Kamāl al-Dīn al-Fārisī's additions to Abharī's "proof" of the parallel postulate

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Abstract: This article is devoted to Kamāl al-Dīn al-Fārisī's (d. 1319) additions to the well-known al-Abharī's "proof" of the parallel postulate. These additions are found in only one codex, the manuscript Tunis 16167/7 also often referred to as one of the units of Tunis, al-Aḥmadiyya 5482 which is usually wrongly attributed to Qāḍī Zāde al-Rūmī.

Keywords: Abharī, Fārisī, Ṭūsī, Qāḍī Zāde, Parallel Postulate, Arabic Euclidian geometry.

Introduction

Out of the ten works contained in the manuscript ms. Tunis 16167, the seventh (folios 74a-75a) has not been studied before, nor edited and discussed by any researcher. It contains two short additions to Athīr al-Dīn al-Abharī's (d. 1263) "proof" of Euclid's parallel postulate attributed to Kamāl al-Dīn al-Fārisī (d. 1319) by an anonymous writer. We present this manuscript with focus on Kamāl al-Dīn al-Fārisī's additions and propose an edition of this text with an English translation and notes.

Al-Abharī's addition to the Parallel Postulate is well known to researchers in the history of Arabic geometry. For example, in his History of non-Euclidean Geometry, Rosenfeld [1988, 85-86] states that al-Abharī reworked Euclid's Elements in a book known under the name *Iṣlāḥ al-Usṭuqusāt* (Improvement of the Elements) and, that his attempt "to prove the parallel axiom enjoyed the greatest popularity in the 13th century as well as subsequent centuries".

This "proof" appears also in the commentary of Qāḍī Zāde al-Rūmī: *Sharḥ ashkāl al-taʾsīs* of al-Samarqandī with slight differences. In his commentary of

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al-Samarqandī's 18th proposition, Qāḍī Zāde [1984] explicitly mentions al-Abharī and quotes his "proof" of the parallel postulate. It is remarkable to note that Qāḍī Zāde seems to ignore Fārisī's additions so that the latter's contribution was absent from all modern published histories of the parallel axiom. In this paper, we aim to fill up the gap.

Fārisī's place in the Arabic Euclidean tradition

Kamāl al-Dīn al-Fārisī was a prominent scientist of the 13th-14th centuries whose fame rested on his *Tanqīḥ al-manāẓir* (The revision of the Optics), a commentary on Ibn al-Haytham's *Kitāb al-manāẓir* (The Book on Optics). He is also known for his work on amicable numbers[1] and his commentary on Ibn al-Khawwām's (d. 1325): *al-Fawā'id al-bahā'iyya fī l-qawāʿid al-ḥisābiyya*, a important textbook in arithmetic, algebra and practical geometry.[2] But, as far as we can find, none of the standard bio-biographical sources has credited him with any substantial work in Euclidean geometry; only some of his short commentaries and glosses are extant. However, his proficiency in theoretical geometry, revealed by his original treatment of geometrical optics must have been preceded by a period of studying and teaching geometry, probably in Tabrīz, when he was a student of the polymath Quṭb al-Dīn al-Shirāzī (d. 1312). His *Risāla fī l-zāwiya* (Treatise on the angle),[3] and two short treatises explicitly attributed to him by an anonymous author, are witnesses of his acquaintance with both renditions of Euclid's *Elements* as presented by al-Ṭūsī's and by al-Abharī's. The first of these two short treatises is *An-naẓar fī qawl al-Ṭūsī fī ākhir al-maqāla al-thālitha ʿashar* (Reflexion on what al-Ṭūsī said at the end of Book XIII), the second is the subject of the present paper, and both are found in the same collection of manuscripts, Tunis 16167, copied in the fifteenth century.

1. Rashed [1982] has edited this treatise and translated it into French. Brentjes [1991] comments on this paper.

2. Mawaldi [1994].

3. Mawaldi [2014].

Abharī's Iṣlāḥ al-Usṭuqusāt

One cannot understand Fārisī's additions without a review of Abharī's "proof" contained in his *Iṣlāḥ al-Usṭuqusāt* (Improvement of the *Elements*).[4] Specialists of the Arabic Euclidean tradition consider this book as part of the "the Arabic secondary transmission of Euclid's *Elements*", that is an edition mixed with additions and comments. This book has not been published but some of its features have been studied. For example, while discussing interpolations in different editions (Greek, Arabic and Latin) of Book I of Euclid's *Elements*, Sonia Brentjes [1997-98] analyzes al-Abharī's version among 12 other texts. Also, we realize from the study by Gregg De Young [2004] of the Latin translation of Euclid's *Elements* attributed to Gerard of Cremona and by his confronting it with primary and secondary Arabic transmissions that Ibn Sīnā's (d. 1036) *Uṣūl al-handasa* and Abharī's *Iṣlāḥ* had a same major source.

While Abharī's *Iṣlāḥ* seems to be retaining the structure of Euclid's *Elements* with its division in thirteen books, the editor introduces some additions and reorganizes the order of some propositions. For example, just after a very short introduction, he starts by listing Euclid's basic definitions. However, when defining parallel lines he inserts an alternate statement using distances between two straight lines:

> Parallel lines are those which are in a same plane and do not meet one another, even if extended linearly in both directions. One may also say that parallel lines are those which are in a same plane and, if extended linearly and indefinitely, the distance to one another is always the same. Distance is the shortest line connecting them. (fol. 2a)

After the last definition, Abharī turns out to Euclid's postulates; however, he retains only the four first ones. He writes: (1) We have to make a connection between any two points by a straight line. (2) To extend any limited straight line rectilinearly. (3) To draw, with any point <as center> and <with> any distance (radius), a circle; (4) All right angles are equal to one another. Then, he

4. Copies of al-Abharī's *Iṣlāḥ kitāb al-Usṭuqsāt fi l-handasa lī Uqlīdis = Iṣlāḥ uṣūl Uqlīdis* can be found in Chester Beatty Ar. Ms. 3424. Some catalogues list two other copies, one in Bursa, Hüsein Celebi 744 and the other in the Museum of Archeology Turkey, 596. For this work, we used a scanned copy of the Chester Beatty manuscript.

adds two statements absent from the *Elements*: (5) Two straight lines do not <together> bound an area. (6) A straight line cannot be continued in a straight line by two <different> straight lines. For each one of these six postulates, their enunciation is followed by a "demonstration".[5] Abharī's fifth and sixth postulates are the same as those listed by al-Samarqandī in *Ashkāl al-taʾsīs*, edited by De Young [2001, 62-63], who notes that the fifth "is found in all manuscripts of the Arabic tradition", while the sixth "is not found in the Arabic translations. Although not included as a "postulate" by Proclus, it also seems to be rooted in Greek discussions of the Euclidean text". After the demonstration of the sixth postulate, Abharī announces that Euclid's parallel postulate shall be proved later:

> Euclid has added to these postulates another one which states that if a line falls on two straight lines and makes the interior angles which are on a same side less than two right angles, then the two lines meet on that side. (fol. 3b)

Abharī omits all Euclid's common notions and starts immediately with proposition 1 of Book I and, considering implicitly that the twenty-eight first propositions do not involve the parallel postulate which in turn has to be proved, he introduces it as an addition (*ziyāda*) placed between the twenty-eighth and the twenty-ninth proposition:

> *Ziyāda*: if a line falls on two straight lines and makes the interior angles which are on a same side less than two right angles then the two lines meet on that side. Before proving <this proposition> we start by an introduction <i.e. a lemma>. (fol. 11b)

