The *Tāj al-azyāj* of Muḥyī al-Dīn al-Maghribī (d. 1283): methods of computation

Carlos Dorce

1. Introduction

This paper is about the first astronomical work of Muḥyī al-Dīn al-Maghribī[1], an Andalusī-Maghribī astronomer who worked in Damascus and Marāgha. His first dated astronomical work is the *zīj* titled *Tāj al-azyāj wa-ghunyat al-muḥtāj* (The Crown of the Astronomical Handbooks and the Satisfaction of the Needy). I know of three copies of this *zīj*: one in the Escorial Library, numbered Árabe 932 (Ms. E), another in the Chester Beatty Library, Dublin (Ms. Ar. 4129, Ms. D) and another in the Department of Arabic Philology of the University of Barcelona (Ms. B). Ms. E (fol. 57v) states that the *zīj* was compiled in Damascus in 656H/1258. This copy was made by Muẓaffar ibn ʿAbd Allāh in Tunis in 797H/1394. The manuscript has 119 folios in which the tables are

[1] The basic references to Muḥyī al-Dīn al-Maghribī are in H. Suter, *Die Mathematiker und Astronomen der Araber und Ihre Werke*, B. G. Teubner, Leipzig, 1900, repr. in idem, *Beiträge zur Geschichte der Mathematik und Astronomie in Islam*, 2 vols., Institut für Geschichte der Arabisch-Islamischen Wissenschaften, Frankfurt am Main, 1986, I, pp. 162-163; J.A. Sánchez Pérez, *Biografías de matemáticos árabes que florecieron en España*, Memorias de la Real Academia de Ciencias Físicas y Exactas, Madrid, 1921, repr. in Sierra Nevada, Granada, 1995, pp. 140-141; S. Tekeli, "Muḥyī l-Dīn al-Maghribī". *Dictionary of Scientific Biography*. Vol. IX, Charles Scribner's Sons, New York, 1974, pp. 555-557, and *Biographical Dictionary of Mathematicians*, Vol. 3, Collier Macmillan Canada, Toronto, 1991, pp. 1767-1769; D.A. King, *A Survey of the Scientific Manuscripts in the Egyptian National Library,* Publications of the American Research Center in Egypt, Catalogs, vol. 5, Winona Lake, Ind., 1986, p. 151 (G21); and M. Comes, "Ibn Abī l-Šukr al-Magribī, Abū ʿAbd Allāh", *Enciclopedia de al-Andalus. Diccionario de autores y obras andalusies.* Tomo I, Granada, 2002, pp. 381-385.

presented on fols. 60v-119v. Ms. B is not dated although we read that the copyist was the otherwise unknown ʿAbd Allāh al-Ṣanhājī al-Dādisī, "the most outstanding figure" in astronomical timekeeping in Marrakesh[2]. This manuscript has 111 folios in which the tables are presented on pages 43v-111v. It is in Maghribī script although the *abjad* used in it is Eastern, as in Ms. E. Finally, Ms. D is dated 1155H/1742. It is a Maghribī manuscript with 94 folios; fols. 28r-94r contain tables.

I have studied the tables of this *zīj* and the calculation procedures used by Muḥyī al-Dīn to compute them, in an attempt to identify what new parameters and methods of interpolation he used. This study has been made using methods established by Mielgo[3], Van Dalen[4] and Van Brummelen[5], and the computer programs designed by the first two. It appears that six tables of the *zīj* have been calculated with a procedure that has first been suggested by Van Brummelen and which I will describe and analyze in more detail in this paper.

The canons of the *zīj* begin with 21 chapters dedicated to chronology and calendars. In all these chapters, four eras are mentioned: Alexander (1st October B.C. 312), Diocletian (29th August A.D. 284), Hijra (15th July A.D. 622) and Yazdijird (16th June A.D. 632). These four epochs represent a wide variety of astronomical systems and societies: the Alexander era was used in the Byzantine calendar, the Diocletian era in the Coptic calendar and Hijra and Yazdijird eras are the beginning of Muslim and Persian calendars. The *Tāj al-azyāj* has tables for the conversion of the Persian, Coptic, Julian and Muslim calendars. The epochs of Philippus (12th November B.C. 324), Augustus (30th August A.D. –30), Antoninus

2 D.A. King suggests the possibility of identifying him with ʿAlī b. Muḥammad al-Dādisī (d. 1683). See C. Brockelmann, *Geschichte der Arabischen Litteretur* II (Berlin, 1902), p. 463; *Supplementband* II (Leiden, 1938), p. 708; H.P.J. Renaud, «Additions et corrections à Suter, "Die Mathematiker und Astronomen der Araber"», *Isis* 18 (1932), pp. 166-183 (cf. p. 180); and D.A. King, *A Survey of the Scientific Manuscripts...*, F48, p. 142.

3 H. Mielgo, "A Method of Analysis for Mean Motion Astronomical Tables" in *From Baghdad to Barcelona, Studies in the Islamic Exact Sciences in Honour of Prof. Juan Vernet*, 2 vols., Instituto "Millás Vallicrosa" de Historia de la Ciencia Arabe, Barcelona, 1996, Vol. I. pp. 159-180.

4 B. Van Dalen, *Ancient and Mediaeval Astronomical Tables: Mathematical Structure and Parameter Values*, Universiteit Utrecht, Faculteit Wiskunde en Informatica, Unpublished doctoral thesis, Utrecht, 1993.

