude $\Delta\bar{\lambda}$ and anomaly $\Delta\bar{\alpha}$ between each pair of subsequent eclipses. The latter quantities can be computed from the reliable pre-existing values for the mean motions, which can be provisionally used without any undesirable consequences in short periods. In what follows, we will see how Kāshī obtained them.

The values of $\lambda_☽$ are directly computed from the solar theory, with the exception that Kāshī does not take a point diametrically opposed to the solar true position ($\lambda_\odot + 180°$) as the lunar longitude, but adjusts it with the «equation of shift» (or the «third lunar equation», as called in Kāshī's table of this equation in his *zīj*) in order to derive the position angle of the Moon on its inclined orb with respect to the vernal equinox point (not as projected onto the ecliptic). Kāshī sees this as an advantage of his lunar measurement, as he emphasizes it in paragraph **[I]**. The equation of shift (or «reduction to the ecliptic» in modern terminology) is a function of the

62. On this instrument, see Mozaffari 2018a, pp. 616-620.



(a)



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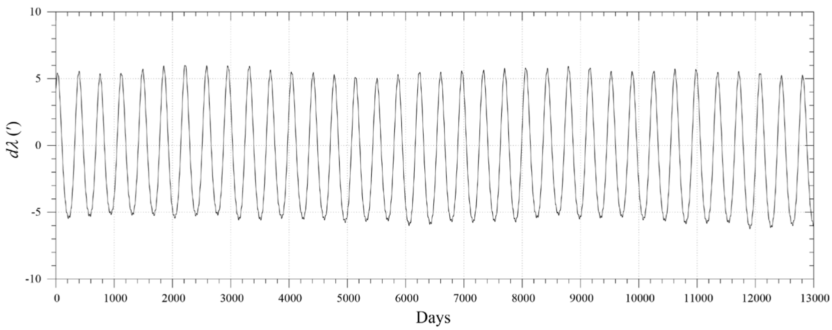
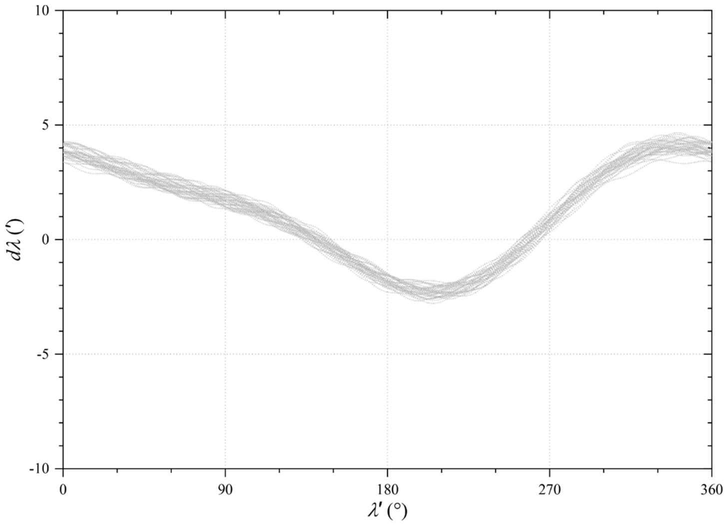
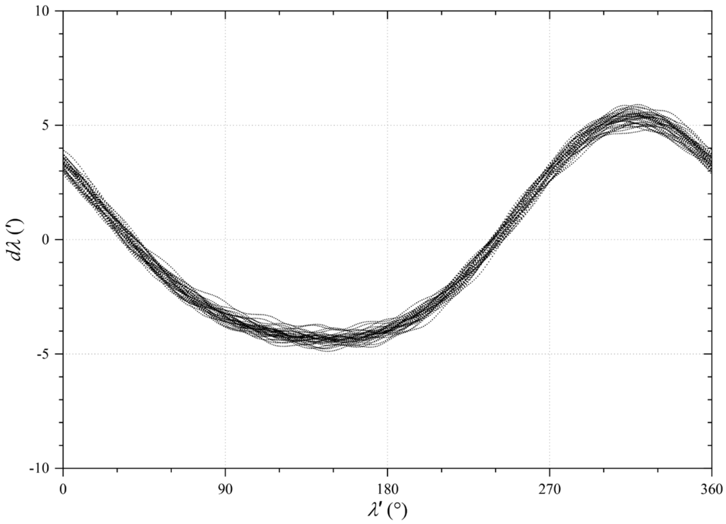


(c)

FIGURE 3: The longitudinal errors in Ibn Yūnus's solar theory for 13,000 days after 980.0 AD (a) and in its revised version used in the *Īlkhānī zīj* for 13,000 days after 1280.0 AD (b) and 1400.0 AD (c).



(a)



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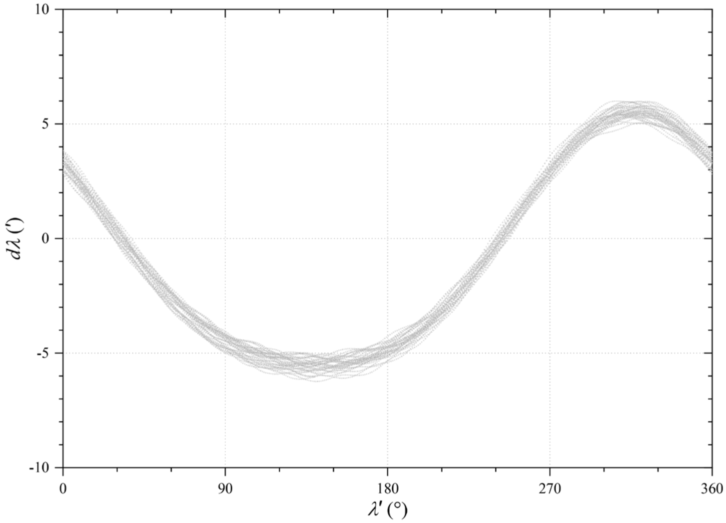
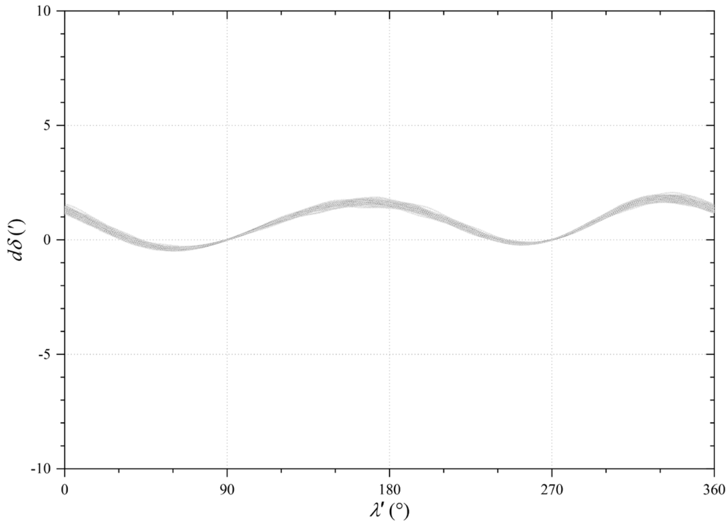
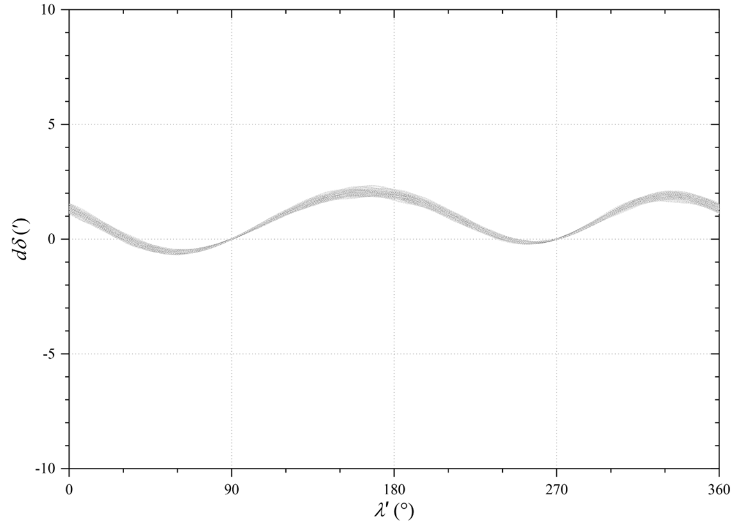


(c)

FIGURE 4: The longitudinal errors in Ibn Yūnus's solar theory for 13,000 days after 980.0 AD (a) and in its revised version used in the *Īlkhānī zīj* for 13,000 days after 1280.0 AD (b) and 1400.0 AD (c), plotted against the modern longitude values.



(a)



(b)

Figure 5: The errors in the solar declination/noon altitude in the revised version of Ibn Yūnus's solar theory as used in the *Īlkhānī zīj* for 13,000 days after 1280.0 AD (a) and 1400.0 AD (b), plotted against the modern longitude values.

inclination of the lunar orbit (i) and the argument of latitude, i.e., the difference between the lunar longitude (taken either on its inclined sphere or on the ecliptic) and the longitude of the lunar ascending/descending node ($\lambda_\beta = \lambda_{☽} - \lambda_{☊/☋}$): $s = \tan^{-1}(\tan \lambda_\beta \cos i) - \lambda_\beta$ (see Figure 6).[63] The formula intrinsically gives the proper sign for s; but if it is taken as always-positive (as it is in a medieval table), the rule is: $s \leq 0$ if $0° \leq \lambda_\beta \leq 90°$ or $180° \leq \lambda_\beta \leq 270°$, and $s \geq 0$ if $90° \leq \lambda_\beta \leq 180°$ or $270° \leq \lambda_\beta \leq 360°$. With $i = 5°$, the equation reaches the extremal values ±0;6,33° for $\lambda_\beta = 44°–46° + k \cdot 90°$ ($k = 0, 1, 2, ...$). Ptolemy considers the effect of the equation in *Almagest* IV.6 and VI.7,[64] but neglects it in his procedure for the computation of the lunar longitude, obviously because of its small size (about a quarter of the apparent angular diameter of the Moon). In medieval Islamic astronomy,

63. It is simply because of the symmetry rule in the trigonometric function that it makes no difference which node's longitude is taken into account.

