Naṣīr al-Dīn al-Ṭūsī on Lunar Crescent Visibility and an Analysis with Modern Altitude–Azimuth Criteria

Hamid-Reza Giahi Yazdi

Introduction

In Islamic astronomy, lunar crescent visibility was an important observational subject of study. As the Muslim calendar is completely lunar, each month begins with the first observation of the lunar crescent above the western horizon. The determination of lunar crescent visibility is a difficult task, because it depends on a variety of astronomical and a variety of atmospheric parameters, some of the most important of which are the moon's angular separation from the sun, the time lag between sunset and moonset, the position of the moon in its orbit (relative to its apogee and perigee), the fraction of the moon's surface that is illuminated, the clearness of the sky, and the observer's power of vision. Therefore, medieval Muslim astronomers sought to develop criteria for determining lunar crescent visibility at the beginning of each month.[1]

Some of the tables or criteria for lunar crescent visibility in medieval Islamic astronomy have been studied by historians of astronomy, but many of their methods have not yet been investigated. A historically important criterion based on two parameters, the ecliptic elongation, (*bu*ᶜ*d-i siwā'*) and the adjusted elongation (*bu*ᶜ*d-i mu*ᶜ*addal*), was used by Naṣīr al-Dīn al-Ṭūsī (597-672 H/1201-1274 A.D.) in his *Īlkhānī zīj,* compiled ca. 640 H

[1] See King, "Ru'yat al-hilāl" [=lunar crescent visibility] in the *Encyclopedia of Islam*; also Kennedy and Janjanian, 1965, or Kennedy, 1983, pp. 151-156. See Kennedy, 1956, for significant *zīj*es containing tables on lunar crescent visibility. For a list of modern works dealing with lunar crescent visibility in Islamic sources, see King, 1993, II, p.219 or III, p.166, on lunar crescent visibility and the regulation of the Islamic calendar.

/ 1242. I have not found this criterion in works prior to his, but it is used in later works such as the *Khāqānī zīj* compiled ca. 820 H/1417 A.D., by the Iranian mathematician and astronomer Ghiyāth al-Dīn Jamshīd al-Kāshī (796-853 H/1393-1449 A.D.), Ulugh Beg's *Sulṭānī zīj* (ca. 1430 A.D., Samarqand), and *Bahādur Khānī zīj,* compiled in 1838 by Gh.-Ḥ. Jawnpūrī (1790-1862 A.D.) in south Bihar in India, with minor modifications.[2]

The astronomical basis of al-Ṭūsī's method

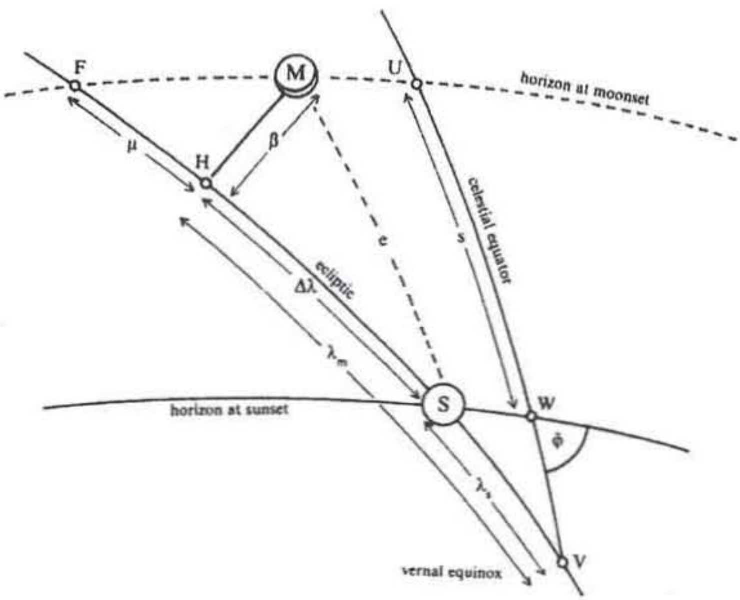

Fig. 1. From King, 1993, III, p. 156.