5 G.R. Van Brummelen, *Mathematical Tables in Ptolemy's Almagest*, Simon Fraser University, February 1993, unpublished doctoral thesis.

(20^{th} July A.D. 137) and Hulaghū (11^{th} January A.D. 1258) are also mentioned, and the first three are widely quoted in the astronomical tradition and *zījes*. The Philippus era is dedicated to Philippus Arrhidaeos, the first Macedonian king in the list of kings of the *Almagest*[6]. Augustus and Antoninus are, respectively, the first and the last Roman emperors in the same list. These three epochs are used in the *Almagest*. The Philippus and Diocletian eras are also found in the *Handy Tables*[7], in al-Khwārizmī's *Zīj*[8], and in the *Chronology* of al-Bīrūnī[9]. We find the Augustus and Antoninus eras in the *Chronology*. Diocletian's era is also used by Yaḥyā ibn Abī Manṣūr (d. c. 833) in his *al-Zīj al-Mumtaḥan*[10] and in the *Toledan Tables*[11], while Philippus' era is also mentioned in Ḥabash al-Ḥāsib's *Zīj*[12] and al-Battānī's *Zīj*[13]. The Hulaghū era is introduced by Muḥyī al-Dīn for political reasons. Muḥyī al-Dīn compiled the *Tāj al-azyāj* in Damascus the same year in which Hulaghū conquered Bagdād (1258) and two years before the fall of Damascus. Naṣīr al-Dīn al-Ṭūsī (1201-1274) dates the beginning of the construction of the Marāgha observatory[14] in Jumāda I of 657H/April-May of 1259 but the works probably began in 1256. Muḥyī al-Dīn was probably aware of this fact and wished to become a member of the staff of the observatory. This might be the reason for his use of Hulaghū's era. The *zīj* also refers to the Jewish

[6] G.J. Toomer, *Ptolemy's Almagest*, Duckworth, London, 1984, p. 11.

[7] O. Neugebauer, *A History of Ancient Mathematical Astronomy*, Springer-Verlag, Berlin, Heidelberg, New York, 1975, pp. 970-971.

[8] O. Neugebauer, *The Astronomical Tables of Al-Khwārizmī*. In *Kgl. Danske Vidensk. Hist. -fil. Skrifter*, 4:2 (1962), p. 82.

[9] See Sachau's edition (Leipzig, 1923) and English translation (London, 1879).

[10] B. Van Dalen, "Ta'rīkh", *Encyclopédie de l'Islam* X, livr. 167-168, Leiden, 1998, pp. 283-290.

[11] G.J. Toomer, "A Survey of Toledan Tables", *Osiris* 15 (1968), pp. 5-174.

[12] M.-Th. Debarnot, "The *Zīj* of Ḥabash al-Ḥāsib: A Survey of MS Yeni Cami 784/2", in D.A. King, & G. Saliba (eds.), *From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E.S. Kennedy*, Annals of the New York Academy of Sciences (vol. 500), 1987, p. 39.

[13] C.A. Nallino, *Al-Battānī sive Albatenii Opus Astronomicum*. Mediolani Insubrum, 1903 & 1907 (vols. I and II), Vol. I, pp. 66-71.

[14] See A.I. Sayili, *The Observatory in Islam and its Place in the General History of the Observatory*, Publications of the Turkish Historical Society, Series VII, nr 38, Türk Tarih Kurumu Basimevi – Ankara, 1988, pp. 187-223.

and Christian calendars, as we find tables for the festivals of both religions.

The treatment of planetary longitudes is Ptolemaic and conventional. There are no Western Islamic innovations such as trepidation, independent motion of the apogees derived from the motion of the solar apogee discovered by Ibn al-Zarqālluh (d. 1100), or the solar model of variable eccentricity[15] except the use of the "Meridian of Water" as the base of geographical longitudes[16]. On the other hand, in the 22nd chapter Muḥyī al-Dīn says that he has found from his own observations[17] that the latitude of Damascus is 33;20°.

Spherical astronomy is the main subject of the following chapters. There are rules for the computation of the day arc, the ascendent, the declination, the second declination, the inverse declination (*al-mayl al-ma'kūs*), and the right ascension. I do not know a previous use of the inverse declination. Muḥyī al-Dīn defines it as $\chi(\lambda) = \delta(90^\circ \pm \lambda)$. It is used as an auxiliary function in some computations like the second declination of the Sun, the true longitude of a planet or the declination of a planet relative to the equator. We can also find the standard method for the determination of the astrological houses[18], the equation of time, and the "Method of the *Zījes*" for the determination of the *qibla*[19].

[15] J. Samsó, "Andalusian Astronomy: Its Main Characteristics and Influence in the Latin West", in J. Samsó, *Islamic Astronomy and Medieval Spain*, Ashgate-Variorum, Aldershot, 1994, Paper I.

[16] The essential references about geographical coordinates and the "Meridian of Water" are: E.S. and M.H. Kennedy, *Geographical Coordinates of Localities from Islamic Sources*, Frankfurt am Main, 1987. M. Comes, "Las Tablas de coordenadas geográficas y el tamaño del Mediterráneo según los astrónomos andalusíes", *Al-Andalus, El Legado Científico*, Palacio de Mondragón, Ronda, 1 de abril-15 de junio, 1995, pp. 22-37 and M. Comes, "The 'Meridian of Water' in the Tables of Geographical Coordinates of al-Andalus and North-Africa", *Journal for the History of Arabic Science* 10 (1994), pp. 41-51, repr. in M. Fierro & J. Samsó (eds.) *The Formation of al-Andalus, Part 2, Languages, Religion, Cultures and the Sciences*, Ashgate-Variorum, Aldershot, 1998, pp. 381-391.