64. Toomer [1984] 1998, pp. 191, 297.

it was taken into account in nearly all astronomical tables composed from the ninth century onwards. Kāshī's term, the «equation of shift», appears to have been borrowed from Muḥyī al-Dīn al-Maghribī,[65] but his table, which correctly gives the abovementioned maximum values, is unprecedented.[66] Bīrūnī's *al-Qānūn al-masʿūdī* VII.8.2 gives the extremal values ±0;6,32° for $\lambda_\beta = 44° + k \cdot 90°$;[67] and al-Khāzinī (fl. 1120 AD) in his *Muʿtabar zīj* and the astronomical tables belonging to the Maragha tradition, including Muḥyī al-Dīn's works, the *Īlkhānī zīj*, and Wābkanawī's *Muḥaqqaq zīj*, all have a table with greatest values ±0;6,40° for $\lambda_\beta = 44°–46° + k \cdot 90°$. Kāshī's table is repeated in Ulugh Beg's *Sulṭānī zīj*.[68] Kāshī first computes the distance in longitude between the Sun and the closest lunar node, $\lambda_\odot - \lambda_{☊/☋}$, at the middle of each eclipse ([Table 1], Col. **[3]**). This distance is equal to the distance between the centre of the Earth's umbra, provisionally taken as marking the longitude of the Moon with reference to the ecliptic, and the opposite node, i.e., $\lambda_{☽}* - \lambda_{☊/☋}$, $= \lambda_\odot \pm 180° - \lambda_{☊/☋}$. With the result, he computes the equation of shift ([Table 1], Col. **[4]**). It is worth noting that he did not calculate his figures simply by interpolation in his table;[69] he might have preferred to compute them accurately, or he might not have had the table when he carried out his lunar measurements. Then, the lunar longitude with reference to its inclined sphere is adjusted as $\lambda_{☽} = \lambda_{☽}* + s$ ([Table 1], Col. **[5]**). Figure 6 shows the situation for Kāshī's first lunar eclipse: at the mid-eclipse, the centre of the Earth's shadow (the highlighted circle) has a longitude of $\lambda_{☽}* = \lambda_\odot + 180°$, while the Moon is located at $\lambda_{☽}$ on its inclined orb, which is marked by drawing an arc from the centre of the shadow perpendicular to the lunar orbit.

In paragraph **[IV]**, Kāshī gives the lunar mean motion in longitude $\Delta\bar{\lambda}$ and in anomaly $\Delta\bar{\alpha}$ together with its longitudinal motions $\Delta\lambda$ in the time intervals Δt between each pair of consecutive eclipses:

65. Muḥyī al-Dīn also has «the equation of the inclined sphere of the Moon» (*taʿdīl al-falak al-māʾil*) as an alternative designation for this equation (al-Maghribī, *Talkhīṣ*, f. 83v; *Adwār*, M: ff. 83v-84r, CB: ff. 81v-82r).

66. Kāshī, *Zīj*, IO: f. 133v, P: p. 133, Q1: f. 111r, Q2: —, S: f. 97v, C: p. 230. Kāshī counts it as his 21st improvement on his predecessors' astronomical tables in the introduction of his *zīj* (IO: f. 3v, P: p. 22, Q1: f. 1v, Q2: f. 3r, S: f. 3v, C: p. 5).

67. Bīrūnī 1954-1956, Vol. 2, pp. 810, 814.

68. For a review and details, see Mozaffari 2014, pp. 94-95.

69. The interpolation in his table gives 0;1,28,57°, 0;0,52,57°, and 0;0,20,56° for the three eclipses.

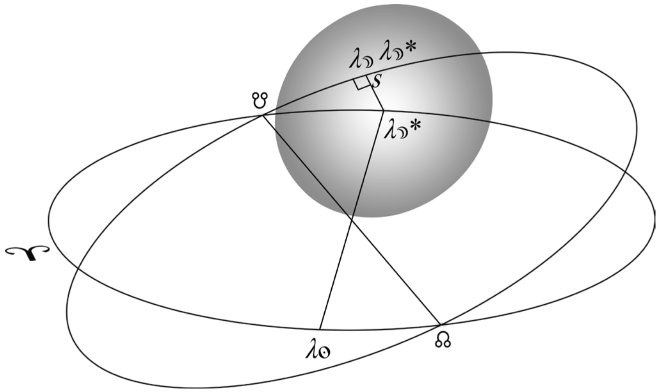


Figure 6: The relation between the Moon's positions on its inclined sphere and on the ecliptic.

	Δt	$\Delta\bar{\lambda}$	$\Delta\bar{\alpha}$	$\Delta\lambda$
1 → 2	176d 21;52,17h	171; 3,13,26°	151;20,36,10	173;18, 6,35°
2 → 3	177d 3;10, 0h	173;57,39,20	154;13,33,34	175;59,11,27

The values $\Delta\lambda$ are derived from Col. **[5]** of [Table 1]. By dividing the values of $\Delta\bar{\lambda}$ and $\Delta\bar{\alpha}$ by those of Δt, we can extract the provisional values our author used for the lunar mean daily motion in longitude, ω_λ, and in anomaly, ω_α:

$$\omega_\lambda \approx 13;10,35,\ 1,52,48^\circ \pm 2^{\mathrm{v}\circ}$$
$$\omega_\alpha \approx 13;\ 3,53,56,30,35^\circ \pm 5^{\mathrm{v}\circ}$$

The value of ω_λ is very nearly equal to Muḥyī al-Dīn's value of 13;10,35,1,52,46,45° measured at Maragha,[70] which he used in his *Adwār*. Muḥyī al-Dīn presents all fundamental motional parameter values underlying the tables of this *zīj*;[71] and it can readily be shown that Kāshī computed his values of $\Delta\bar{\lambda}$ directly by multiplying the values of Δt with Muḥyī al-Dīn's value for ω_λ, i.e., *not* by using the

70. See Mozaffari 2014, esp. pp. 82-83.

71. al-Maghribī, *Adwār*, M: f. 75v, CB: f. 73v.

mean lunar longitudinal motion table in the *Adwār*.[72] The value of ω_α is very close to the originally Babylonian value 13;3,53,56,29,38,38° in *Almagest* IV.3, which Ptolemy inherited from Hipparchus[73] (see Section 4 for an explanation of what might have been Kāshī's reason to use these values). The fact that the initial values for the mean lunar motions in longitude and in anomaly are adopted from Muḥyī al-Dīn and the Babylonians-Hipparchus respectively is the principal reason why Kāshī's final values for the mean lunar daily motions, computed later in paragraph **[IX]** (see below, Section 3.4), also remain very close to the values of his predecessors we have just mentioned.[74]

Figure 7 displays the lunar mean positions in the first (Hipparchian) lunar model in the *Almagest*, adopting a zero-eccentricity deferent, at the maximum phases of Kāshī's trio of lunar eclipses (it is drawn to scale except for the size of the lunar epicycle, which is shown four times larger for the sake of clarity). Note that in eclipse no. 2 the Moon is very near the mean epicyclic perigee, and in eclipses nos. 1 and 3, it occupies very similar positions with reference to the Earth, *T*, which explains why its nocturnal apparent paths with respect to the local horizon were very close to each other (cf. Figure 2). The difference in the mean longitude of the Moon between the maximum points of each pair of consecutive eclipses is shown as $\Delta\bar{\lambda}_{☽1,2}$ and $\Delta\bar{\lambda}_{☽2,3}$, and the difference in the true longitude of the Moon between each pair of subsequent eclipses, as $\Delta\lambda_{☽1,2}$ and $\Delta\lambda_{☽2,3}$. If we transform the lines passing through the Earth, *T*, and the Moon in eclipses nos. 1 and 3 in such a way that each of them occupies its position with respect to the mean moon (i.e., the line drawn from the Earth pointing toward the epicycle centre), then Figure 1 is produced: points *A*, *B*, and *C* sequentially stand for the mean epicyclic positions of the Moon at the times of the mid-eclipse phases of the triple eclipses with respect to the Earth, *D*. Thus,

72. For the period from the first to the second eclipse, the mean motion table in the *Adwār* (M: f. 76v, CB: f. 74v) gives: 176;27,34° (until the end of *Murdād*, the fifth month in the Yazdigird calendar, i.e., in five months of 30 days) + 342;35,11° (in 26 days) + 11;31,46° (in 21 hours) + 0;28,33° (in 52 minutes) + 0;0,9,21° (in 17 seconds, by interpolation between the entry for 52 minutes and that for 54 minutes, 0;29,39°) = 171; 3,13,21°. By a similar procedure for the time interval between the second and third eclipses, which does not need the interpolation for the seconds of time, the value of 173;57,38,0° is derived.

73. Toomer 1980, esp. p. 98; [1984] 1998, p. 179; Neugebauer 1975, Vol. 1, p. 70.

74. Kennedy (1998a, p. 5) mentions this point, without recognizing its main reason.

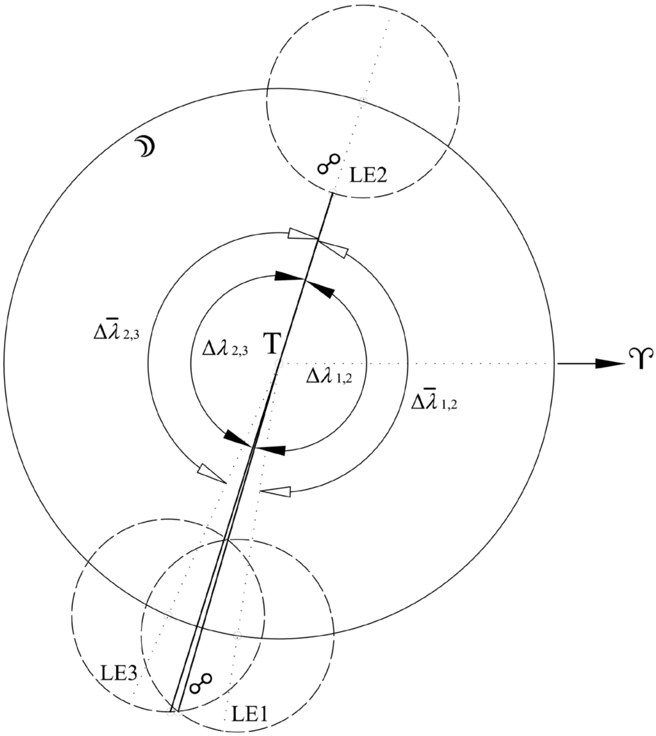


Figure 7: The mean positions of the Moon in the first (Hipparchian) lunar model in the *Almagest* at the times of the maximum phases of the trio of lunar eclipses observed by Kāshī.