A letter written by ʿAlam al-Dīn Qayṣar al-Ḥanafī (d. 1258) to al-Ṭūsī contains the assertion of the sixth century Byzantine scholar, Simplicius, that for every interior point of an angle, one can draw infinitely many bases for the angle such that some of them fall outside the given point. Al-Abharī's takes Simplicius' assertion as a lemma for the proof of the parallel postulate and then he proves it in three steps: (1) a right angle and an acute angle, (2) two acute angles, and (3)

5. De Young [2007] gives a translation of Shirāzī's edition of the six demonstrations. Abdeljaouad [2014-2015] has presented an Arabic edition of these demonstrations with their translation into English.

an acute and an obtuse angle. Rosenfeld [2007, 86] summarizes Abharī's proof of the parallel postulate in this manner:

> Al-Abharī's proof of the parallel postulate for the case of a perpendicular and an oblique line is the same as the proof of Simplicius's proposition 2 (…). Unlike Simplicius, Al-Abharī gives proofs of the two remaining cases of the parallel postulate. The case when the transversal makes two acute angles with two straight lines is proved in the same way as the first case by using an analogous figure. The proof of the case when the transversal makes an acute and an obtuse angle with the two straight lines is the same as Ibn al-Haytham's proof of this case.

As arguments, he uses in his demonstration his own fifth postulate and six Euclidian propositions (nos. 3 – 13 – 15 – 16 – 17 – 26)[6] referring to them in Arabic alphabetical numerals written above the line. For example, in the beginning of the proof of the Lemma, when he writes: "ج لأنّا نفصل ب هـ مثل ب ز" the letter ج refers to the third proposition.

In the next section of this paper, we present in details Abharī's "proof" with Fārisī's additions.

6. (ج) نريد أن نفصل من أطول الخطين مثل اقصرهما. (يج) كل خط وقع على خط آخر فإن الزاويتين عن جنبيه إما قائمتان أو مساويتان لهما. (يه) خطا أ ب، ج د تقاطعا على نقطة هـ، فأقول إن كل زاويتين متقابلتين من الزوايا الحادثة عند نقطة هـ متساويتان. (يو) مثلث أ ب ج أخرج منه خط ب ج، فأقول إن زاوية أ ج د الحادثة أعظم من كل واحدة من زاويتي أ، ب. (يز) كل زاويتين من مثلث، فهما أنقص من قائمتين. (كو) زاويتا ب، ج من مثلث أ ب ج متساويتان لزاويتي هـ، ز من مثلث د هـ ز، كل لنظيره، وضلع ب ج من مثلث أ ب ج مثل هـ ز، أو ضلع أ ز مثل هـ د، فأقول إن المثلثين والأضلاع والزوايا متساو كل لنظيره.

Prop. 3: We want to cut off from the greater a straight line equal to the less.

Prop. 13: If a line stands upon another line, the two angles <formed> on its sides are either two right angles or equal to them.

Prop. 15: Two straight lines AB <and> GD cut one another at point E, then I say that each one of the opposite angles occurring at the intersection, at point E, are equal to one another.

Prop. 16: In triangle ABG, side BG is extended by GD, then I say angle AGD produced is greater than either one of angles A or B.

Prop. 17: In any triangle two angles taken together are less than two right angles.

Prop. 26: The two angles B and G from triangle ABG are equal to the two angles E and Z from triangle DEZ, respectively, and side BG from triangle ABG equals side EZ, or side AZ equals ED, then I say that the two triangles are equal to one another, and their sides and angles are equal to one another, each to its corresponding part.

Abharī's "proof" of the parallel postulate with Fārisī's additions

The treatise starts by an introduction praising Fārisī and reporting that the latter considers Abharī's "proof" of the parallel postulate as the best known to him. He quotes explicitly Abharī's *ziyāda*:

> The best known way to prove the famous Euclid's Postulate is what Athīr al-Dīn al-Abharī's has written in his Amendment to Euclid's Elements. He <i.e. Abharī> writes: "Before proving it <i.e. the postulate> we start by an introduction, that is: Angle ABG is halved by line BH. Then I say: It is possible to draw in it [this angle] infinitely many chords in such a way that they are located one under the other and each of them is the base of an isosceles triangle".

Then he reproduces Abharī's entire text introducing in it two changes: the first is the addition of his own arguments in the proofs of the lemma and the postulate, and the second is the suppression of the numbers of the propositions referred to in the arguments.

We divide Text 7 into eight sections for ease of reference. Section 1 contains a preamble; sections 2, 4, 6, 7 and 8 reproduce the whole treatise of Abharī, while sections 3 and 5 contain additions of al-Fārisī to Abharī's work.

Section 1: It contains a preamble written most probably by one al-Fārisī's students. It is devoted to a long praise of the multiple qualities of his master who is supposed to have added some commentaries to al-Abharī's proof of the parallel postulate found in his *Iṣlāḥ*. Absent from Qāḍī Zāde's *Sharḥ,* this preamble is replaced by a simple notice added after Proposition 23 and saying: "this is the place for the promised proof of the well-known postulate. The philosopher Athīr al-Dīn al-Abharī said:…".

Section 2: It contains a very helpful lemma needed in the demonstration of the parallel postulate. It says that given any angle, "it is possible to draw in that angle infinitely many chords located one under the other". That means that one can construct many isosceles triangles with their bases perpendicular to the bisecting line of the given angle. The steps for the proof are as follow:

The line BH bisects the given angle ABG.

Let EBZ be an isosceles triangle with BE on the side BA of the angle and BR on the side BG.

Let be H the intersection of EZ and BH.

Therefore the two triangles EBH and ZBH are equal and BH is perpendicular to ZR.

Take BT on BA and BK on BG such that BT = BK. Then segment TK does not cut segment EZ.

(diagrams 1 to 4).

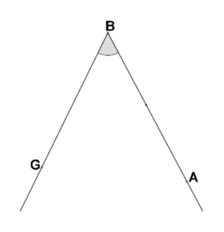

diagram 1

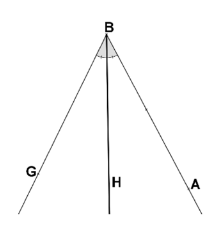

diagram 2

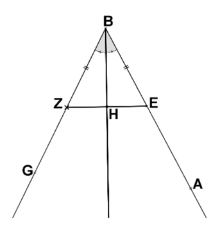

diagram 3

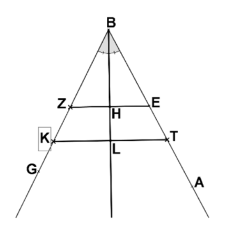

diagram 4

Al-Abharī does not justify the first four steps since he uses implicitly several propositions of Book 1.

For step 5, which is essential in the proof, one has to show that lines TK and EZ meet neither in H nor in any point inside HZ. Indeed, if TK intersected EZ in H there would be two straight lines both perpendicular at H to the straight line BH and this is impossible. And, says al-Abharī, TK does not intersect HZ since if it did, two straight lines would enclose one area; and this is also impossible (figures 5 and 6). This is the place where al-Fārisī adds his own explanation.