[17] Mercè Comes, who is working on a new edition of E.S. & M.H. Kennedy, *Geographical Coordinates of Localities from Islamic Sources*, Frankfurt am Main, 1980, has confirmed to me that this latitude is documented here for first time.

[18] E.S. Kennedy, "The Astrological Houses as Defined by Islamic Astronomers", *From Baghdad to Barcelona*, Vol. II, pp. 538-540 and 555. Also in J.D. North, *Horoscopes and History*, The Warburg Institute, University of London, London, 1986.

[19] See the bibliography quoted by J. Samsó, "Ibn Isḥāq al-Tūnisī and Ibn Mu'ādh al-Jayyānī on the Qibla", *Islamic Astronomy and Medieval Spain*, paper VI, pp. 9-15. See

After a few chapters dedicated to solar and lunar eclipses, the canons end with the "essay about the *tasyīrs* with the incident horizon" (*Risāla fī l-tasyīrāt bi l-ufq al-ḥādith*). In this part of the canons, Muḥyī al-Dīn explains the *tasyīr* problem, the projection of rays and the first vertical method for the division of the houses. In relation to the last subject, Muḥyī al-Dīn seems to have coined the term "incident horizon" (*al-ufq al-ḥādith*) to denominate the six defining great circles[20].

As in the canons, the tables of the *zīj* begin with chronological tables. We see two important Eastern characteristics: the use of Syrian names of the months in the Julian calendar and Muḥyī al-Dīn's interest in the determination of Easter[21]. In relation to the Muslim calendar, it is interesting that Muḥyī al-Dīn used the method of intercalation of years attributed to Ulugh Begh[22].

The mean motion tables are calculated with new underlying parameters. There are tables calculated for periods of 90 years, single years, months, days, hours and fractions of 2' of hour for each planet and there is a table (Ms. E: 80v-81v; Ms. B: 58v-59v; Ms. D: 43r-44r) which explicitly gives the parameters used. The daily parameters for each planet are:

Sun (longitude)	0;59, 8,20, 8, 4,37 °/d.
Moon (longitude)	13;10,35, 1,36,32,17 °/d.
Moon (anomaly)	13; 3,53,56, 9,27, 7 °/d.
Moon (nodes)	0; 3,10,38,58,42,48 °/d.
Saturn (longitude)	0; 2, 0,41,30,59,54 °/d.
Jupiter (longitude)	0; 4,59,14,46,58,13 °/d.
Mars (longitude)	0;31,26,38,16, 2,26 °/d.

also Ahmad S. Dallal "Ibn al-Haytham's Universal Solution for Finding the Direction of the *Qibla* by Calculation", *Arabic Sciences and Philosophy*, vol. 5 (1995), pp. 145-193, and D.A. King, *World–Maps for Finding the Direction and Distance to Mecca. Innovation and Tradition in Islamic Science*. London and Leiden, 1999, pp. 61-64 and 163-168.

20 E.S. Kennedy, "The Astrological Houses as Defined by Islamic Astronomers", pp. 541-543 and 555-556.

21 G. Saliba, "Easter Computation in Medieval Astronomical Handbooks", *Al-Abhath*, Vol. 23, 1970, pp. 179-212, repr. in E.S. Kennedy, Colleagues and Former Students, *Studies in the Islamic Exact Sciences*, American University of Beirut, Beirut, 1983, pp. 677-709.

22 M. Ocaña, *Nuevas tablas de conversión de datas islámicas a cristianas y viceversa*, Instituto Hispano-Árabe de Cultura, Ministerio de Cultura, Madrid, 1981, p. 31.

Venus (anomaly)	0;36,59,28,56,37, 0 °/d.
Mercury (anomaly)	3; 6,24, 8,11, 4, 1 °/d.

I have checked these parameters in each table for days, months and years and I can assert that all the tables have been calculated using roundings or truncations of the daily parameter except for Mercury's mean motion table for days and the table of the nodes for years[23].

The daily parameter for the Sun corresponds to a tropical year of 365;14,30 days, the value that Muḥyī al-Dīn used in his *Talkhīṣ al-Majisṭī* (whose parameters are based on the observations made at the Marāgha observatory between 1262 and 1274)[24]. In this treatise and in his *Adwār al-anwār madā al-duhūr wa-l-akwār* (1275), the daily parameters for the Moon are 13;10,35,1,52,46,45°/d. for the mean motion in longitude, 13;3,53,42,51,59°/d. for the mean motion in anomaly and 0;3,10,37,37,12,20°/d. for the mean motion of the nodes. In both works, the daily parameters for Saturn and Jupiter are 0;2,0,36,45,35,41°/d. and 0;4,59,16,40,55,8°/d., respectively. Another difference between the *Tāj al-azyāj* and these later works is the precession of equinoxes. The motion of precession in the *Talkhīṣ al-Majisṭī* and in the *Adwār al-anwār* is 1°/66 Persian years[25] whereas in the *Tāj al-azyāj* it is 1°/72 Persian years, the same value as the one we find in the *Barcelona Tables* (c. 1381)[26].