arc $AB = \Delta\bar{\alpha}_{1,2}$,

arc $BC = \Delta\bar{\alpha}_{2,3}$,

$\angle ADB = \Delta\lambda_{1,2} - \Delta\bar{\lambda}_{1,2} = \Delta p_{1,2} = 2;14,53,9°$, and

$\angle BDC = \Delta\lambda_{2,3} - \Delta\bar{\lambda}_{2,3} = \Delta p_{2,3} = 2;1,32,7°$,

where p stands for the equation of anomaly (or epicyclic equation) of the Moon ($\angle ADL$, $\angle BDL$, and $\angle CDL$). In paragraph **[V]**, Kāshī states that his figure is the result of the combination of three figures from *Almagest* IV.6. This chapter has seven figures: figures 1, 3, and 4 are related to the trio of the ancient Babylonian lunar eclipses, while figures 5, 6, and 7 represent Ptolemy's triple lunar eclipses observed at Alexandria; each set of three figures shows how Ptolemy gradually put his derivation in a pictorial form. In figure 2, Ptolemy demonstrates his method for determining the epicycle radius on the basis of the eccentric hypothesis.[75] It is surprising that the second figure does not exist in the surviving manuscripts of the two ninth-century Arabic translations of the *Almagest* by Ḥajjāj b. Yūsuf b. Maṭar in 827-828 AD and Isḥāq b. Ḥunayn in 880-890 AD, which was later revised by Thābit b. Qurra (d. 901 AD),[76,77] Kashī's lettering in his figure is also identical to Ptolemy's.

The problem is to find the radius of the lunar epicycle $r = KS$ in terms of the units of which the radius of the deferent $R = KD$ is taken as 60^p, so that the direct distances between the epicyclic positions occupied by the Moon at the times of the maximum phases of the three lunar eclipses are subtended by $\angle ADB$ and $\angle BDC$ as seen from the Earth. The procedure, as explained in paragraph **[VI]**, is as follows. The recomputed values are given within square brackets, and the erroneous digits in Kāshī's numbers are shown in bold-italics.

In [Figure 1], we have: arc $CA = 360° - (\Delta\bar{\alpha}_{1,2} + \Delta\bar{\alpha}_{2,3}) = 54;25,50,16°$, corresponding to the epicyclic equation $\angle ADC = \Delta p_{1,2} + \Delta p_{2,3} = 4;16,25,16°$. Sin $\angle ADC$ = 4;28,16,***32*** [...,$\underline{29}$]. $\angle AEC = 1/2$ arc $CA = 27;12,55,8°$. According to Euclid I.32,[78] in ~+*CED*, $\angle ECD = \angle AEC - \angle ADC = 22;56,29,52°$ (NB. $\angle AEC = 180° - \angle CED$). Sin $\angle ECD$ = 23;23,15,21 [...,$\underline{19}$]. If we take $DE = 60^a$, where «^{a}» stands for a unit:

$$CE = DE \cdot \text{Sin } \angle ADC \,/\, \text{Sin } \angle ECD = 11;28,1\boldsymbol{5},\boldsymbol{3}^{a} \qquad [\ldots,1\underline{4},\underline{55}].$$

Similarly, $\angle AEB = 1/2\,(360° - \text{arc } AB) = 104;19,41,55°$. So, in $\triangle EBD$, $\angle EBD = \angle AEB - \angle ADB = 102;4,48,46°$. Sin $\angle ADB$ = 2;21,12,5***6*** [...,5$\underline{7}$]. Sin $\angle EBD$ = 58;40,16,48. Thus:

75. See Toomer [1984] 1998, pp. 193, 194, 196, 197, 199, 202, and 203.

76. *Arabic Almagest*: Isḥāq-Thabit: (1) Pa1: f. 74v, Pa2: f. 70r, TN: f. 60v, S: f. 47r; (3) Pa1: f. 76r, Pa2: f. 71v, TN: f. 61r, S: f. 47v; (4) Pa1: f. 77r, Pa2: f. 72r, TN: f. 61r, S: f. 48r; (5) Pa1: f. 78v, Pa2: f. 74r, TN: f. 63r, S: f. 49r; (6) Pa1: f. 80r, Pa2: f. 74v, TN: f. 63r, S: f. 49v; (7) Pa1: f. 81v, Pa2: f. 75r, TN: f. 63v, S: f. 50v (the section is not extant in MSS. E1, E2, LO1, and PN). Ḥajjāj: (1) LO2: f. 91v, LE: f. 58r; (3) LO2: f. 92r, LE: f. 58v; (4) LO2: f. 92v, LE: f. 59r; (5) LO2: f. 95v, LE: f. 60r; (6) LO2: f. 96r, LE: f. 60v; (7) LO2: f. 96v, LE: f. 61r.

77. The second figure is also absent from the other commentaries related to the *Almagest*, most notably, al-Ṭusī's *Taḥrīr al-majisṭi* (*Exposition of the* Almagest), P1: pp. 130, 132, 133, 134, 135, P2: ff. 35r-36r, P3: ff. 55r-56r, to which Kāshī refers in the prologue of his *zīj* (IO: f. 2v, P: p. 21, Q1: f. 1r, Q2: f. 2v, S: f. 3r, C: p. 4).

78. *Elements*, pp. 19-20.

$BE = DE \cdot \sin \angle ADB / \sin \angle EBD = 2;24,24,\boldsymbol{48}^{a}$ [...,$\underline{50}$].

Also, $\angle CEB$ = 1/2 arc BC = 77;6,46,47°. $\sin \angle CEB = 58;29,19,2\boldsymbol{2}$ [...,2$\underline{1}$], and $\cos \angle CEB = 13;22,54,1\boldsymbol{4}$ [...,1$\underline{6}$]. Thus, in $\triangle BET$:

$BT = BE \cdot \sin \angle CEB / R = 2;20,46,3\boldsymbol{3}^{a}$ [...,3$\underline{5}$]
$ET = BE \cdot \cos \angle CEB / R = 0;32,12,30^{a}$

Consequently, $CT = CE - ET = 10;56,2,\boldsymbol{33}$ [...,$\underline{25}$]. Thus,

$BC = \sqrt{(CT)^2 + (BT)^2} = 11;10,58,\boldsymbol{36}^{a}$ [...,2$\underline{8}$]

In addition:

$BC = \mathrm{Crd}(\text{arc } BC) = 116;58,38,4\boldsymbol{4}^{b}$ [...,4$\underline{2}$]

in terms of the unit «^{b}» of which the epicycle radius $r = KS = 60^{b}$. Together, the last two equations provide us with a scale to transform all the lengths measured so far in terms of the unit «^{a}» to «^{b}». First of all:

$DE = 627;37,\boldsymbol{13},\boldsymbol{55}^{b}$ [...,21,7]

And

$BE = 25;10,3\boldsymbol{6},\boldsymbol{46}^{b}$ [...,3$\underline{7}$,$\underline{22}$],

from which arc $BE = 24;13,\boldsymbol{19},\boldsymbol{42}°$ [...,$\underline{20}$,$\underline{14}$]. Thus, arc AE = arc AB – arc BE = $127;7,1\boldsymbol{6},\boldsymbol{28}°$ [...,1$\underline{5}$,$\underline{55}$]. Hence,

$AE = \mathrm{Crd}(\text{arc } AE) = 107;26,55,\boldsymbol{18}^{b}$ [...,$\underline{7}$]

Since $AE < 120^{b}$, the centre of the epicycle, by assuming point K, should be located outside the circular segment *ANES*. $AD = AE + DE = 735;4,9,13^{b}$ [...,$\underline{16}$,$\underline{14}$]. On the basis of Euclid III.36 (text: 35),[79] we have:

79. *Elements*, pp. 63-66.

$$MD \cdot LD = ED \cdot AD = 2{,}8{,}9{,}\boldsymbol{4;32{,}22{,}14{,}49{,}15{,}55}^{b2} \qquad [\ldots,\underline{7};\underline{13},\underline{58},\underline{33},\underline{48},\underline{47},\underline{38}]$$

Further, from Euclid II.6:[80]

$$DK^2 = MD \cdot LD + r^2$$

in which $r = 1;0,0^b$. Thus:

$$DK = 681;52{,}\boldsymbol{6{,}18}^{b} \qquad [\ldots,\underline{13},\underline{25}].$$

Therefore, if the radius of the orbit $DK = 60^p$, the length of the epicycle radius is derived from the last result as equal to

$$r = 60 \cdot 60 \,/\, DK = 5;16{,}46{,}3\boldsymbol{6}^{p} \qquad [\ldots,3\underline{3}].$$

As can be seen, all the errors Kāshī committed in the process of calculation, some of which must have arisen from the sine table applied, led to a negligible total error in the final result of just $+3^{iii}$. With the above value for r, he computes the maximum value of the lunar epicyclic equation at syzygies as 5;2,53° (opposite argument 95°) and at quadratures as 7;42,***19***° [7;42,<u>27</u>°] (= the sum of the tabular values of 5;2,26° and 2;39,53° for argument 98°). Note that Kāshī did not measure the lunar eccentricity, but adopts Ptolemy's value of $10;19^p$ (corresponding to the maximum value of 13;8° for the equation of centre tabulated opposite arguments 113°–115°).[81]

3.4. The derivation of the mean motions in longitude and anomaly.

Kāshī first computes (in paragraph **[VII]**) the lunar mean positions in his second lunar eclipse according to his geometrical schema presented in Figure 1, as explained in what follows.

In the first step, we want to compute the mean anomaly of the Moon at the middle of the second lunar eclipse on the basis of the Hipparchian model with a

80. *Elements*, p. 33.

81. The lunar equation tables: Kāshī, *Zīj*, IO: ff. 132v-133v, P: pp. 132, 133, 135, Q1: ff. 109v-110r, 111r, Q2: —, S: ff. 96v-97v, C: pp. 227-228, 230.

zero eccentricity deferent, i.e., $\bar{\alpha}_2^*$ = arc *LEB*. We have drawn *KNS* ⊥ *AE* (in the text, we are told to join *AK*, which is unnecessary, as it was not drawn in any of the consulted MSS). On the basis of Euclid III.3,[82] *AN* = *EN*. *DN* = *DE* + 1/2 *AE* = 681;20,4***1***,***34***^{b} [...,4$\underline{8}$,$\underline{41}$] = 59;57,14,***10***^{p} [...,$\underline{9}$]. *DN* = Sin ∠*SKM*. So, ∠*SKM* = arc *SEM* = 87;45,17,2° [...,1$\underline{6}$,$\underline{31}$]. Thus,

arc *LS* = 180° – arc *SEM* = 92;14,4***2***,***58***° [...,4$\underline{3}$,$\underline{29}$].