[2] The parameters mentioned in the *Khāqānī, Ulugh Beg* and *Bahādur Khānī zīj*es are those that appear in the *Īlkhānī zīj*, but not all the critical values are shown. There is no mention of part 2 of al-Ṭūsī's method in any of them. In the *Khāqānī zīj* all the crescents with Δλ >10° and U'W' >16° are considered "quite visible" and in *Ulugh Beg* and *Bahādur Khānī zīj*es all the crescents with Δλ >10° and U'W'>14° are considered "visible" (cf. table 1). For the *Khāqānī zīj* I used Suleymaniye Library Ms. 2692, (fol. 67v); for Ulugh Beg's *zīj* see Sedillot, pp. 425-426 (for its French translation see section 11, pp. 190-192); for the *Bahādur Khānī zīj* see Jawnpūrī, pp. 556-557.

Fig. 1 shows the western horizon at sunset. In this figure, S is the position of the sun on the ecliptic, M is the position of the moon in relation to the ecliptic vernal point, $\lambda_s = VS$ is the solar longitude, $\lambda_m = VH$ is the lunar longitude, and $\beta = HM$ is the lunar latitude; thus: $SH = VH - VS = \lambda_m - \lambda_s = \Delta\lambda$. W and U are the points on the celestial equator which set at the time of sunset and moonset respectively.

Al-Ṭūsī explained the first lunar crescent visibility by finding the difference in longitudes $\Delta\lambda$, or ecliptical elongation (*buᶜd-i siwā'*), and the difference in the setting times of the moon and the sun (corresponding to the arc UW in degrees), or adjusted elongation (*buᶜd-i muᶜaddal*), for the beginning of each lunar month. He also developed a criterion using these quantities. It should be noted that al-Ṭūsī did not compute UW for the western horizon directly: I will discuss the method he used in the commentary below.

Naṣīr al-Dīn al-Ṭūsī's criterion

Many manuscripts of the *Īlkhānī zīj* exist. Below is a translation of al-Ṭūsī's criterion from Ms. 684:1 (copied 10th c. H/ 16th c. A.D), in the Shahīd Muṭahharī (ex-Sepahsālār) library in Tehran:[3]

Chapter 2, Section 11: On finding the times of lunar crescent visibility, and the appearance and disappearance of the planets

> There are several procedures. The most used method is (as follows). We calculate the true longitude of the two luminaries and the lunar latitude for the sunset on the 29th day of the preceding lunar month. We then take 3/5 of the lunar latitude or we multiply the latitude by 36′: we call it "deviation" (*inḥirāf*). If the lunar latitude is northerly, we add the deviation to the lunar longitude. If it is southerly, we subtract (the deviation) from it. We call the result (in either case) "the adjusted moon" (*qamar-i muᶜaddal*). We then subtract the oblique ascension of the point of the ecliptic opposite the setting sun from the oblique ascension of the point of the ecliptic opposite the adjusted moon. We call the result "the adjusted elongation" (*buᶜd-i muᶜaddal*). We obtain the difference between the true longitudes of the two luminaries, and call it "ecliptical elongation" (*buᶜd-i siwā'*).
>
> If the adjusted elongation is between 10° and 12°, and the ecliptical elongation is greater than 10°, the thin crescent can be observed.

[3] I also used another manuscript of the *Īlkhānī zīj*: Tehran University Library, Ms. 156 (Ḥikmat collection); see Monzavi 1348 H.S/1969, p. 300, for a list of manuscripts of the *Īlkhānī zīj*.

Otherwise, it is not visible, unless the adjusted elongation amounts to 12° or more. If the adjusted elongation is between 12° and 14°, the crescent will be observed moderate(ly). If it is between 14° and 16°, the crescent will be bright. If it is between 16° and 18°, the crescent will be visible. If it is greater (than these values), the crescent will be quite visible. The direction of the crescent is (always the same as) the (direction of the lunar) latitude.