The table of the equation of Sun is calculated with $e = 2;5,59^p$ as the underlying eccentricity. In his *Talkhīṣ al-Majisṭī*, Muḥyī al-Dīn determines the values $e = 2;5,55^p$ or $e = 2;5,57^p$ or $e = 2;5,59^p$ by observation which confirms the value found in the *Tāj al-azyāj*[27].

The tables of the equations of the anomalies of the Moon and the planets are the same as in the *Almagest*, except for six tables which have

[23] See Appendix.

[24] G. Saliba, "An Observational Notebook of a Thirteenth-Century Astronomer", *Isis*, 74 (1983), p. 396, repr. in idem, *A History of Arabic Astronomy, Planetary Theories during the Golden Age of Islam*, New York University Press, New York and London, 1994, p. 171.

[25] G. Saliba, "An Observational Notebook… ", pp. 396 and 399. *A History of Arabic Astronomy*, pp. 171 and 174.

[26] J. Chabás, "Astronomía andalusí en Cataluña: las Tablas de Barcelona", in *From Baghdad to Barcelona*, Vol. II, pp. 477-525.

[27] G. Saliba, "An Observational Notebook… ", p. 390. *A History of Arabic Astronomy*, p. 165.

been calculated with the "new" method which I describe in the following section.

2. Definition of homothetic table[28]

If we look at astronomical tables with the purpose of knowing which is the method used for computing them, we will not suppose that the author of the *zīj* calculated each value using the correct formula. The mean motion tables are easy to compute but the planetary equation tables must have been computed by calculating a few values of the table and interpolating the intermediate values or by using an easier method than calculating each value. In this sense, we have a lot of works explaining and analysing methods of computation of tables that are the basis for the study of any *zīj*[29].

Then, two tables will be homothetic if one of these is the other multiplied by a constant. Thus, for any given astronomic table T_1, I use the term **homothetic table of parameter *K*** to refer to another table T_2 which is calculated as follows:

$$T_2(\alpha) = K \cdot T_1(\alpha) \text{ for each argument of the table}$$

Under these conditions, I call T_1 the **model table** and the constant K the **factor**.

[28] I have chosen this name because of the similitude between this kind of tables and the relation between two homothetic geometrical figures.

[29] Important papers are the works of E.S. Kennedy, "A Medieval Interpolation Scheme Using Second Order Differences, *A Locust's Leg: Studies in Honour of S.H. Taqizadeh*, London (Percy Lund, Humphries) 1962, pp. 117-120, repr. in *Studies in the Islamic Exact Sciences (SIES)*, American University of Beirut, Beirut 1983, pp. 522-525; and "The Motivation of al-Bīrūnī's Second Order Interpolation Scheme", *Proceedings of the First International Symposium for the History of Arabic Science*, Vol. II, Alepo, 1978, pp. 126-132, repr in *SIES*, pp. 157-163; B. Van Dalen, "A Statistical Method for Recovering Unknown Parameters from Medieval Astronomical Tables" in *Centaurus* 32, pp. 85-145 and *Ancient and Mediaeval Astronomical Tables*...; J. Hamadanizadeh, "A Survey of Medieval Islamic Interpolation Schemes", *From Deferent to Equant: a Volume of Studies in the History of Science and Medieval Near East in Honor of E.S. Kennedy*, The New York Academy of Sciences, New York, 1987, pp. 143-152; and G.R. Van Brummelen, "The Numerical Structure of al-Khalīlī's Auxiliary Tables", *Physis*, 28 (1991), pp. 667-697; "Mathematical Methods in the Tables of Planetary Motion in Kūshyār ibn Labbān's *Jāmi' Zīj*", *Historia Mathematica*, 25 (1998), pp. 265-280 and his doctoral thesis *Mathematical Tables in Ptolemy's Almagest.*

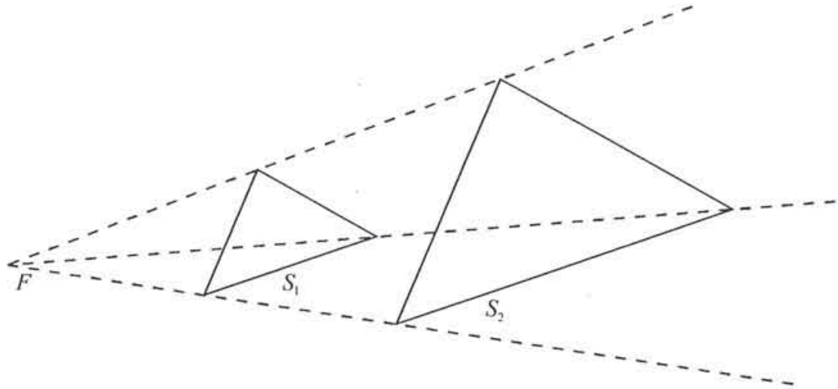

Figure 1

It is clear to see that this procedure preserves the order of the interpolation polynomy with which the model table is calculated as well as any other characteristics or attributes of the model table.