Also, from the values we have already computed for arcs *AE* and *BE*, we have:

arc *SB* = 1/2 arc *AE* + arc *EB* = 87;46,5***7***,***56***° [...,5$\underline{8}$,$\underline{12}$].

As a result,

$\bar{\alpha}_2^*$ = arc *LEB* = arc *LS* + arc *SB* = 180;1,4***0***,***54***° [...,4$\underline{1}$,$\underline{41}$].

Next, we want to know the mean longitude of the Moon at the maximum point of the second lunar eclipse $\bar{\lambda}$☽$_2$. To this end, we first need to calculate the epicyclic equation at the time, i.e., p_2 = ∠*LDB*, and then subtract it from the longitude of the Moon at the time, λ☽$_2$, already given in [Table 1], Col. [5]. We have calculated ∠*SKD* earlier. Hence, the epicyclic equation of the Moon at the middle of the first lunar eclipse is computed in △*SKD* as $|p_1|$ = ∠*ADK* = 90° – ∠*SKD* = 2;14,4***2***,***58***° [...,4$\underline{3}$,$\underline{29}$] (we know that it is subtractive/negative). Also, $\Delta p_{1,2} = p_2 - p_1$ = ∠*ADB* = 2;14,53,9°. Thus,

$p_2 = \Delta p_{1,2} - |p_1|$ = +0;0,***10***,***11*** [...,$\underline{9}$,$\underline{40}$].

As a result,

$\bar{\lambda}$☽$_2$ = λ☽$_2 - p_2$ = 72;14,36,7° [...,38]. (1)

In the third, and final, stage, we want to adjust $\bar{\alpha}_2^*$ with the equation of centre according to Ptolemy's lunar model. To do so, we should first derive the mean longitude of the Sun at the time, $\bar{\lambda}$⊙$_2$, which Kāshī gives as 252;52,28,57°; we know

82. *Elements*, p. 43.

that it has been derived from the *Īlkhānī zīj*. Consequently, the *centrum*, i.e., the mean double elongation, is computed as $2(\bar{\lambda}_{\text{☽}2} - \bar{\lambda}_{\odot 2}) = 358;44,14,20^\circ$. Kāshī gives the equation of centre for this value as $q_2 = +0;11,6,45^\circ$. Therefore, the mean anomaly of the Moon would amount to:

$$\bar{\alpha}_2 = \bar{\alpha}_2^* + q_2 = 180;12,4\boldsymbol{7},\boldsymbol{39}^\circ \qquad [\ldots,4\underline{8},\underline{26}] \qquad (2)$$

The date and time of his second lunar eclipse is equal to

t_2 = 774 years 355 days 13;13,5 absolute hours or 12;48,46 equated hours

reckoned from the beginning of the Yazdigird era for the longitude of Kāshān, as already given in **[II]** (see Table 3).

Kāshī requires a set of trustworthy data for the lunar mean longitudinal and anomalistic positions at a time sufficiently far from his own to be compared with his derived quantities. To do so, he chooses Ptolemy's second lunar eclipse presented in *Almagest* IV.6, which took place on 2/3 Choiak (the fourth month) 19 Hadrian (= 882 Nabonassar) = 20/21 October 134 (JDN 1770294/5); Ptolemy gives the time as 1 equinoctial hour before midnight (actually, ~ **22:42** MLT), and correctly estimates the magnitude as 5/6 of the apparent diameter of the lunar disk (~ **0.8**) from the north.[83] The beginning of the Yazdigird era is 16 June 632 (JDN 1952063). So, the date and time of the eclipse would be equal to 181769 days minus 11 hours or

t_1 = 497 years 363 days 13 absolute hours (at the longitude of Alexandria)

before the Yazdigird era, as Kāshī precisely calculates. He converts the time to the local meridian of Kāshān as

11;23,36 absolute hours,

by taking a longitudinal difference of 24;6° between the two cities (error ~ +2;35°), corresponding to a difference of 1;36,24 hours between their local times

83. 5MCLE: #05156. Toomer [1984] 1998, p. 198. On this eclipse, see Steele 2000a, pp. 102-103; 2000b, pp. 93, 103-104.

(the table of the geographical coordinates in the *Khāqānī zīj* gives $L = 61;54°$ for Alexandria and $L = 86;0°$ for Kāshān).[84] Ptolemy's data quoted by Kāshī are:

$$\begin{array}{ll} \text{the mean longitude} & \bar{\lambda} = 29;30°, \text{ and} \\ \text{the } \textit{equated} \text{ anomaly} & \alpha = 64;38°.^{85} \end{array} \qquad (3)$$

Ptolemy uses these data for comparison with the corresponding quantities he derived from the second of the three ancient Babylonian lunar eclipses mentioned at the beginning of IV.6 in order to correct the mean lunar motions in IV.7. Ptolemy never says that the latter quantity is the *equated* (not *mean*) anomaly. In *Almagest* IV.6, he works with the simple Hipparchian lunar model with a zero-eccentricity deferent, according to which the Moon is assumed to have only one inequality; the difference between the mean and true epicyclic apogee/perigee is introduced later in V.5, but Ptolemy never returns to his derivation of the mean lunar motions in order to revise it on the basis of his completed lunar model. In contrast, Kāshī intends to compare the two values for the mean lunar epicyclic anomaly in order to derive the velocity of its mean anomalistic motion. To do so, he needs to compute the mean lunar anomaly at the time of the maximum phase of Ptolemy's second lunar eclipse, just as he did for the maximum point of his second lunar eclipse earlier. He thus adds another quantity to Ptolemy's data: the double mean elongation 2= 3;38°, from which he calculates the equation of anomaly $q = -0;32,40°$.[85] Therefore,

84. Kāshī, *Zīj*, IO: f. 73r, P: p. 102, Q1: f. 67r, Q2: f. 28r, S: f. 55r, C: f. 131r.

85. Entering the double elongation = 3;38° in the table of the first lunar equation in the early version of the *Khāqānī zīj* (S: f. 96v; with $q(3°) = 0;26°$ and $q(4°) = 0;35°$) gives $q = 0;31,42°$ and that in the final edition (IO: f. 132v, P: p. 135, Q1: f. 109v, Q2: —, C: p. 227): 0;31,58° (NB. $q(3°) = 0;26,24°$ and $q(4°) = 0;35,12°$). From Ptolemy's table of the lunar equations in *Almagest* V.8 (Toomer [1984] 1998, p. 238): 0;32,6°. None of these values agrees with the value of 0;32,40° given in the text. In the *Īlkhānī zīj* (C: p. 73, P: f. 25v, M: f. 44v, T: f. 34v, B: f. 38r, F: f. 33r, L: f. 40r, Fl: f. 36v, O: f. 37v, Ca: f. 41v), the tables of the lunar equations are drawn up at intervals of 0;12° and are asymmetrical and displaced. In the table of the lunar first equation, an additional value of 13;8° was added to all entries in order to make the table user-friendly and always-additive. The tabular entries for the arguments of 3;36° and 3;48° are respectively 13;40° (i.e., 0;32°) and 13;42° (i.e., 0;34°); thus, $q(3;38°) = 0;32,20°$, which is also incompatible with Kāshī's value. However, the entries for the arguments of 3° and 4° are respectively 13;35° (read: 0;27°) and 13;44° (read: 0;36°), which yields $q(3;38°) = 0;32,42°$, very close to Kāshī's result.

the mean anomaly $\bar{\alpha} = 64;5,20°$. (4)

The mean lunar longitude of 29;30° with the double mean elongation of 3;38° clearly indicates that our author took the mean solar longitude at the time of the maximum phase of Ptolemy's second lunar eclipse to be 207;41°. It is curious that the latter value exceeds by about a single degree the value of 206;42° that can be derived from Ptolemy's mean solar motion tables, and from which he must have computed his value of 205;10° for the true longitude of the Sun at the time (modern value: **206;24**°). All this may incidentally imply that Kāshī was aware of the error of ~ –1° in Ptolemy's solar theory. In all likelihood it seems that he found this error by making a simple comparison between the mean longitude of the Sun as computed from the *Almagest* and the value given by the solar theory used in the *Īlkhāni zīj*: computing backwards in time from the solar tables in the latter work a value of 207;48° is obtained, which is not too far from the value of 207;41° that Kāshī used in practice.

The time interval between the maximum point of Ptolemy's second lunar eclipse and that of Kāshī's second lunar eclipse amounts to

Δt = 1272 years 354 days 0;36,41 absolute hours or 0;46,23 equated hours.

Kāshī does not give the time of Ptolemy's second lunar eclipse in equated hours, but from the above value it is easy to deduce that he took the mean time of Ptolemy's second lunar eclipse to be 11;57,37 hours, and thus, he must have adopted the equation of time E = +0;34,1 hours for this time.

The difference in the lunar mean longitude ($\Delta\bar{\lambda}$) is 42;44,36,7° in the period between Ptolemy's and Kāshī's lunar eclipses (the difference between (1) and (3)), during which the Moon also performed 17006 complete revolutions around the Earth. By dividing the accumulated lunar motion in longitude into the above time span in days, its daily rate ω_λ is computed (to the seventh sexagesimal fractional place) as

$$\omega_\lambda = \Delta\bar{\lambda} / \Delta t = 13;10,35,1,52,47,50,50°.$$

Analogously, the mean lunar motion in anomaly (Δ) in the course of the period since Ptolemy's day is equal to 16862 complete revolutions plus 116;7,27,39° (the difference between (2) and (4)), and so the daily mean lunar anomalistic motion is derived as

$$\omega_\alpha = \Delta\bar{\alpha}/\Delta t = 13;3,53,56,30,37,19,59^\circ \text{ or } ...37,20^\circ,$$

as Kāshī rounds it to the sixth sexagesimal fractional place.