Commentary

The calculation of the ecliptical elongation is easy to understand from the text, but that of the adjusted elongation is more complicated and needs to be done after the computation of the "deviation" and the "adjusted moon". The deviation (d=3/5β) is an approximate value for *HF* which was used instead of *HF* (the arc between the perpendicular projection of the moon on the ecliptic and the point on the ecliptic which sets simultaneously with the moon). In the *zījes* of *Ulugh Beg* and *Bahādur Khānī, HF* is called "the setting equation"(*ta*ᶜ*dīl al-ghurūb*) and calculated according to an exact method. In the *Bahādur Khānī zīj*, the computation corresponds to the following formulae:

$$\text{(1)} \qquad \sin(HF) = \text{tg}(\beta) / \text{tg}(90° - \theta).$$

where β is the lunar latitude and θ is the altitude of the pole of the ecliptic, which was called "the latitude of the visible climate" (*ᶜarḍ-i iqlīm al-ru'ya*), in (1) if $\beta = \pm 2.65°$, (the mean value of the moon's ecliptic latitude), $\theta = 13.9°$ and $\theta = 60.9°$ (the minimum and the maximum values of θ for the location of al-Ṭūsī's observatory in Marāgha, φ =37;24°), the value of *HF* is calculated = $\pm 0.66°$ and $\pm 4.77°$ respectively. Therefore the mean value of *HF* is $\pm 2.72°$. For this value of β, $d = 3/5\ \beta = \pm 1.59°$, obviously near *HF*. It seems that the deviation (d) was chosen near the mean value of the setting equation *HF*, to simplify the calculation of latitudes near Marāgha. Then the adjusted moon *VF* could be calculated as below:

$$\text{(2)} \qquad VF = \lambda_m \pm d \approx VH \pm HF. \quad (\beta \lesseqgtr 0)$$

Part	Difference inlongitudes ($\Delta\lambda$)	Difference in setting times ($U'W'$)	Condition of crescent
1	>10°	10°-12° 12°-14° 14°-16° 16°-18° >18°	thin moderate bright visible quite visible
2	<10°	≥12°	visible

Table 1

Consequently, the adjusted elongation is the difference $U'W'$ between the oblique ascension of the point of ecliptic opposite the setting sun and the oblique ascension of the point of the ecliptic opposite the adjusted moon. This difference is equal to *UW* in Fig. 1 and is approximately equivalent to the true difference in the setting times in degrees.[4]

Today, we can compute the lunar coordinates exactly, up to the moon's setting time; but in Islamic astronomy the computation of the change in lunar coordinates for small consecutive intervals was not easy. Muslim astronomers computed the lunar position for the time of sunset; then they computed the approximate time of moonset. Since the moon moves toward the eastern sky after sunset, the real difference in setting times in degrees is slightly more than the "adjusted elongation".[5] However, after calculating these parameters, al-Ṭūsī determined the lunar crescent visibility or invisibility by using the critical values given in Table 1.

This criterion is similar to that of Battānī (244-317 H/858-929A.D.), who presents the same parameters but different critical values.[6]

[4] See King, the article "al-Maṭāli^{c} " in the Encyclopedia of Islam.

[5] See Iḥiaee, 1993.

[6] Al-Battānī considers this criterion: $\Delta\lambda \geq 13\ 2/3°$ and UW $\geq 10\ 5/6°$ for the lunar crescent visibility. See Nallino, 1899, pp. 132-133 and Bruin, 1979.

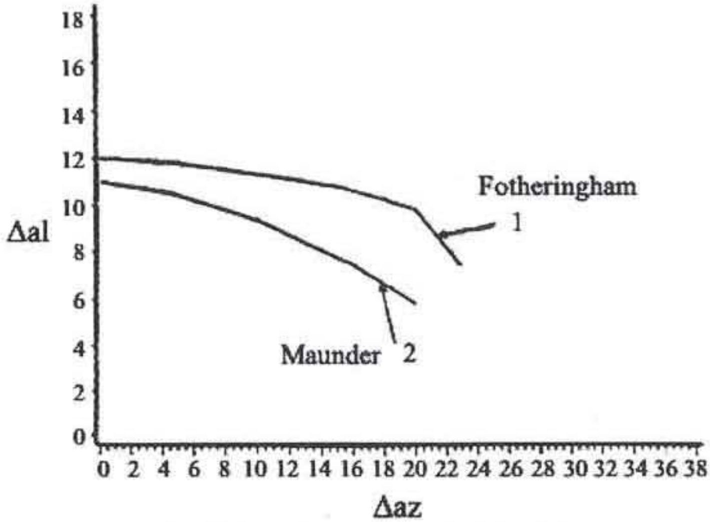

Fig. 2. From Ilyas (Penang, 1995) p. 23.