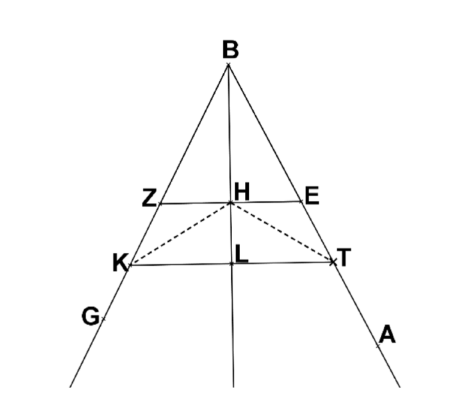

diagrama 5

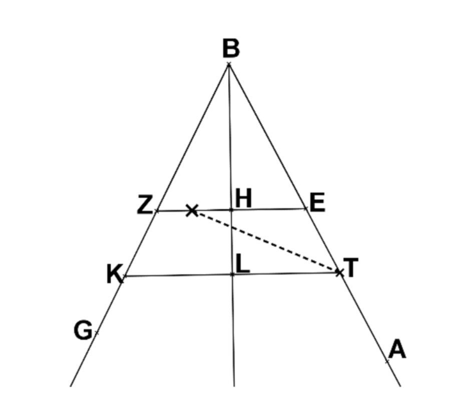

diagrama 6

Section 3: Al-Fārisī’s first addition.

It starts by a question, al-Fārisī speaking: “I say: why line TK does not meet line EZ at a point other than H?” His answer is that the situation is not different from the first one when TK intersects EZ in H. When the straight line TK falls on EK at Z, it either coincides with the straight line ZK, which is impossible, or it does not coincide with it and then two straight lines would enclose a surface, and this too is impossible (diagram 6). Here ends al-Fārisī’s explanation.

Section 4: The sections that follow are devoted to the proof of the parallel postulate when different hypotheses are considered. Al-Abharī wants to prove the following general statement “A line falling on two straight lines and making the two interior angles on one side less than 180°, when extended indefinitely on that side the two lines will meet”. Three cases are possible: either one interior angle is acute and the other right, or both are acute or one is acute and the other obtuse.

Section 5: Proof of the case where one angle is right and the other acute.

Let the two lines be AG and BD, and the connecting line AB and suppose ∠BAG acute and ∠ABD right. Al-Abharī constructs an isosceles triangle EAT with T on AG, admitting AB as a bisecting line for the angle ∠EAT and such that line DB is entirely contained in EAT. Here are the steps of this construction:

Construct an angle ∠BAE equal to ∠BAT with AT on line AG and AE = AT.

Join points E and T. Let Z the intersection of ET and AB. Then AZ is perpendicular to ET.

In angle ∠EAT choose chord ET such that AZ > AB, (as consequence of the preceding lemma). Then line DB is entirely contained inside triangle ETZ.

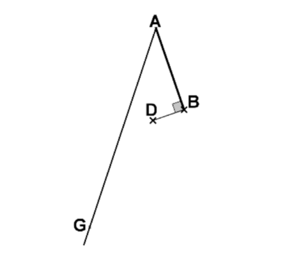

diagram 5-1

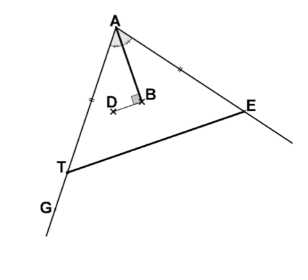

diagram 5-2

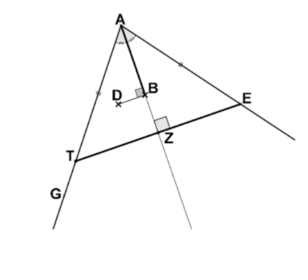

diagram 5-3

At this point of the proof, al-Fārisī adds some arguments justifying step 3.

Section 6: Al-Fārisī’s arguments.

Al-Fārisī starts by asking some questions: "Why should one of the chords fall under point B, and why is it not possible that all chords fall between <points> A and B since AB may be indefinitely divided?".

He is implicitly using the infinite divisibility of a magnitude, a principle attributed to Aristotle. He objects that, in the lemma, it is said that one can construct an infinite number of chords in angle ∠ABG, but what if all these chords intersect line AB inside segment AB? To this objection, al-Fārisī points out that the infinite divisibility of a segment is unsubstantial (immaterial) and can only be thought virtually while cutting off segments from AB is concrete and their total length is infinite and cannot be limited by the length of line AB.

Section 7: Al-Abharī ends his proof of the first case of the parallel postulate in three steps.

Line AZ is perpendicular to ET (as a bisecting line of the isosceles triangle EAT).

Line RT does not intersect line BD (if it did there would exist a triangle with two interior right angles).

Line BD, when extended linearly, intersects necessarily AT.

For step 3, al-Abharī is implicitly using the so-called Pasch axiom. It says that "A straight line lying inside a triangle and cutting one of its sides not at a vertex intersects one other side of the triangle". Euclid used it implicitly in several proofs. So when extended indefinitely, in the side of C and D, the two lines AG and BD such that ∠BAG is acute and ∠ABD right, meet necessarily.

Section 8: Al-Abharī now supposes that both angles ∠BAG and ∠ABD are acute and he proves that, when extended indefinitely on the side of G and D, the two lines AG and BD meet. Five steps are needed:

(1) Construct angle ∠BAE equal to ∠BAG; then AB is a bisecting line of angle ∠EAG.
(2) Make chord ER cutting AB in H such that AH > AB, with R on AG and AR = AE.
(3) ∠HBD is obtuse since ∠ABD is acute.
(4) Line ER does not intersect BD on the side of D and R, (since if it did, it would result a triangle with interior angles one right and the other obtuse).
(5) Then line BD when extended meets AR.

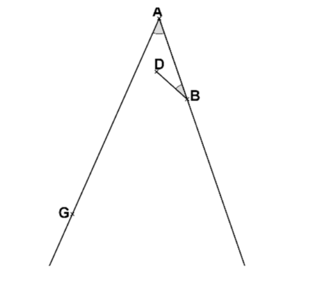

diagram 8-1

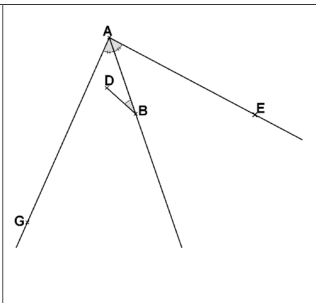

diagram 8-2

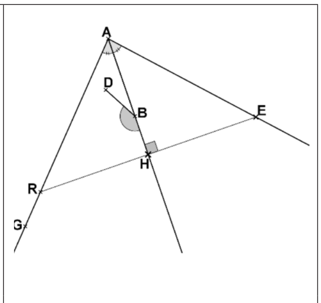

diagram 8-3

So when extended indefinitely in the side of G and D and when both ∠BAG and ∠ABD are acute, the two lines AG and BD meet necessarily.

Section 9: Al-Abharī now assumes that line EZ falls on lines AB and GD making angles ∠BEZ acute and ∠DZE obtuse, with ∠BEZ + ∠DZE = 180°. He proves that, when extended indefinitely on the side of B and D, the two lines AB and GD meet. Seven steps are needed:

(1) Bisect line EZ.
(2) Drop a perpendicular HT from point H on line GD.
(3) From H, extend linearly HT toward M.
(4) ∠THZ is acute since ∠BEH is right.
(5) ∠EHM and ∠BEH are acute.
(6) The two lines AE and HM meet on the side of B and M (as shown in section 8). Let K be the intersection point.
(7) Angle ∠EKH is obtuse; otherwise it would be right or acute.