The value of ω_λ is larger than Muḥyī al-Dīn's corresponding value, which Kāshī has deployed as his provisional value, by the trivial amount of $\sim 1^{v}$. Similarly, the value of ω_α is more than Hipparchus' corresponding value, which Kāshī uses as his initial estimation, again by the small value of $\sim 1^{iv}$.

From the two values for ω_λ and ω_α, he computes in sequence the mean motions in one month (30 days), one Egyptian/Persian year (of 365 days), 100 years, and 600 years, and also derives his radix positions for the beginning of 781 Y (= 4 December 1411, JDN 2236763). His base meridian is Shīrāz, which is located 2° (actually, ~ 1°) east of Kāshān. So the difference in local time between the two zones reaches about eight minutes. Therefore, the time span between the maximum point of lunar eclipse no. 2 and the epoch is 5 years + 9 days + 11;3,14 hours. From [Table 2] in **[IX]**, the mean motion in longitude in this interval is derived as $51;34,55,1,45,23,55,\underline{55}^\circ$ and the mean motion in anomaly as $207;11,52,9,56,13,1\underline{4}^\circ$. Kāshī made a minor mistake in the computation of the last fractional sexagesimal place of both. These quantities should be added to the corresponding mean positions at the time of the maximum phase of the second lunar eclipse (as given above, (1) and (2)) in order to yield the radices:[86]

the epoch mean longitude	$\bar{\lambda}_0 = 123;49,31,\ 8,45,23,55,\underline{55}^\circ$ and
the epoch mean anomaly	$\bar{\alpha}_0 = 27;24,39,48,56,13,1\underline{4}^\circ$.

Remark: As already mentioned in Section 3.2, in the prologue to his *zīj*, Rukn al-Dīn al-Āmulī offers some improvements on the *Īlkhānī zīj* ascribed to Quṭb al-Dīn al-Shīrāzī, and states that the latter wrote his corrections in the form of scattered notes in the margins of the *Īlkhānī zīj*. He also states that «in this era [...] before Ulugh Beg established the observatory at Samarqand», *only* the correction in the mean longitude of the Moon (+30′) was taken into account in the *Īlkhānī zīj*. This correction can also be found in a marginal gloss left in the upper right-hand corner of the table of the mean lunar yearly positions on folio 31v of an early fourteenth-century manuscript of the *Īlkhānī zīj*, which is designated by the siglum T in

86. The mean lunar motion tables: Kāshī, *Zīj*, IO: ff. 127v, 128v-130r, P: pp. 107, 109-112, Q1: ff. 104v, 105v-107r, Q2: —, S: ff. 92v, 93v, 94r-v, C: pp. 217, 219-222.

the bibliography. The table contains the lunar mean positions from 703 Y (1333 AD) to 803 Y (1433 AD) (see Figure 8), and is different from that in the original version, which was drawn for the period 601-701 Y (1232-1331 AD). In fact the correction was *not* applied to the table, because we can, indeed, find the values expected from the epoch positions and mean motions used in the *Īlkhānī zīj* in each entry without any addition. Thus, the comment appears to be an *instruction* for practitioners to take this correction into account, but since it is in the past tense, no one can recognize it without additional effort or without having access to the original version of the *Īlkhānī zīj*. Further, there are two comments in the left margin, in which we are correctly told that the table is based on the radix and parameter values (*uṣūl*) of the *Ḥākimī zīj*.[87] What is curious is that Kāshī's radix value $\bar{\lambda}_0$ given above is nearly +30′ larger than the value extracted from the *Īlkhānī zīj* (123;19,8°). It seems that, by a mere coincidence, Kāshī's lunar measurements confirmed an already known correction to the *Īlkhānī zīj* and then, perhaps, justified its continued use. However, neither value is better than the other (modern: **123;36**°).

FIGURE 8: The table of the mean lunar annual positions in MS. T of the *Īlkhānī zīj* (f.31v). The marginal comment in the upper right-hand corner reads: «we added thirty minutes [of arc] to the mean longitude of the Moon». The glosses in the left-hand margin read: upper: «this [table] was considered with caution; it is on the basis of the parameter values (*uṣūl*) of the *Ḥākimī* [*zīj*]»; lower: «this [table] is, indeed, [on the basis of] the parameter values of the *Ḥākimī* [*zīj*]».

87. See Mozaffari 2014, pp. 110, 112.

4. Discussion and Conclusion

We are now in a position to judge Kāshī's abilities and practical skills as an observational astronomer. We know that the time of the maximum phase of his second lunar eclipse is the only evidence in this respect, because, as we have seen in Section 3.1, he could not have determined the corresponding times of his two other lunar eclipses without any recourse to theory, and so these cannot be taken as pure empirical data. He committed an error of only ~ –8 minutes in the time of the maximum phase of the second eclipse (see Table 3), which is of the same order of magnitude as those in Muḥyī al-Dīn's eclipse times (≤ +5 minutes). This, indeed, demonstrates that he must have been competent in making tolerably precise observations, and that the simple timekeeping devices available to him at the time (the water and sand clocks mentioned in Section 3.1) had a high degree of accuracy. The errors in his two other eclipses compare favourably with those in Taqī al-Dīn's eclipse timings (~ –1 hour).[88]

We have seen above that Kāshī measured the radius of the epicycle and mean motions of the Moon. His value for the first is only slightly larger than Ptolemy's $5;15^p$. The lunar inequality at syzygies amounts to about $\pm 5;1^\circ$,[89] which can be produced from Ptolemy's value. It is significant that Kāshī's value of ~ $5;17^p$ is more precise than the Banū Mūsā's $5;22^p$, Ibn al-Aʿlam's $5;4^p$, Ibn Yūnus' $5;1^p$, Jamāl al-Dīn Muḥammad b. Ṭāhir b. Muḥammad al-Zaydī's $5;3^p$ (a Persian astronomer from Bukhārā and the first director of the Islamic Astronomical Bureau founded at Beijing in 1271 AD by the Mongolian Yuan dynasty of China), Ibn al-Shāṭir's $5;10^p$, Taqī al-Dīn's ~ $5;24^p$, and the values of ~ $5;12^p$ obtained three times by Bīrūnī, Muḥyī al-Dīn, and Ulugh Beg and his team of astronomers at Samarqand (probably, with a contribution made by Kāshī himself).[90]

88. See Mozaffari and Steele 2015, p. 355.

89. See Neugebauer 1975, Vol. 3, pp. 1106-1107.

90. For the sources of the values quoted above, see Mozaffari 2014, pp. 105-106; Mozaffari and Steele 2015, p. 348. For Jamāl al-Dīn's table of the lunar equation of anomaly, see Sanjufīnī, f. 37r (the maximum value in the table, $4;50^\circ$, is only by a single arc-minute less than Ibn al-Aʿlam's corresponding value of $4;51^\circ$). Ibn al-Shāṭir constructed a lunar model consisting of a hypocycle (called «the rotator, *al-mudīr*» or «the carrier of the [Moon's] body, *ḥāmil al-jirm*») of radius $r_1 = 1;25^p$, whose centre rotates on the circumference of an epicycle of radius $r_2 = 6;35^p$. The Moon is located in the perigee of the rotator at syzygies and in its apogee at quadratures, so that the linear distance between the Moon and the centre of the epicycle (which Ibn al-Shāṭir calls, in a general sense, «the

We present the values of ω_λ and ω_α measured by Ptolemy and the eminent Islamic astronomers[91] in Figures 9 and 10 respectively (as the filled circles •), along with the corresponding graphs drawn on the basis of modern theories.[92] The limited objectives of the present paper do not permit us to analyse them in depth. It suffices to say at this point that in the case of ω_λ, all the values measured by the Islamic astronomers (mostly, with errors of the order of 10^{-6}) are notably more accurate than Ptolemy's (with an error of the order of 10^{-5}). Also, Muḥyī al-Dīn's value in the *Tāj al-azyāj* (*Crown of the zījes*, which he wrote in Syria before joining the Maragha team) has the highest degree of precision (with an error of the order of 10^{-7}). Furthermore, and more importantly for the present study, it can clearly be seen that Kāshī's preference for Muḥyī al-Dīn's value in the *Adwār* rather than Ibn Yūnus' value in the *Īlkhānī zīj* is wholly justified. In contrast, Muḥyī al-Dīn's value of ω_α in the *Adwār* is a clear outlier.[93] The other values of ω_α are very nearly of the same degree of accuracy, and the Babylonian-Hipparchian value is slightly better than the others. It is very interesting that Kāshī appears to have found the latter safer and more reliable for provisional use in his lunar measurements than all the other values available to him from Ptolemy and his Muslim predecessors.

Finally, a note concerning the relation between the astronomical observations and the development of theoretical astronomy in the late Islamic period is in order. As is well known, some members of the Maragha team, including al-Ṭūsī, Mu'ayyad al-Dīn al-ʿUrḍī (1200-1266 AD) and Quṭb al-Dīn al-Shīrāzī (1236-1311 AD), devised non-Ptolemaic lunar models for the purpose of resolv-

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apparent radius of the epicycle») varies between 5;10^p (corresponding to the epicyclic equation of 4;56,24°) and 8;0^p (corresponding to the anomalistic equation of 7;39,45°). The first value is identical to the epicycle radius in the Ptolemaic model (see the references cited in notes 96-98 below). It is curious that the same value is found in some Indian and pre-Islamic Persian sources, was in use by the Iranian astronomers in the earliest stages of the rise of astronomy in Islam (see Pingree 1968, p. 104; 1970, p. 112; Chabás and Goldstein 2003, pp. 252-253), and is deployed in both the Castilian and Parisian versions of the *Alfonsine Tables* (see Chabás and Goldstein 2003, p. 157; 2004, pp. 225-226).

91. Most of these values are due to the meticulous work of Dr. Benno van Dalen who derived them from the lunar mean motion tables in the various *zījes*.