The first example of a homothetic table that I am familiar with is the table of the equation of anomaly of the Moon in the *Almagest*[30]. The tables of the anomaly of the Moon are given in two columns: c_5 that tabulates the equation of anomaly at apogee and c_6 that tabulates the difference between the equations of the anomaly at perigee and at apogee. So, if we define p_1 as the equation of anomaly at apogee and p_2 as the equation of anomaly at perigee, we can define a table for the equation of anomaly of the Moon at perigee (which I call $c_{5,6}$) in this way:

$$c_{5,6} = c_5 + c_6 = p_1 + (p_2 - p_1) = p_2$$

where these tables consider the true anomaly α_v as the argument. The errors of the first nine values of the table $c_{5,6}$ were already discussed by Toomer[31]. He also comments that the first seven values of the table fit a ratio (radius of epicycle / geocentric distance of epicycle centre) of 0.136 instead of 0.133 ≈ 5;15 / 39;22) which Ptolemy's parameters require and

[30] O. Pedersen, *A Survey of the Almagest*, Odense University Press, Odense, 1974, pp. 184–198.

[31] G.J. Toomer, *Ptolemy's Almagest*, p. 237, note 30.

which underlies all values from argument 60 onwards. Toomer says that this could be derived from a distance of 38;36^p units or an epicycle radius of 5;21^p, neither of which would have any justification.

	c_5		c_6		0;30 $\cdot c_5$	0;30 $\cdot c_5$ truncated
6	0;29		0;14	(+1)	0;14,30	0;14
12	0;57	(-1)	0;28	(+1)	0;28,30	0;28
18	1;25	(-1)	0;42	(+2)	0;42,30	0;42
24	1;53		0;56	(+3)	0;56,30	0;56
30	2;19	(-1)	1;10	(+4)	1; 9,30	1; 9
36	2;44	(-1)	1;23	(+5)	1;22, 0	1;22
42	3; 8	(-1)	1;35	(+5)	1;34, 0	1;34
48	3;31		1;45	(+4)	1;45,30	1;45
54	3;51		1;54	(+2)	1;55,30	1;55
60	4; 8	(-1)	2; 3	(+1)	2; 4, 0	2; 4
66	4;24	(-1)	2;11		2;12, 0	2;12
72	4;38		2;18	(-1)	2;19, 0	2;19
78	4;49	(+1)	2;25	(-1)	2;24,30	2;24
...	...	...	...	...	...	...
144	3;10		1;51		1;55, 0	1;55

Table 1

Toomer cannot explain this discrepancy but he says that it is too consistent to be the result of mere inaccurate calculation. In his detailed study of the tables of the *Almagest*, Van Brummelen[32] says that the recomputation of the table casts serious doubt on Toomer's suggestion that an errant parameter is to blame.

Van Brummelen computes all these ratios and we can see how the result of the ratio changes continuously rather than staying at a fixed level of 0.136. This leads Van Brummelen to the hypothesis that this table[33] $c_{5,6}$ (in its first values) is calculated by multiplying c_5 by 1;30. This hypothesis is equivalent to saying that c_6 is a homothetic table of parameter $K = 0;30$ whose model table is c_5 (see table 1, where the differences appearing in parenthesis correspond to an accurate recomputation of c_5 and c_6). The rest

[32] G.R. Van Brummelen, *Mathematical Tables in Ptolemy's Almagest* (doctoral thesis), pp. 176-179.

[33] This table is not actually found in the *Almagest*.

of the values of the table are not calculated using the same technique: we can see this in the example of the argument 144 in table 1.

3. Homothetic tables in the *Tāj al-azyāj*

The first homothetic table that we can find in the *Tāj al-azyāj* is the table c_6 of the Moon whose maximum is 2;48° (Ms. E: 83v-86r; Ms. B: 61v-64r; Ms. D: 46r-48v). I discovered that this table was homothetic because of the distribution of the differences between the table and the correct values that we can obtain with the exact formula. After checking that the linear and second order interpolations were not the method of computation of the table, I noticed that the distribution of these differences was the same as the one that we obtain from the table of the *Almagest*. Furthermore, the biggest positive and negative differences of the tables were placed in the same arguments and with the same sign. Then, I thought in an homothetic function as the correct method used for computing the table. In table 2, the values under *Rec.* are differences between the exact computation and tabular values; under *error* we have the differences between columns $K \cdot T_1$ *Rounded* and T_2.

T_1 = *Almagest*, T_2 = *Tāj al-azyāj*, K = 1; 3,20											
α_v	T_2	Rec	$K \cdot T_1$	$K \cdot T_1$ Round.	error	α_v	T_2	Rec	$K \cdot T_1$ Round.	$K \cdot T_1$	error
6	0;15	+1	0;14,47	0;15	0	**117**	2;44		2;43,47	2;44	0
12	0;28		0;29,33	0;30	+2	**120**	2;40	-1	2;40,27	2;40	0
18	0;43		0;44,20	0;44	+1	**123**	2;35	-3	2;36,13	2;36	+1
24	0;58	+3	0;59, 7	0;59	+1	**126**	2;31	-3	2;32, 0	2;32	+1
30	1;13	+4	1;13,53	1;14	+1	**129**	2;28	-1	2;27,47	2;28	0
36	1;27	+6	1;27,37	1;28	+1	**132**	2;23	-1	2;23,34	2;24	+1
42	1;40	+6	1;40,17	1;40	0	**135**	2;18	-1	2;18,17	2;18	0
48	1;51	+5	1;50,50	1;51	0	**138**	2;12	-1	2;11,56	2;12	0
54	2; 0	+3	2; 0,20	2; 0	0	**141**	2; 5	-1	2; 4,34	2; 5	0
60	2;10	+2	2; 9,50	2;10	0	**144**	1;57	-2	1;57,10	1;57	0
66	2;16	-1	2;18,16	2;18	+2	**147**	1;49	-2	1;48,43	1;49	0
72	2;24	-2	2;25,40	2;26	+2	**150**	1;40	-3	1;40,17	1;40	0
78	2;31	-2	2;33, 4	2;33	+2	**153**	1;32	-2	1;31,50	1;32	0
84	2;38	-2	2;39,23	2;39	+1	**156**	1;23	-2	1;23,24	1;23	0
90	2;44		2;43,37	2;44	0	**159**	1;15		1;14,57	1;15	0
93	2;46		2;45,43	2;46	0	**162**	1; 5		1; 5,27	1; 5	0
96	2;47	-1	2;46,47	2;47	0	**165**	0;55		0;54,54	0;55	0
99	2;48		2;47,50	2;48	0	**168**	0;45	+1	0;44,20	0;44	-1
102	2;48	-1	2;47,50	2;48	0	**171**	0;35	+2	0;32,44	0;33	-2
105	2;49		2;47,50	2;48	-1	**174**	0;24	+2	0;22,10	0;22	-2
108	2;48		2;46,47	2;47	-1	**177**	0;13	+2	0;10,34	0;11	-2
111	2;47		2;46,47	2;47	0	**180**	0; 0		0; 0, 0	0; 0	0
114	2;46		2;45,43	2;46	0						