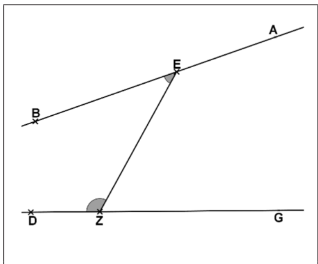

diagram 9-1

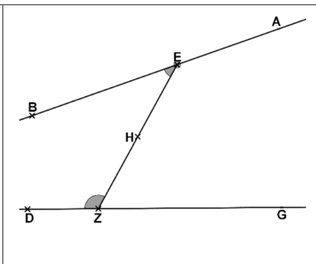

diagram 9-2

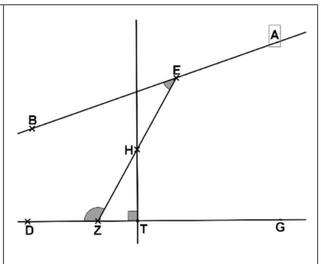

diagram 9-3

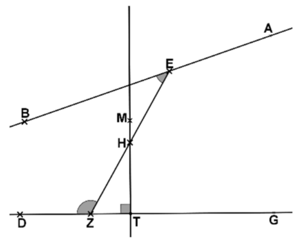

diagram 9-4

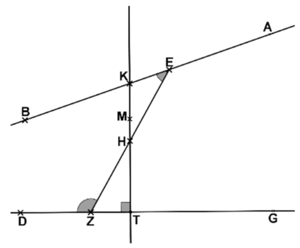

diagram 9-5

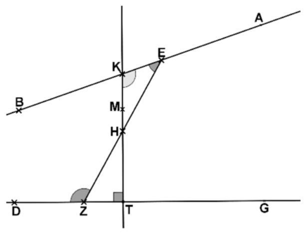

diagram 9-6

The seventh and last step needs to be proven on its own.

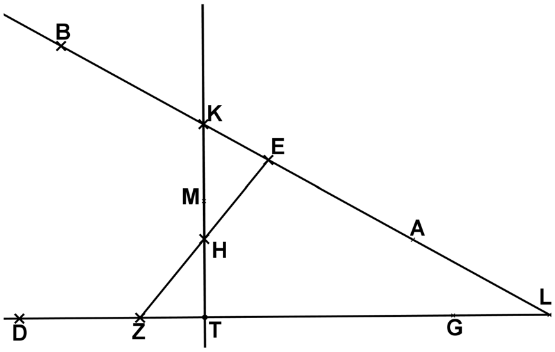

diagram 9-7

(*a*) 7.1 Assume that $\angle EKH$ is a right angle.

$\Rightarrow \angle EKH + \angle EHK = \angle HTZ + \angle THZ$.

$\Rightarrow \angle KEH = \angle HZT$.

$\Rightarrow \angle DZE + \angle EZT = \angle DZE + \angle KEH$.

$\Rightarrow \angle DZH + \angle HZT < 180°$, since $\angle DZE + \angle KEH$ less than 180° (by hypothesis).

$\Rightarrow$ This is impossible, since $\angle DZH$ and $\angle HZT$ are the two angles in Z.

$\Rightarrow \angle EKH$ cannot equal a right angle.

(*b*) 7.2 Assume that $\angle EKH$ is an acute angle.

$\Rightarrow \angle EKH$ acute and $\angle KTC$ right.

$\Rightarrow$ Lines AB and CD meet on the side of A and C in a point L. (as shown in §6).

But $\angle BEZ + \angle DZE < 180°$ (by hypothesis) and $\angle AEZ + \angle KEZ = 180°$ (by construction), so angle $\angle DZE$ is smaller than angle $\angle AEZ$. This is impossible, since an exterior angle of a triangle cannot be smaller than an interior angle opposite to it. Then, $\angle EKH$ is not an acute angle.

Therefore, $\angle EKH$ is neither right nor acute, it is obtuse and angle $\angle DTK$ is acute; and since $\angle EKH$ is a right angle, lines AB and CD meet on the side of B and D (as shown in §6).

This is the end of the proof of the third case. The parallel postulate is supposed to have been proved.

Final remarks

Al-Abharī is one among several pre-Islamic, Islamic and post-Islamic mathematicians who tried to deduce this postulate from the preceding ones. In his *History of non-Euclidean Geometry*, Rosenfeld (1988, viii) gives "a detailed account of the attempts to prove Euclid's fifth postulate, the so-called parallel postulate; these attempts led directly to Lobachevski's discovery".

While exposing al-Abharī's "proof" of the parallel axiom, al-Fārisī adds two remarks, one at the third section and one at the sixth. Each time, he begins by introducing the subject by an interrogation.

> Section 3: "I say: why line TK does not meet line EZ at a point other than H?".
> Section 6: "Why should one of the chords fall under point B, and why is it not possible that all chords fall between <points> A and B since AB may be indefinitely divided?".

For section 3, al-Fārisī says that whether lines TK and EH share point H or have a segment in common, that would mean that two straight lines could bound a plane figure. This contradicts Abharī's axiom 6.

In section 5, al-Abharī writes "So draw these chords <one under the other> until one of them falls under point B". This statement, notes Rosenfeld [1988, 86], is "the same as Simplicius' proposition 1 and al-Jawharī's proposition 30, and al-Abharī's proof differs little of other proofs of these proposition". In fact, for al-Abharī, the fact that TK falls outside of AB is evident and needs no argument. He does not give an answer to the objection contained in the letter of ʿAlam al-Dīn Qayṣar ibn al-Qāsim al-Ḥanafī to al-Ṭūsī pointing out that it is not proved that chord TK will fall outside AB for "every chord that subtends the angle ZBD will fall between points

A and B, for AB is infinitely divisible".[7] In section 6, Al-Fārisī wants to remove this objection by opposing imagination and virtuality "that the infinite divisibility of a segment is unsubstantial (immaterial) to a physical concrete argument, cutting off segments from AB is concrete and their total length is without end, so it cannot be limited by the length of line AB". Al-Fārisī, who combined theoretical investigations with practical experimentation in his works on optics, has —in this text— no reluctance in introducing in his mathematical proofs unorthodox arguments, as motion and concrete division of segments.

The manuscript Tunis Mss-16167/7

The short work under consideration here belongs to the codex Tunis Mss-16167 (also known as Aḥmadiyya 5482) and is the seventh unit (ff. 74a-75a) among ten, all devoted to commentaries on Euclid's Elements.[8] Rashed [2002, 736] presents a short description of the codex and lists this particular Fārisī's treatise as the sixth unit instead of the seventh, for he mixes two works of Fārisī ignoring the existence of Fārisī's addition to Book xiii of al-Ṭūsī's *Taḥrīr*. (cf. Abdeljaouad [2014-2015])

The Tunis codex is composed of 90 folios, 13x21,5 cm, 23 lines each, and copied with *nastaʾliq* script by a single hand: Darwīsh Aḥmad al-Karīmī who ended copying it in 869/1464. Fārisī's treatise contains four diagrams placed at the end of the proofs. Most of the treatises of this collection of manuscripts have been analyzed, and some have even been edited and translated into French, English or Persian.[9] Our text seems to be the only known extant copy of the treatise.

For the critical edition, we use as comparing source the Chester Beatty Ar. Ms. 3424 copy of Abharī's *Iṣlāḥ*. When editing and reproducing Abharī's proof of the

7. Sabra [1969, 9].

8. This volume also contains the well known Ibn al-Haytham's (d. 1038) *Sharḥ muṣādarāt Uqlīdis l-Ibn al-Haytham* [Commentary on the Premises of Euclid's Elements] (ff. 1b-59b), Al-ʿAbbās ibn Saʿīd al-Jawharī's (d. 835) *Ziyādāt al-ʿAbbās ibn Saʿīd fī l-maqāla al-khāmisa min Uqlīdis* [Additions to the Fifth Book of Euclid's Elements] (ff. 60b-61a) and Thābit b. Qurra's (d. 901) *Fi l-ʿillati l-latī lahā rattaba Uqlīdis ashkāl kitābihī dhālika l-tartīb* [Treatise on the Cause of why Euclid disposed Propositions of his book in such order] (ff. 86b-90b).