92. On the basis of the formulae given in Meeus 1998, p. 338.

93. In the case of Muḥyī al-Dīn's value in his *Adwār*, the error is solely due to a serious error he made in the final step of calculation: the division of the accumulated mean anomalistic motion by the time interval (see Mozaffari 2014, p. 84). He has a precise value for ω_α in his earlier work, the *Tāj*, as illustrated in Figure 10.

ing the serious difficulties in Ptolemy's model arising from the latter's negligence of certain philosophical facts: the two blatant contradictions were the uniform motion of the lunar eccentric around the Earth, instead of its centre, and the prosneusis point. Nevertheless, none of these models replaced Ptolemy's in the calculation of the lunar positions and eclipse predictions. The construction of alternative lunar models was continued and elaborated a century later by Ibn al-Shāṭir (1306-1375/1376 AD). He not only objects to Ptolemy's lunar model, but is also critical of those of his Maragha predecessors in the second chapter of the prolegomenon to Book I of his *Nihāyat al-su'l fī taṣḥīḥ al-uṣūl* (*A final inquiry on the rectification of* [*astronomical*] *hypotheses*);[94] he afterwards puts

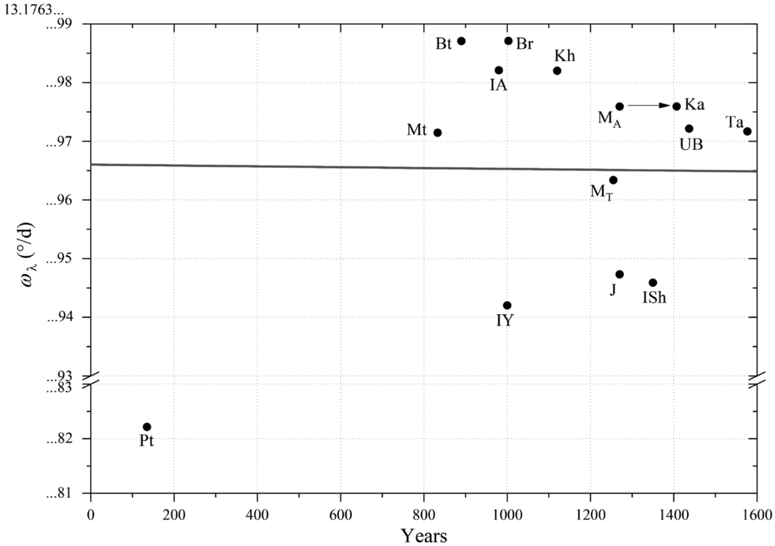

Figure 9: A number of values measured for the mean motion of the Moon in longitude in the medieval Islamic period: Mt = Mumtaḥan team, Bt = al-Battānī, IA = Ibn al- Aʿlam, IY = Ibn Yūnus, Br = Bīrūnī, Kh = al-Khāzinī, M = Muḥyī al-Dīn al-Maghribī (the subscript «A» denotes his *Adwār al-anwār*, and the subscript «T» stands for his *Tāj al-azyāj*), J = Jamāl al-Dīn Muḥammad b. Ṭāhir b. Muḥammad al-Zaydī of Bukhārā, ISh = Ibn al-Shāṭir, Ka = Kāshī, UB = Ulugh Beg, and Ta = Taqī al-Dīn, together with Ptolemy's (Pt), are shown as filled circles (●), along the graph of this parameter drawn on the basis of modern theories, with the smoothly *descending* curve in the middle.

94. Ibn al-Shāṭir, *Nihāya*, O1: ff. 3r-5r, O2: ff. 3r-5r, O: f. 22r-23v. The whole text was edited and translated into French in Penchèvre 2017.

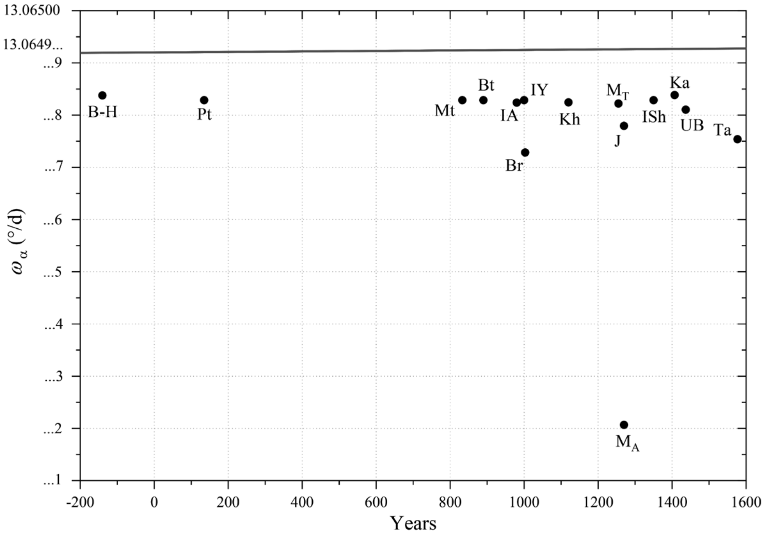

FIGURE 10: A number of values measured for the mean motion of the Moon in anomaly in the medieval Islamic period, together with Ptolemy's and the Babylonian-Hipparchian (B-H) values (for the abbreviations, see the caption to Figure 9). The modern graph of this parameter is displayed as the smoothly *ascending* curve.

forward in I.9-11[95] his double-epicyclic concentric model, one of whose characteristic features was to resolve a serious empirical failure in Ptolemy's model. Whereas Ptolemy's variation in the Moon's distance from the Earth entails that its apparent angular diameter at quadratures is almost twice as large as that at oppositions, Ibn al-Shāṭir reduced this ratio to a reasonable value less than 1.3.[96] What distinguishes Ibn al-Shāṭir from his Maragha precursors is the fact that he puts his model in practical use for positional astronomy in his *Jadīd zīj*.[97] As we have seen in this study, Kāshī based his astronomical tables on Ptolemy's

95. Ibn al-Shāṭir, *Nihāya*, O1: ff. 13r-19r, O2: ff. 14r-21r, O3: ff. 29v-34r.

96. For a survey of these models, see Saliba 1996, pp. 92-104 and, especially, for Ibn al-Shāṭir's one, see Roberts 1957; Saliba 1987.

97. Ibn al-Shāṭir, *Zīj*, Section 6 («On the knowledge of the longitude of the Moon»): K: ff. 18r-v, O: ff. 23v-24r, L1: ff. 17r-v, L2: ff. 21v-22r, PR: ff. 23r-v; Section 59 («On the knowledge of the Moon's distance from the Earth»): K: ff. 37v-38v, O: ff. 85v-86v, L1: ff. 39r-40r, L2: ff. 49r-50r, PR: ff. 49v-50r; Lunar equation tables: K: ff. 55v-57v, 125r-v, O: ff. 35v-40r, L1: ff. 55r-57r, L2: ff. 71r-73r, PR: —.

models, which reinforces the opinion that he was not acquainted with Ibn al-Shāṭir's legacy. It is curious to note that the Samarqand observers some years later, and Taqī al-Dīn about two centuries afterwards, also adhered to Ptolemy's lunar model in their lunar measurements.

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Appendix: Edition of the Persian Text

We have not noted minor and trivial variants. Most of them are:

(1) simple orthographical differences in writing Persian and Arabic words;

Examples: the conjunction *kih*/که («that») is written as *kiy*/کی in MS. S. ‖ *mā'il*/مائل and *māyil*/مایل («inclined») ‖ *siyum*/سیم, *siyu*^{w}*m*/سیوم, *siwum*/سوم («third») ‖ *si*^{h}*gānih*/سه گانه, *sigānih*/سگانه («triple») ‖ *hamčunān*/همچنان, *hamčunīn*/همچنین («also») ‖ *ānkih*/آنکه and *ānk*/آنک («that»): in only one place in the text; the latter form can be found in MSS. S and L as well as in MS. IO. ‖ *činānkih*/چنانکه and *činānk*/چنانک («as»): the latter form only occurs in MS. S and in a single place in MS. IO. ‖ etc.

(2) due to the confusion over writing the Persian words in their original or Arabicized forms;

Example: the date of the third eclipse, for which MS. IO has *hiždahum*/هژدهم, while the other MSS have *hijdahum*/هجدهم.

and so on. Dropped words are noted only if they are absent from more than one manuscript.

سخن در تصحیح اوساط قمر از رصد خسوفات

سه خسوف رصد کردیم در بلدهٔ کاشان و اوساط قمر از آن استخراج کردیم، چنانکه بطلمیوس کرده است، الّا آنکه او مرکز تدویر را در خسوفات در سطح فلک البروج فرض کرده است و نظیر تقویم آفتاب در وسط خسوف تقویم قمر گرفته است. و ما موضع تقاطع منطقهٔ مائل با عظیمهٔ که بمرکز ظلّ گذرد و بر سطح مائل بر زوایای قائمه باشد موضع قمر گرفته‌ایم، چه عبارت از[1] «وسط خسوف» آن وقت است که قمر بر آن نقطه باشد {وسط خسوف دائماً به عقده اقرب بود از موضع اجتماع حقیقی}.[2] و بطلیموس[3] این معنی را در شکل دوم از فصل ششم از مقالهٔ ششم مجسطی اثبات کرده است، امّا بجهت سهولت بعمل نیاورده است.[4]

و خسوف اوّل در شب سی ام شهریورماه قدیم سنهٔ خمس و سبعین و سبعمائه یزدجردی بود؛ از نصف اللّیل مذکور تا وسط خسوف **ج ید ل** ساعات مطلقه {، یعنی حقیقیّه،}[5] گذشته بود و **ب نو کط** ساعات معدّله، {یعنی وسطیّهٔ معدّله بتعدیل الأیّام،}[6] یعنی بحسب مبدأ مفروض که ذکر آن در مقالهٔ ثالثه کرده شود.[7]