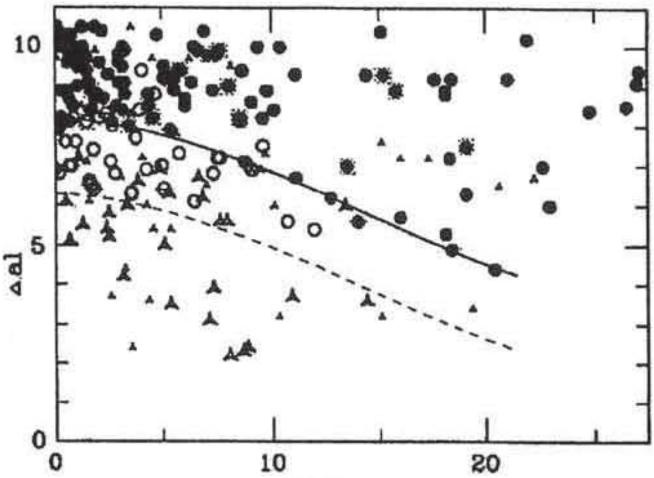

Fig. 3. Filled circles: successful sightings with the naked eye; open circles: visible through telescopes or binoculars, but not visible to the naked eye; large triangles: denote that the crescents were invisible for both optical-aided and the naked eye; small triangles: represent unsuccessful sightings with the naked eye without optical-aid; circles with small halo of small dots: distinguish events observed at high latitudes (at least 45°). The diagram is from http://www.saao.ac.za .

Comparison of Ṭūsī's method with altitude-azimuth criteria

In the early 20^{th} century, lunar crescent visibility was determined on the basis of altitude-azimuth diagrams, in which the azimuth difference between the sun and the moon at sunset, Δaz, is drawn versus the altitude of the moon at sunset, Δal. On the basis of various observations, a series of limiting curves for lunar crescent visibility is obtained.

In Fig. 2, curve 1 is based on the research of Fotheringham (1910), and curve 2 is obtained from the study by Maunder (1911).[7] According to Fotheringham's criterion, the smallest angular separation of the moon from the sun (to be observed by the naked eye) is about 12°, but Maunder considered it to be 11°. Fig. 3 shows a new altitude-azimuth criterion established by *SAAO* (the South African Astronomical Observatory). It presents two boundary curves; below the solid one, visual sighting is improbable.[8] Clearly, the naked eye boundary is lower than Maunder's and Fotheringham's limits. The dotted line shows the limit of optically-aided sightings. These curves are intentionally optimistic and both curves are given in ideal viewing conditions. Since other factors such as the moon's distance from the earth, atmospheric conditions, and the observer's power of vision would affect lunar crescent visibility, in fig. 3 each limit is practically a band rather than an exactly linear curve. For instance, some open circles indicating cases which were just visible through telescopes or binoculars overlap to some extent with the successful sightings with the naked eye (filled circles).

In the altitude-azimuth diagram, as the azimuth difference between the sun and the moon increases, the angular separation and the moon's illuminated fraction will also increase. So, the crescent will be visible for a smaller value of altitude. It is obvious that al-Ṭūsī's method is based mainly on computation whereas the altitude-azimuth method has an observational foundation.

It is interesting to define the observational interpretation of al-Ṭūsī's criterion in terms of altitude-azimuth diagrams. This interpretation has not been made before either on al-Ṭūsī's criterion or on other lunar crescent visibility tables or criteria in Islamic astronomy. By interpretation of al-Ṭūsī's criterion in this way we may determine its exactness and obtain information about his observations. For example, we may consider the

[7] See Ilyas, 1994; di Cicco, 1989.

[8] See Caldwell and Laney, 2001.

determination of a crescent altitude and azimuth difference at sunset with critical values of parameters such as $\Delta\lambda$ = 10° and $U'W'$ = 10°. The angle between the ecliptic and the horizon depends on the geographical latitude φ, time of the observation, and obliquity of the ecliptic ε. So, for the specific values of $\Delta\lambda$ and $U'W'$, different values of lunar azimuths and altitudes are possible. Since the geographical latitude is an important parameter that connects al-Ṭūsī's method to the altitude-azimuth method, we should first define the latitude of the observation site. The *Īlkhānī zīj* was prepared in the observatory at Marāgha in Iran, with φ = 37;24°. Therefore I used this modern value for φ in my calculations. The variation of ε does not have a noticeable effect on the calculations, so I used ε = 23;30°.