9. See for example: [Rashed 2002, 736]. A complete list of references can be found in Rosenfeld & Ihsanoğlu [2003, §§43-103-181-193-194-328-674].

parallel postulate Qāḍī Zāde introduced, in his *Sharḥ ashkāl al-taʾsīs*, several commentaries which somehow polluted the initial text; we preferred not to refer to this text.[10]

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10. A complete edition of Qāḍī Zāde's references to al-Abharī's proof in Arabic can be found in [Jaouiche 1988, 237-244], while its French translation with notes can be found in [Souissi 1982, 261-265] and in [Jaouiche 1986, 116-118].

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Edition and translation of manuscript Tunis_16167/7

[§1] In the name of Allah, the most merciful, the most gracious.

The greatest, the supreme, the knowledgeable, the head of the wise verifiers, the chief of the geometers, the perfect in faith and belief, al-Ḥasan al-Fārisī, may God cover him with His satisfaction, said:

"The best known way to prove the famous Euclid's Postulate is what Athīr al-Dīn al-Abharī, God's generosity to him, has written in his *Amendment to Euclid's Elements*".

/تـ: 74و/ بسم الله الرحمن الرحيم

قال[11] المولى الأعظم الحبر الأعلم رئيس الحكماء المحققين سلطان المهندسين كمال الملّه والدين الحسن الفارسي تغمّده الله برضوانه:

«أحسن وجه ذكر في بيان القضية المشهورة لاقليدس ما حرره أثير الدين الأبهري (نعمة له)[12] في إصلاحه لكتاب الأسطقسات».[13]

[§2] He writes: Before delving in its proof, we start by a premise, that is: Angle ABG is halved by line BH. Then I say: It is possible to draw in it [this angle] indefinitely chords located one under the other and each of them being the base of an isosceles triangle.

For we take BE equal to BZ and connect E and Z. Then EB and BH are equal ZB and BH, and the two angles B are equal. There-

قال:[14] ولنقدم على بيانها مقدمة وهي هذه: زاوية ا ب ج مُنَصّفَةٌ بخط ب ح، فأقول إنه يمكن أن يخرج لها أوتار[15] إلى غير النهاية يقع بعضها تحت بعض وتكون كل واحدة منها قاعدة لمثلث متساوي الساقين.

لأنَّا نفصل ب هـ مثل ب ز[16] ونصل هـ، ز. فـ هـ ب، ب ح مثل ز ب، ب ح وزاويتا ب متساويتان. فزاويتا /هـ: 10 ظ/ ح متساويتان.[17] فـ ب ح

11. في هوامش هذا التحقيق، نستعمل علامة(هـ) للإشارة إلى نص «الإصلاح لكتاب الأسطقسات» لأثير الدين الأبهري، مخطوط شستر بيتي، عربي عدد 3424. ونستعمل علامة (تـ) للإشارة إلى مخطوط تونس 7/16167 الذي نحققه هنا.

12. في (هـ): «نعمة له»، هذا اقتضاب للدعاء: «نعمة الله له».

13. في (هـ): «زيادة: في تقرير أنه إذا وقع خط على خطين وصيّر الزاويتين الداخلتين في جهة أقل من قائمتين، فإنهما يلتقيان في تلك الجهة». (ص. 10أ).

14. يعني: «قال الأبهري».

15. في (هـ): «أوتاراً».

16. في (هـ): زيادة «ج» أي الحرف الذي يرمز إلى الشكل الثالث المستعمل في هذا البرهان.

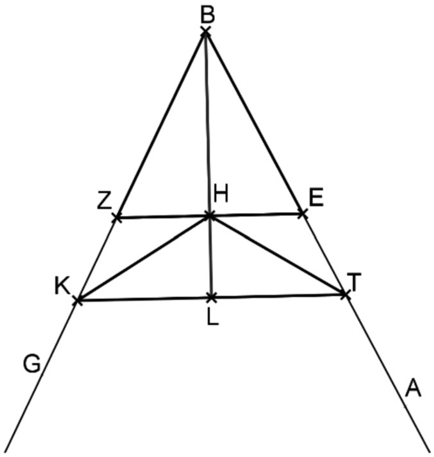
17. في (هـ): زيادة «ز» أي الشكل السابع.

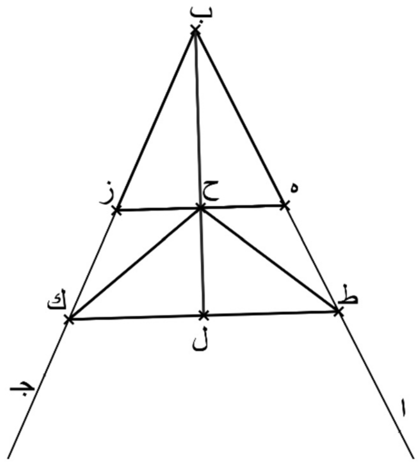
fore the two angles H are equal and BH is perpendicular to EZ. We take BT equal to BK and connect T and K. Line TK does not pass by point H; otherwise angles BHT and BHK would be equal to two right angles, they are also equal to angles BHE and BHZ. This is impossible.

<Line TK> does not cut HZ; otherwise the two lines would encompass one surface. This is impossible.

عمود على هـ ز. ونفصل ب ط مثل ب ك[18] ونصل ط، ك. فخط ط ك لا يمرّ بنقطة ح. وإلا لكان زاويتا ب ح ط، ب ح ك مثل قائمتين،[19] وقد كان ب ح هـ، ب ح ز مثل قائمتين. هذا خلف.

ولا يقطع (خط ح ز)[20] وإلا لأحاط خطّان مستقيمان بسطح واحد. هذا خلف.





Therefore we can draw infinitely many chords.

وهكذا يمكن إخراج /74ظ/ الأوتار إلى غير النهاية.

[§3] I say: And line TK does not meet line EZ at a point other than H; because if that point lies between E and Z, it follows from the impossibility what follows from its passage through H.

(أقول و لا يَـمُرُّ خط ط ك بنقطة أخرى من خط هـ ز غير ح، لأن تلك النقطة إن كانت فيما بين هـ، ز لزم من المحال ما لزم من مروره بـ ح.

18. في (هـ): زيادة «ج» أي الشكل الثالث.
19. في (هـ): زيادة «يجـ» أي الشكل الثالث عشر.
20. في (تـ): «هـ ط» لا معنى لهذه الحروف. وفي (هـ): «خط هـ ز» وهو خط ح ز.

And if the endpoint <K> is Z, then when TK passes through point Z forward to K, either it gets superposed on ZK connecting K and Z rectilinearly by both the lines ZB and ZT, or they do not coincide and form together a surface. The two cases are impossible.

وإن كانت الطرف مثل ز، فلأن ط ك بعد مجاوزته بنقطة ز إلى ك إما أن ينطبق على ز ك فيتصل ك، ز على استقامته بخطي ز ب، ز ط معًا، أو لا ينطبق، فيحيط مع ز ك بسطح. وكلاهما مح)[21].[22]

[§4] If that has been checked, we say: If a line falls on two lines and makes the interior angles which are on a same side less than two right angles, then the two lines if extended meet on that side, because these two angles can only be both acute, or one is acute and the other either right or obtuse.