Table 2

The model table is its homologue in the *Handy Tables* or a derived Islamic *zīj* like, for example, al-Battānī, and the factor of this table is the ratio between the maximum of the table and the maximum of the model table $K = 2;49 / 2;39 \approx 1; 3,20$. The rest of tables of the Moon are the same as their homologues in the *Almagest* except c_5 whose maximum is 4;51,0°. In fact, if we calculate table $c_{5,6}$ (rounding c_5 to one sexagesimal fraction), the resulting table is almost the same as the table in the *Almagest*. I have recalculated both tables (c_5 and c_6) and the underlying parameters of the tables are shown in table 3, considering two deferent radii: the normalised $R = 60^p$ and the Ptolemaic $R = 49;41^p$.

I have not found the method used to calculate table c_5, but I think that it is possible to say that its maximum was calculated as the difference between the maximums of $c_{5,6}$ and c_6: 7;40° – 2;49° = 4;51, 0°.

	***Tāj al-azyāj* $R = 60^p$**	***Tāj al-azyāj* $R = 49;41^p$**	***Almagest* $R = 49;41^p$**
	Table c_5		
Radius of deferent	$60;\ 0^p$	$49;41^p$	$49;41^p$
Eccentricity	$12;29^p$	$10;19^p$	$10;19^p$
Radius of epicycle	$6;\ 7,42^p$	$5;\ 4,22^p$	$5;15^p$
	Table c_6		
Radius of deferent	$60;\ 0^p$	$49;41^p$	$49;41^p$
Eccentricity	$12;29^p$	$10;19^p$	$10;19^p$
Radius of epicycle	$6;41^p$	$5;33^p$	$5;15^p$

Table 3

Before studying the planetary tables, it is useful to remember here how the Ptolemaic tables are presented. The first table is the equation of the centre of the planet (called c_3). After a table for an interpolation function (c_4), we have three more columns (c_5, c_6 and c_7):

- c_5 contains the difference between the equation of anomaly at apogee and at mean distance;
- c_6 contains the equation of anomaly at mean distance;
- c_7 contains the differences between the equations of the anomaly at perigee and the corresponding equation at mean distance.

Now we can calculate the equation of anomaly of each planet from these tables with the well-known rule:

$$\gamma(a_v, c_m) = c_6 - c_4 \cdot c_5 \quad \text{if } c_m < c_0$$
$$\gamma(a_v, c_m) = c_6 + c_4 \cdot c_7 \quad \text{if } c_m > c_0$$

where a_v is the true anomaly, c_m is the mean centre and c_0 is the position of the centre of the epicycle at the mean distance of the centre of the epicycle from the centre of the Earth (60^p).

The planetary equation tables of the *Tāj al-azyāj* are based on those of the *Handy Tables*[34] or on those of another derived Islamic *zīj*: the planetary models underlying the tables are Ptolemaic and almost every table is the same as its homologue in the *Handy Tables*. The differences between these tables (except in the case of c_3 of Venus) are due to homothetic tables:

Table c_6 of Jupiter	homothetic table of parameter $K = 1;0,30$
Table c_3 of Mars	homothetic table of parameter $K = 1;1,20$
Table c_6 of Mars	homothetic table of parameter $K = 1;0,45$
Table c_3 of Mercury	homothetic table of parameter $K = 1;7$
Table c_6 of Mercury	homothetic table of parameter $K = 1;1,22$

Another important characteristic of homothetic tables is that they can be calculated by adding in spite of multiplying. If we multiply different values by a constant, we can see that the results are proportional to the values and then, the difference between the result and the original value is also proportional to the values. So, if we calculate the difference between the bigger values of the model table and the table that we want to compute, the homothetic table is the same that adding to each value the proportional corresponding difference. This method is easier than the multiplying method for factors in the proximity of 1 (like in the case of Jupiter and Mars).

Table c_3 of Venus is the rounding of the table of the equation of Sun to one sexagesimal fraction. The underlying eccentricity of the solar equation table is $e = 2;5,59^p$ with a maximum equation of $2;0,20^\circ$.