Computational method

I prepared a computer program which computes azimuths and altitudes of lunar crescents for which $\Delta\lambda$ =10° and $U'W' \geq$10°, to examine the first part of al-Ṭūsī's criterion. With some changes, the program also computes altitude, azimuth and the values of $\Delta\lambda \leq$ 10° for crescents with $U'W'$=12°. In this section I examine the second part of al-Ṭūsī's criterion.[9] For this purpose the inputs of the program are latitude, right ascension, and declination of the sun as independent variables. The right ascension and declination of the sun also define the time of the observation, and are considered for the beginnings of the four seasons: spring (0 h, 0 °), summer (6 h, +23.5°), autumn (12 h, 0°), winter (18 h, -23.5°). The program begins with the computation of the sun's azimuth at the time of sunset (for the given right ascension and declination). It continues searching in two loops, in the positive or negative direction of the sun's azimuth (adjustable in the program). The outer loop computes the lunar azimuth, and the inner loop computes the lunar altitude. The accuracy of computations may be adjusted in the program.

At first, the program moves one step in a defined direction, in relation to the sun's azimuth. Then the inner loop increases altitude with respect to the horizon. By changing the altitude, the program first computes the right ascension and declination, and then the ecliptic longitude and latitude of a given lunar crescent. If $\Delta\lambda$ is greater than 10.1°, the program moves to the

[9] The source of the formulae is Duffett-Smith, 1983, pp. 42-43, 46-47, 54-55.

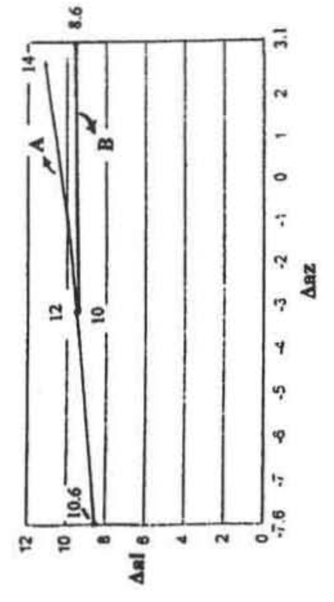

Fig. 4a: beginning of spring

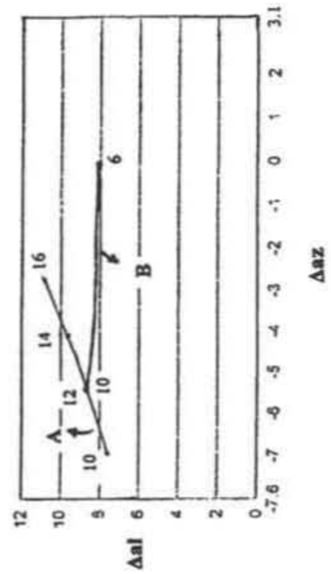

Fig. 4b: beginning of summer

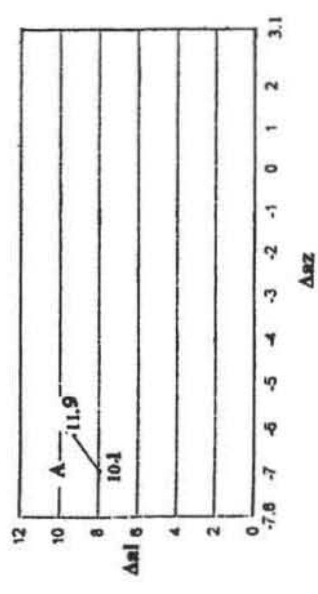

Fig. 4c: beginning of autumn

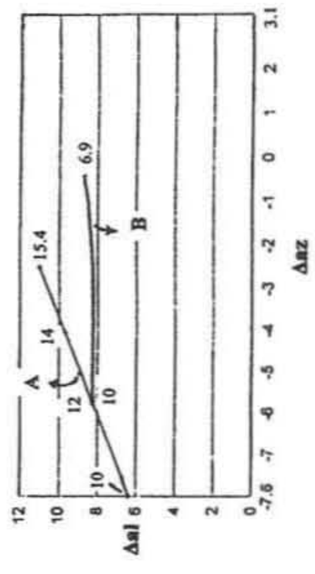

Fig. 4d: beginning of winter

next azimuth value; but if it is less than 10.1°, it increases the altitude. If 9.9°< $\Delta\lambda$ <10.1°, then it checks the lunar latitude.[10]