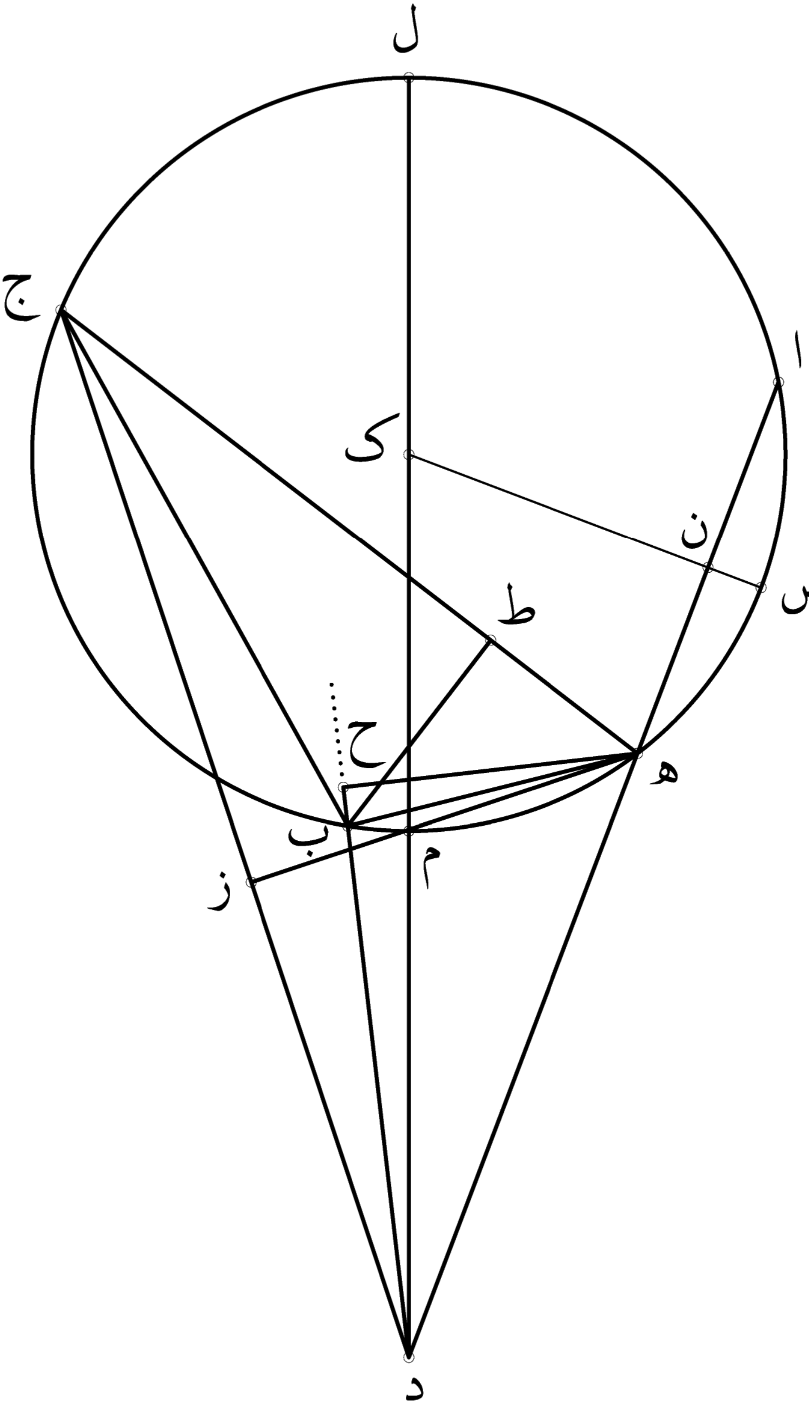
امّا خسوف دوم در شب بیست و هفتم اسفندارمذ ماه قدیم سنهٔ مذکوره بود؛ ساعات ماضیهٔ وسط خسوف از نصف اللّیل: مطلقه **ا یج ه**، معدّله **θ مح مو.**

امّا وسط[8] خسوف سوم در شب هجدهم شهریور ماه قدیم سنهٔ ستّ و سبعین و سبعمائه یزدجردی[9] بود؛ ساعات ماضیّه از نصف اللّیل تا وسط خسوف:[10] بالاطلاق **د یح ل**، بالتّعدیل **ج نح مو.**

تقویم آفتاب و وسط جوزهر را[11] در اوساط خسوفات ثلثه بحساب حاصل کردیم؛[12] بود:[13]

	تقویم آفتاب در وسط خسوف	وسط جوزهر	بعد آفتاب از عقده	تعدیل نقل(3) باشد:	تقویم قمر در وسط خسوف بفلک مایل باشد:
اوّل	**ب یح نه ی ما**	**ط د لب مو**	**و لب ج**	{[زا]ید}(4) ا کط ب	ح یح نو لط مج
دوم(1)	**ح یب یج نج لح**	**ط یج ند نا**	**ج نا یه**	{[زا]ید}(4) θ نب م	ب یب ید مو یح
سوم(2)	**ب ح ید یح مج**	**ط کج یز م**	**ا لا نط**	{[نا]قصه}(4) θ ک نح	ح ح یج نز مه

(1) L and P: ثانی. (2) L and P: ثالث. (3) L and P: بعد. (4) Additions to MSS. IO, Q2, and C.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و حرکت {تقویمی}[14] قمر از وسط خسوف اوّل تا وسط خسوف دوم **قعج یح و له** است. و مدّت مابین الخسوفین **قعو** روز و **کا نح له** ساعت مطلقه، {یعنی حقیقیّه،}[15] و **کا نب یز** معدّله {، یعنی وسطیّه}.[16] و حرکت وسط درین مدّت {، یعنی مدّت معدّله،}[17] **قعا ج یج کو** باشد، و حرکت خاصّه: **قنا ک لو ی.** و ظاهر است که این قوس **ب ید نج ط** در تعدیل افزوده است {، چه وسط کمتر از تقویم است}.[18] و همچنین حرکت {تقویمی}[19] قمر از وسط خسوف دوم تا وسط خسوف سوم **قعه نط یا کز** است و مدّت مابین الخسوفین: **قعز** روز و **ج ه که** ساعت مطلقه و **ج ی θ** معدّله. و حرکت وسط درین مدّت **قعج نز لط ک** باشد، و حرکت خاصّه: **قند یج لج لد.** و ظاهر است که این قوس **ب ا لب ز** در تعدیل افزوده است.

پس، اینجا شکل سوم و چهارم و پنجم از فصل پنجم از مقالهٔ چهارم کتاب مجسطی بیک شکل بیاریم.[20] وفرض کنیم که دایرهٔ ا ب ج فلک تدویر است و نقطهای ا ب ج مواضع قمراند در اوساط خسوفات سِگانه بر ترتیب. و قوسهای سِگانه که میان این نقطهاست معلوم‌اند، چه قوس ا ب حرکت خاصّه است در مابین خسوف اوّل و دوم و زاید است در تعدیل، و قوس ب ج حرکت خاصّه است در مابین خسوف دوم و سوم و همچنان[21] زاید است در تعدیل. پس، قوس ج ا باقی ماند: **ند که ن یو.** و نقصان تعدیل بحسب آن قوس **د یو که یو** است {که مجموع هر دو تعدیل مذکور است}.[22] و پوشیده نماند که بعد اَبعد بر دو قوس ا ب ب ج واقع نشده، چه هر یک اقلّ از نصف دوراند و زایداند در تعدیل. پس بر قوس ج ا واقع باشد. و فرض کنیم که د مرکز بروج است. و خطوط دا د ب دج وصل کنیم. و دا قاطع تدویر است بر نقطهٔ ه. و خطوط ب ج ه ج ه ب وصل کنیم. و از نقطهٔ ه دو عمود ه ز ه ح بر خطّ د ج د ب قائم گردانیم و از نقطهٔ ب عمود ب ط بر خطّ ج ه.

و جیب زاویهٔ ا د ج {– که مقدار مجموع هر دو تفاوت تعدیل مذکور است –}[23] **د کح یو لب** است و آن مقدار خطّ ه ز است باجزائی که د ه شصت جزء باشد. و چون زاویهٔ ا ه ج بر مرکز **ند که ن یو** است و بر محیط **کز یب نه ح** باشد و زاویهٔ ا د ج **د یو که یو** بود، پس زاویهٔ ه ج د[24] باقی ماند بشکل سی و دوم از مقالهٔ اوّل کتاب اصول: **کب نو کط نب**؛ جیبش: **کج کج یه کا.** و این مقدار ه ز است باجزائی که ه ج شصت جزء باشد. پس مقدار ه ج باجزائی که د ه شصت جزء باشد **یا کح یه ج** بود.[25] و نیز جیب زاویهٔ ا د ب {– که زاویهٔ تفاوت تعدیل است در مابین خسوف اوّل و دوم –}[26] **ب کا یب نو** است، و آن مقدار ه ح است باجزائی که د ه شصت جزء باشد. و چون زاویهٔ ا ه ب بر مرکز **۲۰۸ لط کج ن** است و بر محیط **قد یط ما نه**، و زاویهٔ ا د ب: **ب ید نج ط**، پس زاویهٔ ه ب د باقی **قب د مح مو** باشد و تمام آن تا قائمتین، یعنی زاویهٔ ه ب ح، **عز نه یا ید** باشد؛ جیبش: **نح**

م یو مح. و این مقدار ه ح اســت باجزائی که ه ب شــصت جزء باشد. پس ه ب باجزائی که د ه شــصت جزء گیرند **ب کد کد مح** باشــد. و نیز زاویهٔ ج ه ب {– و این [قوس ب ج] قوس خاصّه است مابین خسوف دوم و سوم –}[27] بر مرکز **قند یج لج لد** است و بر محیط **عــز و مــو مز**؛ جیبش: **نح کط یط کب**؛ جیب تمامش: **یج کب ند ید**. و این هر دو مقدار دو خطّ ب ط ه ط اند باجزائی که ه ب شصت جزء باشد. پس باجزائی که ه ب **ب کد کد مح** باشد، یعنی باجزائی که د ه شصت جزء باشد، مقدار ب ط **ب ک مو لج** باشد و مقدار ه ط **θ لب یب ل** و ه ج بآن اجزاء **یا کح یه ج** بود. پس ج ط باقی **ی نو ب لج** باشد؛ مربّعش: **ا نط لج یا مه مب ل ط** سادســه. و مربّع ب ط **ه ل یز ن و ند ط** سادســه. مجموع المربّعین: **ب ه ج کط له مط کد یح**. و این مربّع ب ج است؛ جذرش: **یا ی نح لو**. و این مقدار ب ج است باجزائی که د ه شصت باشد و ه ب[28] بآن اجزاء **ب کد کد مح** بود. ولیکن وتر ب ج باجزائی که نصف قطر تدویر شصت باشد **قیو**[29] **نح لح مد** است. پس، د ه بآن اجزاء **۶۲۷ لز یج**[30] **نه** باشد. و وتر ب ه بآن اجزاء **که ی لو مو** باشد. پس، قوس ه ب **کد یج یط مب** باشد و قــوس ا ه ب **قنــا ک لو ی** بود. پس قوس ا ه **قکز ز یو کح** باشــد. وتر او،[31] یعنی خطّ ا ه، **قز کو نه یح** باشــد؛ و این از قطر اقصر اســت. پس، مرکز تدویر خارج از قطعهٔ ا ه باشــد. و فرض کنیم که آن نقطه ک اســت. و از نقطهٔ د خطّی اخراج کنیم که بنقطهٔ ک گذرد و تدویر را بر ل م قطع کند. پس، آن هر دو بعد اَبعد و اَقرب باشــد. پس، مســطّح ه د[32] که آن **۶۲۷ لز یج نه** است در جمیع ا د که آن **۷۳۵ د ط یج** است **ب ح ط د**[33] **لب کب ید مط یه نه** سادسه باشد. و این مساوی مسطّح م د است در ل د، چنانکه از شکل سی و پنجم از مقالهٔ ســوم کتاب اصول معلوم می شــود. پس، چون مربّع م ک، که آن **ا θ θ** اســت، بر آن زیادت کنیم مربّع د ک حاصل شــود بشکل ششم از مقالهٔ دوم[34] اصول و آن **ب ط ط د لب کب ید مط یه نه** سادســه شود؛ جذرش **۶۸۱ نب و یح** باشد؛ و این مقدار د ک است باجزائی که ک م شــصت باشــد. پس، مقدار ک م، نصف قطر تدویر، باجزائی که د ک شصت باشد **ه یو مو لو** باشد.