قال:[23] وإذا ثبت هذا، فنقول: إذا وقع خطٌّ على خطّين وصيّر الزاويتين في جهة أقل من قائمتين، فإن الخطين[24] يلتقيان في تلك الجهة، لأنه لا يخلو إما أن تكونا حادتين، أو إحديهما حادة والأخرى قائمة أو إحديهما حادة والأخرى منفرجة.

[§5] First let one of the two [angles] be acute and the other right, as line AB falling on two lines AG and BD and making angle ABD right and angle BAG acute, as shown in the second diagram. We take angle BAE equal to angle BAG and extend AB straightly toward Z. Then AZ bisects angle EAG and it is possible to draw in it [i.e. in angle EAG] chords, one under the other, as shown previously. So draw these chords until one of them falls under the point B.

فليكن أولا (إحديهما حادة)[25] والأخرى قائـمة، مثل خطي ا جـ ب د وقع عليهما خط ا ب وصيّر زاوية ا ب د قائـمة وزاوية ب ا ج حادة، كما في الصورة الأولى.فنعمل زاوية ب ا هـ مثل ب ا جـ ونخرج ا ب بالاستقامة إلى ز. فزاوية هـ ا جـ منصّفة بخط ا ز. فيمكن أن يخرج لها أوتار تقع بعضها تحت بعض، (كما سبق).[26] فيخرج لها أوتار إلى أن تقع تحت نقطة ب.

[§6] I say: If someone asks why should some <of the chords> fall under <point> B and

(أقول فإن قيل لم يجب ان يقع بعضها تحت ب ولا يجوز أن تقع جميعًا فيما بين ا،

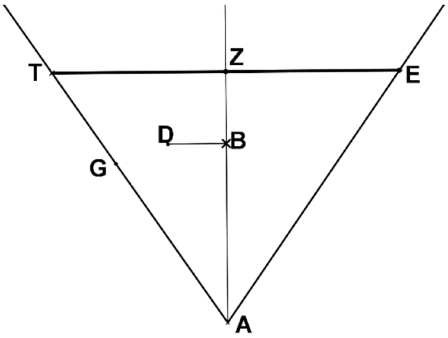
21. «مح»، هذه الحروف رمز لكلمة «محال».
22. في (هـ): غابت هذه الفقرة: «أقول ولا يمرّ... وكلاهما مح».
23. يعني: «قال الأبهري». وهذه الكلمة غائبة من (هـ).
24. في (هـ) «فالخطان».
25. «إحديهما حادة»، زيادة في هامش الورقة يستقيم بها المعنى.
26. في (هـ): «لماmرّ».

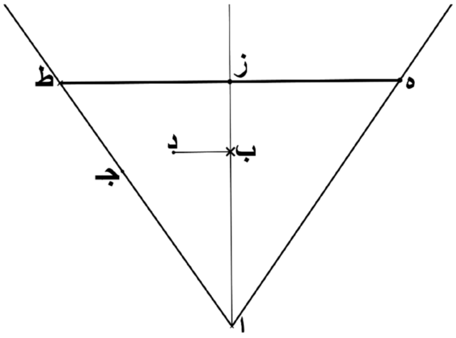
ب، (فإن ا ب)[27] **تقبل القسمة إلى غير نهاية؟**
فلنا الانقسام إلى غير نهاية[28] أمر فرضي ولا يقع بالفعل أبدًا وخروج الأوتار المذكورة تقتضي انقسامات لخط ا ب إلى أقسام غير متناهية بالفعل، فلو كانت جميعًا وهي غير متناهية مارة بين ا، ب لزم أن تكون تلك الأقسام الغير المتناهية عددًا مستلزمًا[29] لكونها خطًا غير متناه محصورًا بين حدي ا، ب. هذا خلف).[30]

why is it not possible that all the chords fall between <points> A and B since AB may be indefinitely divided?
And since dividing <a line> infinitely is a virtual <essential> thing and never becomes actual, while the production of these chords require cutting off the segment AB into an infinite number of actual parts, and if all the infinite <number of> chords fall between A and B, it is necessary that the infinite <number> of parts <of AB> be a number that is an unlimited line limited between the two ends of AB. This is impossible.

قال:[31] وليكن ﻫ ط مارًا تحت نقطة ب.فلأن ا ز عمود على ﻫ ط لما مر، فـ ز ط لا يلقى ب د، وإلا لحدث في مثلث قائمتان.[32] و هذا خلف.
فـ ب د إذا أخرج بالاستقامة يقطع خط[33] ا ط.

[§7] He said: Let HT be [the chord] passing trough point B. Then, since AZ is perpendicular to ET, as shown previously, ZT will not meet BD, for if it did a triangle with two right angles would result. This is impossible; then if BD is extended

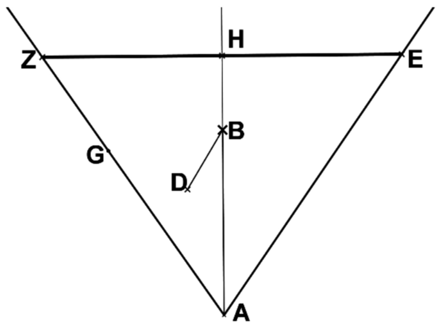


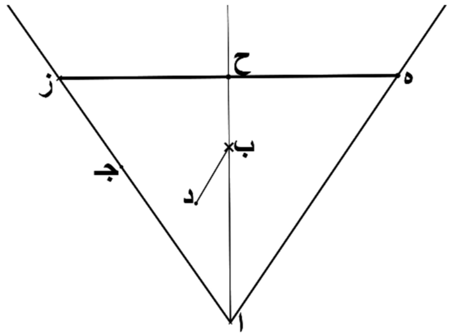


27. في (تـ): «المستلزم»، وهذا لا يصح نحويًا.
28. في (تـ): «ما نهاية»، و ولا يستقيم بها المعنى.
29. «فإن ا ب»، زيادة في هامش الورقة يستقيم بها المعنى.
30. في (هـ): غابت الفقرة: «أقول فإن يمر خط... هذا خلف».
31. يعني «قال الأبهري». وفي (هـ): غابت العبارة: «قال».
32. في (هـ): زيادة «يز» أي الشكل السابع عشر.
33. في (تـ): بهامش الورقة: «ح ط»، ولا يستقيم بها المعنى.

[§8] Let the two angles be acute, as line AB falling on two lines AG and BD and making the two angles GAB and ABD <both> acute, as shown in the second diagram.

ولتكن الزاويتان حادتين.
مثل خطي ا جـ، ب د وقع عليهما خط ا ب وصيّر زاويتي جـ ا ب، ا ب د حادتين كما في الصورة الثانية.





We take angle BAE equal BAG. Then angle EAG is bisected by line AB. We draw in it [i.e. in angle EAG] chords some of them being located one under the other, as shown before.

Let <line> EZ be passing through a point located under point B. And since angle ABD is acute, angle HBD is obtuse, and AHZ was proved right, then line EZ does not meet BD on the side of D and Z, for if it did a triangle would result with two angles, one right and the other obtuse. This is impossible; therefore, when extended, line BD will meet AG.

فنعمل زاوية ب ا ه مثل ب ا جـ. فزاوية ه ا جـ منصّفة بخط ا ب. فنخرج[34] لها أوتار يقع بعضها تحت بعض لما مر.

وليكن ه ز مارًا بنقطة تحت نقطة ب. فلأن زاوية ا ب د حادة فزاوية ح ب د منفرجة و ا ح ز قائمة لما مر، فخط ه ز لا يلقى ب د، (يعني في جهة د، ز)[35] وإلا يحدث في مثلث واحد[36] زاويتان أحديهما /75و/ قائمة والأخرى منفرجة.[37] هذا خلف.(فخط بد)،[38] إذا أخرج، يقطع ا جـ.