In addition to the presumably new planetary parameters that we can see in table 4, we have notice of other new parameters in the *Talkhīṣ al-Majisṭī*[35]:

[34] W.D. Stahlman, *The Astronomical Tables of Codex Vaticanus Graecus 1291*, unpublished doctoral thesis, June 1959, Brown University.

[35] G. Saliba, "An Observational Notebook… ", pp. 400-401, repr. in *A History of Arabic Astronomy*, pp. 175-176.

1. The observations related to Saturn were made on 25th October 1263, 9th December 1266 and on an unknown day in 1273. From these, Muḥyī al-Dīn determines the double eccentricity $2e = 6;30^p$. From another observation made in 1271, Muḥyī al-Dīn determines the radius of the epicycle as $r = 6;31^p$ but accepts the Ptolemaic value ($r = 6;30^p$).

2. The observations related to Jupiter were made on 19th October 1264, 28th December 1267 and 12th August 1274. Muḥyī al-Dīn determines the eccentricity $2e = 5;30^p$ as in the *Almagest*. From the observation made on the 7th September 1264, he determines the radius of epicycle $r = 11;28^p$ but he accepts the Ptolemaic value $r = 11;30^p$, which is the same as in the *Adwār al-anwār*.

3. For the planet Mars, the determination of the eccentricity was made from the observations on 5th November 1264, 19th December 1266 and 26th February 1271. The final value is $2e = 11;53,46^p$ or $11;58,24^p$ or $11;58,48,6^p$: Muḥyī al-Dīn accepts the Ptolemaic value $2e = 12^p$. From the observation made on 14th October 1270, he determines the radius of the epicycle as $r = 39;37,30^p$.

4. The observational determination of new parameters

As we have seen, the homothetic table procedure found in Muḥyī al-Dīn's *Tāj al-azyāj* is only used in the tables of the equation of centre and tables of the equation of anomaly at mean distance. In each homothetic table, the maximum of the table means a new parameter and we must see if this parameter is the result of new observations. For this purpose, we will see how the parameters of a Ptolemaic model are determined. In the case of the equation of anomaly at mean distance, I take the example of Jupiter because we have references of observations made by Muḥyī al-Dīn in his later *Talkhīṣ al-Majisṭī*[36].

In book XI of the *Almagest*, Ptolemy determines the eccentricity of Jupiter from three true oppositions of the planet[37]. After some geometrical

[36] The exact calculations are in G. Saliba, "The Determination of New Planetary Parameters at the Maragha Observatory", in idem, *A History of Arabic Astronomy*, pp. 208-230.

[37] G.J. Toomer, *Ptolemy's Almagest*, pp. 507-517.

and trigonometrical calculations and an iteration procedure, Ptolemy concludes that $e = 2;45^{p}$. In the same way, in his *Talkhīṣ al-Majisṭī* Muḥyī al-Dīn also uses three observations of Jupiter for determining the eccentricity of Jupiter.

Table	Maximum value in		Ptolemaic Parameters	Parameters in *Tāj al-azyāj*
	Tāj al-azyāj	*Handy Tables*		
SATURN				
c_3	6;31°	6;31°	$R = 60;\ 0^{p}$ $e = 3;25^{p}$ $r = 6;30^{p}$	
c_5	0;21°	0;21°		
c_6	6;13°	6;13°		
c_7	0;25°	0;25°		
JUPITER				
c_3	5;15°	5;15°	$R = 60;\ 0^{p}$ $e = 2;45^{p}$ $r = 11;30^{p}$	In c_6: $r = 11;36^{p}$
c_5	0;30°	0;30°		
c_6	11; 9°	11; 3°		
c_7	0;33°	0;33°		
MARS				
c_3	11;40°	11;25°	$R = 60;\ 0^{p}$ $e = 6;\ 0^{p}$ $r = 39;30^{p}$	In c_3: $e = 6;\ 8^{p}$ In c_6: $r = 39;53^{p}$
c_5	5;38°	5;38°		
c_6	41;40°	41;10°		
c_7	8; 3°	8; 3°		
VENUS				
c_3	2, 0°	2,24°	$R = 60;\ 0^{p}$ $e = 1;15^{p}$ $r = 43;10^{p}$	c_3 is the table of the equation of Sun rounded to one sexagesimal fraction.
c_5	1;42°	1;42°		
c_6	45;59°	45;59°		
c_7	1;52°	1;52°		
MERCURY				
c_3	3;24°	3; 2°	$R = 60;\ 0^{p}$ $e = 3;\ 0^{p}$ $r = 22;30^{p}$	In c_3: $e = 3;21^{p}$ In c_6: $r = 23;\ 0^{p}$
c_5	3;12°	3;12°		
c_6	22;32°	22; 2°		
c_7	2; 1°	2; 1°		

Table 4

Following Ptolemy, he determines the angular motion of the planet and, noting the obvious difference between the observed angles and the mean angular motion, concludes that the centre of the mean motion did not coincide with the centre of the universe. After some geometrical and trigonometrical computations and an iteration procedure, Muḥyī al-Dīn determines the eccentricity of Jupiter as $2e = 5;30,39^{p}$, which is truncated to the Ptolemaic $2e = 5;30^p$ as final value. When we have the value of the double eccentricity, the maximum equation of centre is very easy to calculate[38].