پس، از نقطهٔ ک عمود ک ن س بر ا ه اخراج کنیم و ا ک وصل کنیم. پس، بشــکل ســوم از مقالهٔ سوم اصول ا ن ه ن متساویین باشند. پس، چون نصف ا ه، که آن **نج مج کز لط** باشد، بر د ه، که آن **۶۲۷ لز یج نه** اســت، افزاییم، حاصل شــود: **۶۸۱ ک ما لد**، و این مقدار د ن اســت باجزائی که د ک **۶۸۱ نب و یح** باشــد، یعنی باجزائی که ک م شــصت باشد. پس، د ن باجزائی که د ک شــصت گیرند **نط نز ید ی** باشــد. و این جیب زاویهٔ س ک م بود.[35] قوسش:[36] **فز مه یز ب**؛ و این قوس س ه م است. تمامش تا قائمتین **صب ید مب نح** بود؛[37] و این قوس ل س بود.[38] و مجموع هر دو قوس س ه، که نصف ا س ه است، و ه ب مذکور **فز مو نز نو** باشد. پس، قوس ل ه ب، بُعد قمر از ذروهٔ مرئی، **قف ا م ند** باشد؛ و این خاصّهٔ

معدّله اســت[39] در منتصف زمان خسوف اوسط. و زاویهٔ ن د ک، یعنی تمام زاویهٔ ن ک د تا ربع دور، **ب يد مب نح** باشد. و زاویهٔ ا د ب **ب يد نج ط** بود. پس، زاویهٔ ک د ب باقی **θ θ ی یا** باشــد؛ و این[40] تعدیل جزئی اســت که بآن موضع وســطی قمر از حقیقی ناقص می شود. آن را از تقویم قمر بفلک مائل در وسط خسوف اوسط، که آن **ب یب يد مو يح** اســت، نقصان کردیم؛ باقی ماند **ب یب يد لو ز**. و این حاصل وســط قمر اســت در وسط خسوف اوسط. و وسط آفتاب در این وقت **ح یب نب کح نز** است. بعد مضاعف **یا کح مد يد ک** باشــد. تعدیل خاصّه {بازاء مرکز}:[41] **θ یا و مه**. خاصّهٔ معدّله: **و θ ا م ند** بود.[42] پس، خاصّهٔ وسطی **و θ یب مز لط** باشد.

و وســط قمر در منتصف اوســط خســوفاتی که بطلمیوس در اســکندریه رصد کرده است — چنانکه در مجسطی مذکور است — **θ کط ل** است. و خاصّهٔ معدّله: **ب د لح**. بعد مضاعف: **θ ج لح**. تعدیل خاصّه: **θ لب م**. خاصّهٔ وســطی: **ب د ه ک**. و آن پیشــتر از تاریخ یزدجردی بوده اســت **بچهار صد و نود و هفت** ســال و **٣٦٣** روز و **ســیزده** ساعت مطلقه بطول اسکندریه که **یا کج لو** ساعت باشد بطول کاشان.

پــس، مابیــن الرّصدین **١٢٧٢** ســال و **٣٥٤** روز و **θ لو ما** [ســاعت] مطلقه و **θ مو کج** [ساعت] معدّله باشد. و حرکت وسط در مابین الرّصدین بعد از**هفده هزار و شش**دور **ا یب مد لو ز** اســت و حرکت خاصّه بعد از **١٦٨٦٢** دور **ج کو ز کز لط**.[43] پس هر یک از حرکت وســط و خاصّه مذکور مع الأدوار بر مجموع ایّام و کســور مدّت مابین الرّصدین قسمت کردیم تا حرکت یک روزه از هر یک از آن حاصل شد. از تضاعیف آن حرکت ایّام و شهور و ســنین حاصل کردیم. پس از وســط خســوف اوســط تا نصف نهار روز اوّل[44] سال **ذ فا** یزدجردی پنج ســال و نُه روز اســت و **یا یا**[45] **يد** ساعت بنصف نهار زیج بطول کاشان، که **یا ج يد** ساعت باشد هم بنصف نهار زیج بطول شیراز.[46] هر یک از حرکت خاصّه و وسط درین مدّت حاصل کردیم و بر حاصل هر یک در وسط خسوف اوسط افزودیم تا حاصل خاصّه و وســط در نصف نهار روز اوّل[47] ســال **ذ فا** بنصف نهار زیج بطول **فح** حاصل شد. و از وســط آفتاب و وســط قمر حاصل بعد مضاعف و حرکت آن در سنین و شهور و ایّام حاصل کردیم.

	خاصّـــــــه							
	برج	درجه	دقیقه	ثانیه	ثالثه	رابعه	خامسه	سادسه
در یک روز	θ	یج	ج	نج	نو	ل	لز	ک
در یکماه	ا	ا	نو	نح	یه	یح	م	θ
در یکسال	ب	کح	مج	ح	مو	یز	و	م
در صد سال	ز	کا	ند	لز	ح	لا	و	م
در ششصد سال	ی	یا	کز	مب	نا	و	م	θ
در اوّل ذ فا	θ	کز	کد	لط	مح	نو	یج	یب

	وســـــــط								
	برج	درجه	دقیقه	ثانیه	ثالثه	رابعه	خامسه	سادسه	سابعه
در یک روز	θ	یج	ی	له	ا	نب	مز	ن	ن
در یکماه	ا	ه	یز	ل	نو	کج	نه	که	θ
در یکسال	د	ط	کج	و	کو	یا	د	ید	ی
در صد سال	یا	ح	ل	مج	لح	کز	ج	لو	م
در ششصد سال	ز	کا	د	کا	ن	مب	کا	م	θ
در اوّل ذ فا	د	ج	مط	لا	ح	مه	کج	نه	لد[ند؟]

این اســت حاصل رصد خســوفات که ذکر کردیم.[48] اگر در عمر مَهلی باشــد و دولت پادشاه جهان یاور گردد،[49] رصد باقی کواکب بکنیم و بر آن زیجی[50] وضع کنیم. حالیا بقدر وُسع این مقدار ایراد کردیم.

Notes

1. MSS. S and L: —.
2. A marginal gloss in MSS. P and C.
3. MSS. IO, Q1, Q2, P, and C: —.
4. The underlined sentences read a bit differently in MSS. S and L:

و بطلیموس این معنی را در فصل ششم از مقالهٔ ششم مجسطی ذکر کرده است، امّا بعمل نیاورده است تساهلًا.

5. A marginal comment in MSS. IO, Q2, and P.
6. A marginal addition in MSS. IO, Q2, and P.
7. The phrase «یعنی بحسب مبدأ مفروض که ذکر آن در مقالهٔ ثالثه کرده شود» can be found in MSS. Q1, S, and L (in MS. Q1: «کرده خواهد شد [...]»).
8. The term «وسط» is found in MSS. IO, Q1, Q2, P, and C, is crossed out in MS. S, and is not extant in MS. L (in the latter two MSS, it is added in the end of the sentence; see below, note 10).
9. MSS .S and L.— :
10. The phrase «تا وسط خسوف» is not extant in MSS. IO, Q1, Q2, and P, because the term «وسط» is already given in the beginning of the sentence.
11. MSS. S and L: —.
12. MS. P: + برین دستور.
13. MS. Q2: اینست; MS. IO: + این.
14. A marginal comment in MSS. IO, Q2, and C.
15. A marginal note in MSS. IO, Q2, and P.
16. A marginal gloss in MSS. IO, Q2, and P.
17. A marginal comment in MSS. IO, Q2, P, and C.
18. A marginal gloss in MSS. IO, Q2, P, and C.
19. A marginal note in MSS. IO, Q2, and P.
20. MSS. IO and Q2: بیاوریم. MS. C: سازیم.
21. MSS. S, L: همچنین.
22. A marginal comment in MSS. IO, Q2, and C.
23. A marginal note in MSS. IO, Q2, P, and C.
24. MSS. IO and Q2:
25. MSS. S and L: باشد.
26. A marginal addition in MSS. IO and Q2.
27. A marginal gloss in MSS. IO, Q2, and C.
28. MSS. S and L: ب ه.
29. MSS. IO and Q2: قنو.

30. MSS. IO and Q2: لج.
31. MSS. S and L: آن.
32. MSS. IO and Q2: د ه.
33. Only in MS. S, a vertical thick line «|» separates the integral part of the sexagesimal number from its fractions.
34. MSS. S, L, and Q1: سوم.
35. MSS. S and L: باشد.
36. MSS. S and L: قوس او.
37. MSS. S and L: —.
38. MSS. S and L: باشد.
39. MS: Q1: باشد.
40. MSS. IO and Q2: —.
41. An addition in MSS. IO and Q2.
42. MSS. Q1, S, and L: —.
43. In MSS. IO and Q2, the motion in anomaly is added in the margin. In MS. S, both motions are given in the margin. In MS. Q1, the motions in longitude and anomaly have erroneously been replaced by each other, maybe, because of reading the numbers from the margin in the prototype.
44. MS. S: + از.
45. MSS: IO and Q2: ج.
46. MS. S has given only the longitude value 88°/فح without mentioning which city it refers to.
47. MS. S: + از.
48. End of the passage in MS. L.
49. MS. S: – بدولت پادشاه جهان – خلّد الله ملکه
50. MS. S: + دیگر (inserted above the line).