34. في (هـ): «فيمكن أن يخرج».
35. في (هـ): غابت الجملة «يعني في جهة د، ز».
36. في (هـ): غابت الكلمة «واحد».
37. في (هـ): زيادة «يز» أي الشكل السابع عشر.
38. ي(هـ): «فـ ب د».

[§9] Let one of the angles be acute and the other obtuse, as line AB falling on two lines AG and BD and making the two angles BEZ and DZE less than two right angles, with angle DZE obtuse and angle BEZ acute, as shown in the third diagram.

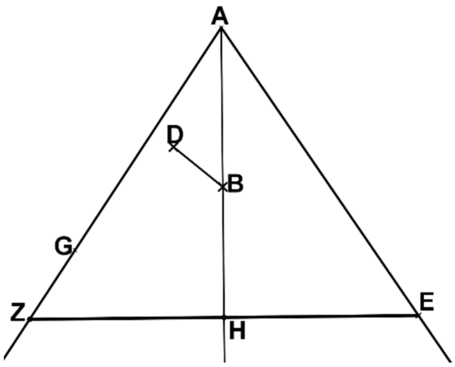
We bisect line EZ at point H and draw a perpendicular HT from point H to GD and extend it straightly.

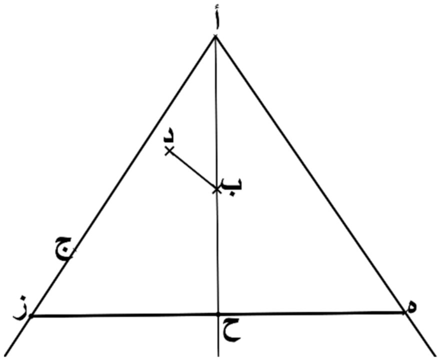
And since angle HTZ is a right angle, angle THZ is acute. Therefore angles EHM and BEH are acute. Then the two lines AE and HM will meet, as shown previously.

وليكن إحديهما حادة والأخرى منفرجة، مثل خطي اب، جـ د وقع عليهما خط هـ ز وصيّر زاويتي ب هـ ز، د ز هـ[39] أصغر من قائمتين وزاوية د ز هـ منفرجة و ب هـ ز حادة كما في الصورة الثالثة.

فننصّف خط هـ ز على نقطة ح ونخرج من نقطة ح خط ح ط عمودًا على جـ د ونخرجه بالاستقامة.

فلأن زاوية ح ط ز قائمة، فـ ط ح ز /هـ: ١١و/ حادة[40] (فـ هـ ح م حادة[41] و ب هـ ح حادة. فخطّا ا هـ، ح م يلتقيان لما سبق.





Let K be their meeting point. Then the angle EKH is obtuse; otherwise it would be right or acute.

For if it were a right angle, both the two angles EKH and EHK would equal the two

وليكن التقاؤهما)[42] على ك، فزاوية هـ ك ح منفرجة، لأنها لو لم تكن منفرجة لكانت إما قائمة أو حادة.

فإن كانت [زاوية هـ ك ح] قائمة، فزاويتا هـ ك ح، هـ ح ك مثل زاويتي ح ط ز، ط ح ز و هـ ح مثل

39. في نسختنا: «د هـ ز»، وهذا غير صحيح.
40. في (هـ): زيادة «يز» أي الشكل السابع عشر.
41. في نسختنا: «ز ح م حادة»، وهذا غير صحيح.
42. في (هـ): سقطت الجملة (فـ ر ح م حادة و ب هـ ح حادة. فخطّا أ هـ، ح م يلتقيان لما سبق. فليكن التقاؤهما).

angles HTZ and THZ, and EH equal HZ, then angle KEH would equal EZT. Then we consider DZE as a common angle, then the two angles Z are equal to the two angles DZE and KEH and then the two angles Z are smaller than two right angles. This is impossible.

And if it <angle EKH> were acute and angle KTG right, the two lines AB and DG would meet in the same side as the two points A and C, as shown before.

Let L be their meeting point; then since the two angles BEZ and DZE are smaller than two right angles and both angles AEZ and KEZ are equal to two right angles, angle DZE is smaller than angle AEZ. This is impossible.

So, angle EKH being neither acute nor right is obtuse, then angle BKT acute and angle DTK right. Therefore AB and GD meet on the side of points B and D.

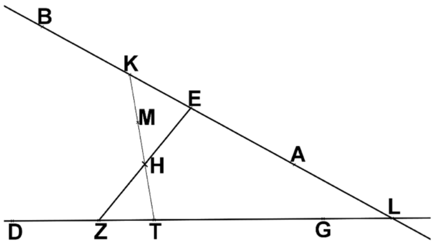
And that is what we wanted <to prove>.

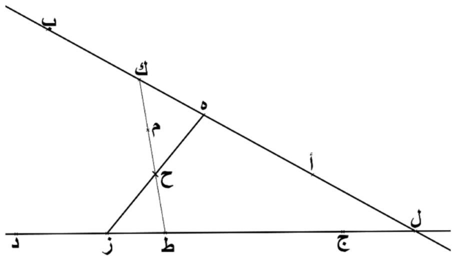
ح ز. فزاوية ك هـ ح مثل هـ ز ط.[43] فنجعل د ز هـ مشتركًا. فزاويتا ز مثل ز اويتي د ز هـ، ك هـ ح. فزاويتا ز أصغر من قائمتين.[44] هذا خلف.

وإنكانت [زاوية هـ ك ح] حادة وزاوية ك ط ج قائمة، فخطّا ا ب، د ج يلتقيان في جهة نقطتي ا، ج، لما مرّ.

وليكن التقاؤهما على نقطة ل. فلأن زاويتي ب هـ ز، د ز هـ أصغر من قائمتين، وزاويتا أ هـ ز، ك هـ ز مثل قائمتين،[45] فزاوية د ز هـ أصغر من زاوية أ هـ ز. فالخارجة أصغر من الداخلة.[46] هذا خلف.

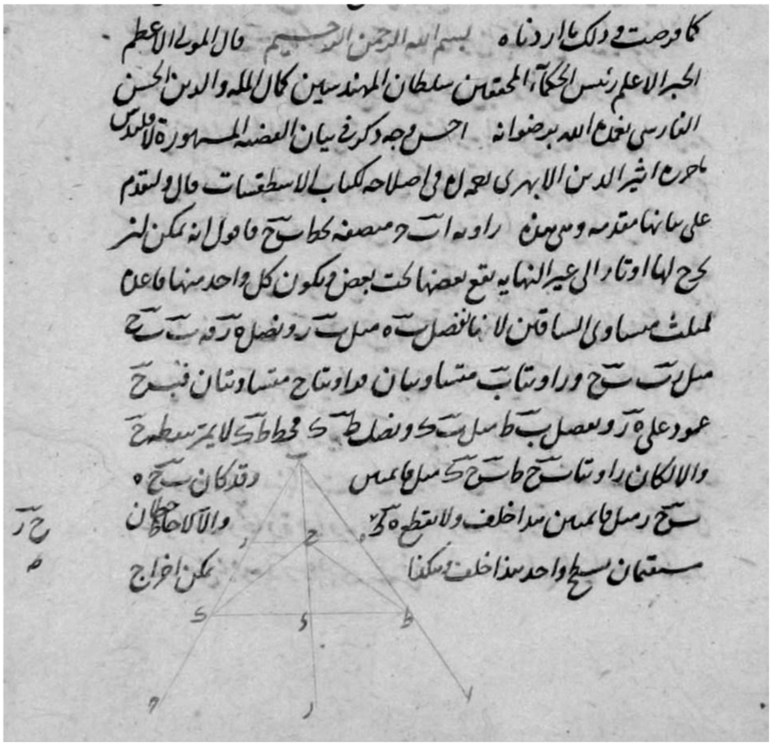
فزاوية هـ ك ح ليست بحادة ولا قائمة فهي منفرجة. فزاوية ب ك ط حادة وزاوية د ط ك قائمة. فخطّا أ ب، ج د يلتقيان في جهة نقطتي ب، د لِمَاتبيّن.[47] وذلك ما أردناه.[48]