If the lunar latitude is outside the interval of (- 5.3°, +5.3°), the program returns to the azimuth loop and changes the azimuth; otherwise, the program computes *U'W'*. If it is less than 10°, the program increases altitude; otherwise it prints the values of azimuth, altitude, difference in longitudes $\Delta\lambda$, difference in setting times *U'W'*, and the ecliptic latitude of the crescent β. The program then returns to the azimuth loop.

Conclusion

On the basis of computations and interpretations of the diagrams, my findings are as follows:

The computations of altitudes and azimuths of the points for which $\Delta\lambda$ = 10° and *U'W'*≥10° show a variety of curves (series *A*). At different times, the numbers above the curves *A* show different *U'W'* values for $\Delta\lambda$ = 10°. It is obvious that the first visibility, the point on the curve *A* where $\Delta\lambda$ =10° and *U'W'*=10°, begins at an altitude near 8°, which corresponds with to the *SAAO* criteria.

The computations showed that with a noticeable change of latitude, the curves also change considerably. The series *B* curves are related to the points where *U'W'*= 12° and $\Delta\lambda \leq$ 10°. These show the altitude-azimuth form of the second part of al-Ṭūsī's method. The numbers below the curves *B* show different $\Delta\lambda$ values for *U'W'*=12°. Curves *A* and *B* have one common point. Curves *B* also show an straight boundary near 8°. Computations show that at the beginning of autumn, there is no crescent observable for the condition mentioned in the second part (Fig.4a - 4d). In his *Risāla-ye Muʿīniyya,* al-Ṭūsī mentions another method :[11]

Chapter 14: On the conjunction, appearance and disappearance of the planets.

In these climates the dominant (criterion) is (the possibility of observing the new moon crescent) when the altitude (of the moon) at sunset is (at least) 8°, or when the time lag between the setting (of the moon) and sunset is at least 4/5 of an hour, (the moon) is visible.

[10] The range 9.9-10.1° is for steps of 0.2°. If we choose smaller searching steps, the range can also be smaller.

[11] See al-Ṭūsī, *Risāla-ye Muʿīniyya*, p. 56-58.

This criterion originated in Babylonian astronomy and seems to have been transmitted to Islamic astronomy through Indian astronomy. Since 4/5 of an hour is the time in which a Hindu *pinkan* water-clock sinks twice, the time lag between moonset and sunset has been considered equal to two *pinkans*.[12] Obviously, with this criterion, 4/5 of an hour is equivalent to $U'W'=12°$. Comparing A and B curves with this older criterion we find that the lunar crescents with $U'W'=12°$ have altitudes near 8°. Therefore, al-Ṭūsī considered them to be equivalent.

It seems that al-Ṭūsī's method in the *Īlkhānī zīj* was devised for middle geographical latitudes and was probably made to define the first visibility point ($\Delta\lambda=10°$, $U'W'=10°$). Of course it is hard to believe that a crescent would be observed near an altitude of 6° (at the beginning of winter). However this criterion applies to many times of the year, and it provides an acceptable approximate method for determining the lunar crescent visibility.

Bibliography

1. F. Bruin, " The first visibility of the lunar crescent ", *Vistas in astronomy*, 21 (1979), 331-58.

2. J.A.R. Caldwell and C. D. Laney, "First visibility of the lunar crescent", in *SAAO* home page: **http://www.saao.ac.za**, page updated 16 February 2001.

3. D. di Cicco, Observer's page: "Breaking the new-moon record", *Sky &Telescope* (September 1989), pp. 322-323.

4. P. Duffet-Smith, *Practical Astronomy with your Calculator*, 2nd edition, Cambridge, 1983.

5. EI^2 : The Encyclopedia of Islam, new ed., Leiden, 1960-2002.

6. M. A. Iḥiaee, "Muḥāsaba-ye buᶜd-i muᶜaddal az awqāt-i ghurūb-i māh wa khurshīd", *Taḥqīqāt-i Islāmī*, vol. 8, nos.1&2 (1993), pp. 145-164.