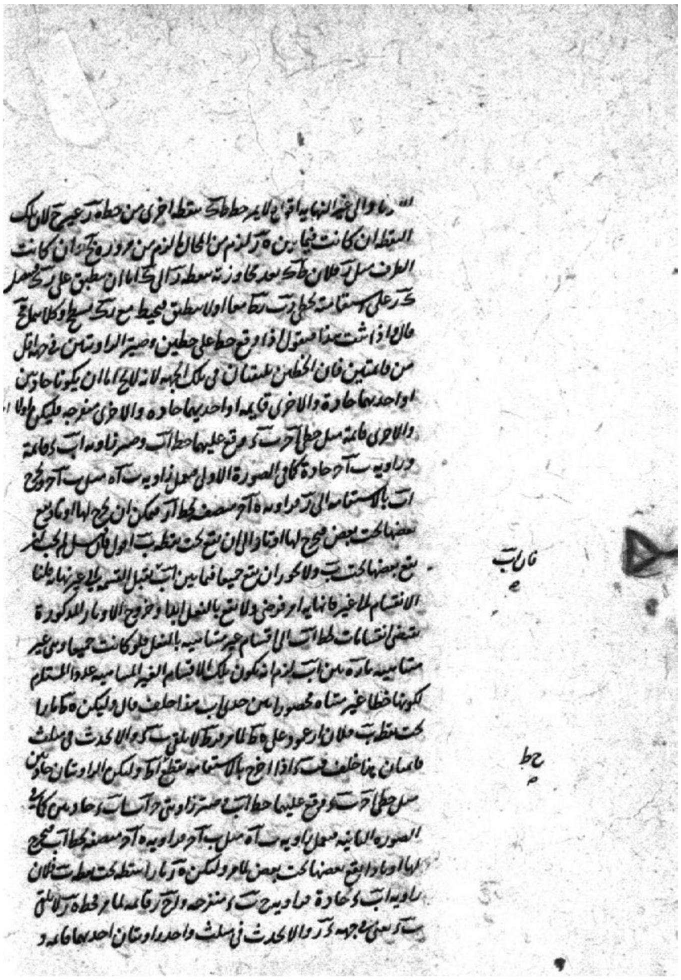


43. في (هـ): زيادة «كو» أي الشكل السادس والعشرين.
44. في (هـ): زيادة «يجـ» أي الشكل الثالث عشر.
45. في (هـ): زيادة «يجـ» أي الشكل الثالث عشر.
46. في (هـ): زيادة «يو» أي الشكل السادس عشر.
47. في (هـ): «لما سبق».
48. في (هـ): «ما أردنا بيانه».

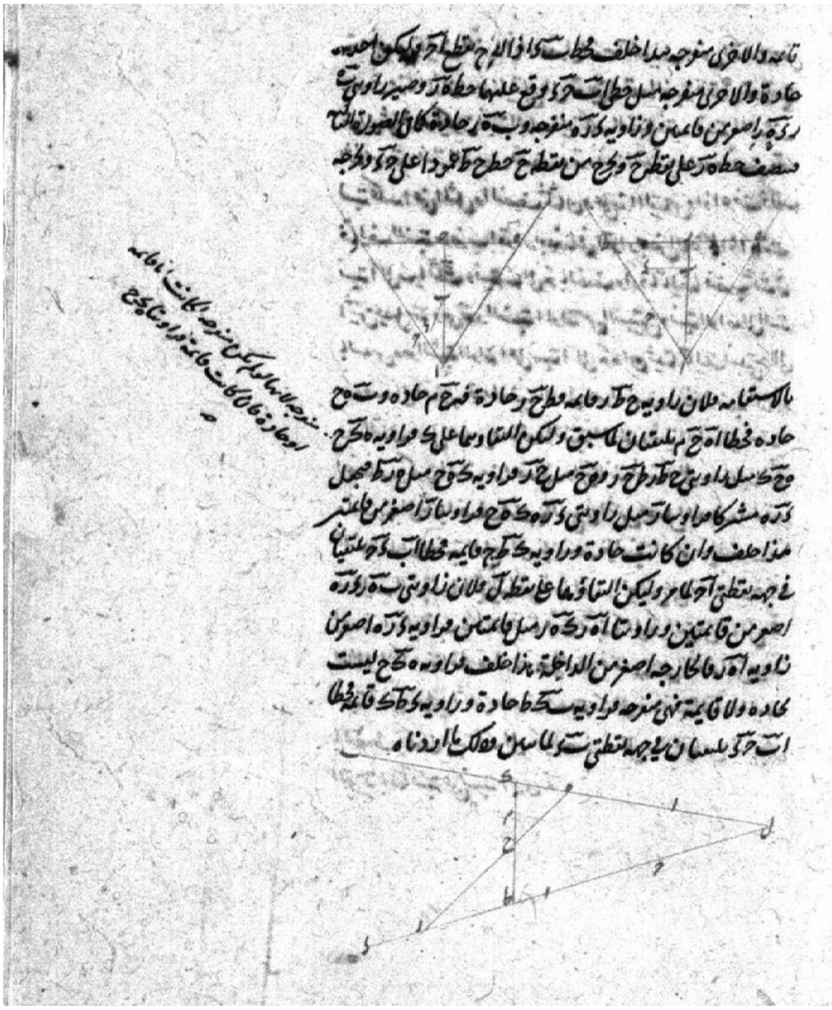
Photographs of the pages of the manuscripts


كما وصفت وذلك ما اردناه   بسم الله الرحمن الرحيم   قال المولى الاعظم
الحبر الاعلم رئيس الحكماء المحققين سلطان المهندسين كمال الملة والدين الحسن
الفارسي تغمده الله برضوانه   احسن وجه ذكر في بيان القضية المشهورة لاقليدس
ما حرره اثير الدين الابهري تغمده الله في اصلاحه لكتاب الاسطقسات قال ونقدم
على بيانها مقدمة وهي هذه   زاوية ا ب ج منصفة بخط ب ح فاقول انه يمكن ان
نخرج لها اوتارا الى غير النهاية يقع بعضها تحت بعض ويكون كل واحد منها قاعدة
لمثلث متساوي الساقين لانا نفصل ب ه مثل ب ز ونصل ه ز فـ ه ب ب ح
مثل ز ب ب ح وزاويتا ب متساويتان فزاويتا ح متساويتان فب ح
عمود على ه ز ونفصل ب ط مثل ب ك ونصل ط ك فخط ط ك لا في سطح
والا لكان زاويتا ب ح ط ب ح ك مثل قائمتين   وقد كان ب ح ه
ب ح ز مثل قائمتين هذا خلف ولا تقطع ه ز   والا لاحاط خطان
مستقيمان سطحا واحدا هذا خلف فيمكننا   يمكن اخراج

Folio 74a



Folio 74b



Folio 75a