7. M. Ilyas, "Lunar Crescent Visibility and Islamic Calendar", *The Quarterly Journal of the Royal Astronomical Society* (U.K), vol. 35, no. 4 (1994) pp. 425-461; reprinted Penang, 1995.

8. Gh.-H. Jawnpūrī, *Zīj-i Bahādur Khānī*, Benares, 1855.

9. E. S. Kennedy, *A Survey of Islamic Astronomical Tables*, Transactions of the American Philosophical Society, N.S., vol. 46, Pt.2, 1956.

10. E. S. Kennedy, colleagues and former students, *Studies in the Islamic Exact Sciences,* Beirut, 1983. It includes a reprint of E. S. Kennedy and M. Janjanian,

[12] See Bruin, 1979.

"The Crescent Visibility Table in al-Khwārizmī's *zīj*", *Centaurus*, 11 (1965), pp. 73-78; repr. on pp. 151-156.

11. D. King, *Astronomy in the service of Islam*, Aldershot, 1993.

12. A. Monzavi, *A Catalogue of Persian Manuscripts*, vol. 1, Tehran, 1348 H.S/1969.

13. C. A. Nallino, *Al-Battani sive Al-Batenii opus astronomicum* (publicazioni del Reale Osservatorio di Brera in Milano,XL), 3 vols. Milan and Rome, 1899-1907; reprinted Frankfurt 1969.

14. L. A. Sédillot, *Prolégomènes des tables astronomiques d'Oloug-Beg*, 2 vols., Paris, 1847-53.

15. Al-Ṭūsī, Naṣīr al-Dīn, *Risāla-ye mu^ciniyya dar hay'a*; with an Introduction by Moḥammad Taqī Dānesh Pazhūh, Tehran (no. 300 in the series), 1335 H.S./1956.

Appendix:

The Persian text of al-Ṭūsī's criteria in the *Īlkhānī zīj* and the *Risāla-ye Mu'iniyya*

مقالت دوم ، فصل یازدهم
در معرفت وقت رویت هلال و ظهور و اختفاء کواکب متحیره

آن را طریقهاست وآنچه مستعمل است در اکثر اوقات آن است که روز بیست و نهم ماه گذشته به وقت غروب آفتاب تقویم نیرین وعرض قمر استخراج کنیم وسه خمس عرض بگیریم یا عرض را در سي وشش دقیقه ضرب کنیم وحاصل را انحراف خوانیم. اگر عرض شمالي بود انحراف بر تقویم قمر افزائیم واگر جنوبی بود از او بکاهیم حاصل را قمر معدل خوانیم. پس مطالع نظیر آفتاب غروب را از مطالع نظیر قمر معدل هر دو ببلد بکاهیم حاصل را بعد معدل خوانیم وبعد میان تقویم نیرین در وقت غروب بگیریم و آن را بعد سوی خوانیم پس اگر بعد معدل میان 10 درجه و 12 درجه باشد و بعد سوا از 10 درجه بیشتر باشد هلال بتوان دید باریك والا نتوان دید تا بعد معدل دوازده وزیادت نشود و اگر بعد معدل میان دوازده وچهارده باشد هلال معتدل باشد واگر میان چهارده و شانزده باشد، هلال روشن باشد واگر میان شانزده وهجده باشد هلال ظاهر باشد واگر بیشتر

باشد سخت ظاهر باشد و جهت هلال جهت عرض باشد.

باب چهاردهم
در قرانات و ظهور و خفاء کواکب

... در این اقالیم اغلب آنست که چون ارتفاع [ماه] از وقت غروب آفتاب هشت درجه می باشد یا میان غروب [ماه] و غروب آفتاب چهار خمس ساعتی [باشد، ماه] ظاهر میشود...

Acknowledgements

It is a pleasure to thank Prof. David King, Dr. Hushang A'lam, Dr. Benno Van Dalen and Dr. Mohammad Bagheri for their comments on preliminary drafts of this paper.