After the determination of the eccentricity, Ptolemy determines geometrically, from a new observation, the size of Jupiter's epicycle[39]. He obtains the value $r = 11;30^p$. With this new parameter, Ptolemy constructs three new tables (c_5, c_6 and c_7). For the determination of the size of the epicycle, Ptolemy uses an observation of the planet made on 10th/11th July A.D. 139. We know that Muḥyī al-Dīn uses an observation of Jupiter made on 7th September A.D. 1264 to determine r in the *Talkhīṣ al-Majisṭī*[40]. So, in the *Tāj al-azyāj*, the determination of the new maximum of the table of the equation of anomaly at mean distance for Jupiter could have been computed from only one new observation. From this observation, Muḥyī al-Dīn determines a new radius of the epicycle that must be $r = 11;36^p$ (a value with which we can obtain the maximum value for the table 11; 9°).

Thus, it appears that new observations were made to establish the maximum equation of anomaly for the planets Mars and Mercury, whose respective tables underlie the new parameters: $r = 39;53^p$ for Mars and $r = 23^p$ for Mercury. On the other hand, we have seen that the determination of Jupiter's eccentricity has been made from observations of three true oppositions of the planet and the same technique was probably used to determine the eccentricities of Mars and Mercury, for which we lack the evidence of the *Talkhīṣ al-Majisṭī*. In the *Tāj al-azyāj*, the maximum equations of the centre of Mars and Mercury are apparently new[41], and this means that Muḥyī al-Dīn could have made observations to establish them. The new eccentricities obtained in the *Tāj al-azyāj* correspond to the inner planets and to Mars, whose period of revolution is only about two years; planets like Saturn and Jupiter have much longer periods of revolution. Only in the case of Jupiter we find a new maximum equation of anomaly which could have been obtained from only one observation. We also have new parameters (solar eccentricity and radius of the lunar epicycle) for the

[38] We can see the exact formula in O. Pedersen, *A Survey of the Almagest*, pp. 279-280.

[39] G.J. Toomer, *Ptolemy's Almagest*, pp. 520-522.

[40] G. Saliba, "An Observational Notebook…", p. 401. *A History of Arabic Astronomy*, p. 176.

[41] I have checked the data file compiled by E.S. Kennedy, available through http://www.rz.uni-frankfurt.de/~dalen/params.htm, as well as all the sources on medieval Islamic astronomy we have at our disposal in Barcelona.

Sun and the Moon, as well as new mean motion parameters. Each new parameter, with the exception of the solar eccentricity, can be derived from only one new observation and the corresponding comparison with an old one made before and quoted in another source. Finally, new observations for the determination of the solar equation[42] and the equation of anomaly of the Moon are also probable. The maximum of the table of the solar equation is 2;0,20° and the table underlies an apparently new eccentricity: e = 2;5,59^{p}, the value used by Muḥyī al-Dīn in the homologous table of the *Talkhīṣ al-Majisṭī* whose maximum is[43] 2:0,21°. On the other hand, the apparently new tables of the equation of anomaly of the Moon of the *Tāj al-azyāj* may also have been calculated from new observations. Furthermore, Muḥyī al-Dīn says in the *Tāj al-azyāj* that he obtained the latitude of Damascus by his own observations. So, we are sure that Muḥyī al-Dīn made observations before living in the Marāgha observatory.

It is, therefore, possible that the new parameters in the *Tāj al-azyāj* derive from a limited set of new observations.

5. Conclusion

Muḥyī al-Dīn wrote the *Tāj al-azyāj* and computed its tables in bad times in Damascus. The Mongols had invaded Bagdad and their arrival to Damascus was imminent when Muḥyī al-Dīn finished his work perhaps with the intention of dedicating it to Hulaghū Khān. Maybe, Muḥyī al-Dīn had to finish the *Tāj* quickly looking at the future new political times and trying to get the gratitude of the new leader. Muḥyī al-Dīn worked in Damascus where we know that he made observations (because he mentions one of them). The *Tāj al-azyāj* is probably the result of a limited set of observations made before the Mongols arrived to Damascus. Muḥyī al-Dīn did not have time to compute his tables using any complicated method of interpolation and decided to use one of the easiest methods of computing: the homothetic table. Each "new" table of the *Tāj al-azyāj* is computed by multiplying the values of the homologue table of the *Handy Tables* or a derived Islamic *zīj*. I do not know if this kind of tables were also used in any *zīj* of the Islamic tradition or if Muḥyī al-Dīn used this

[42] The determination of the eccentricity of the solar model requires three new observations in the same year.

[43] G. Saliba, "An Observational Notebook…", p. 396. *A History of Arabic Astronomy*, p. 171.

method during his later stay in Marāgha, but we must recognise that this is an easy way to compute a long set of interpolated values.

Appendix

The fact that Muḥyī al-Dīn gives the parameter corresponding to one day, one month (of 30 days), one lunar year (of 354 days) and one period of 30 lunar years multiplying the daily parameter by the number of days of each period, makes easy the work of finding the correct procedure of the computation of the mean motion tables. Each table is calculated adding the corresponding parameter for each argument of the table. So we only have to investigate which is the step used in each table and compare with the parameter given by Muḥyī al-Dīn. For each parameter, I am going to use the following notation:

d = parameter corresponding to one day
m = parameter corresponding to one month of 30 days
m'= parameter corresponding to one month of 29 days.
y = parameter corresponding to one lunar year of 354 days.
y' = parameter corresponding to one lunar year of 355 days.
q = parameter corresponding to one period of 30 lunar years.
p = parameter corresponding to one period of 90 lunar years.

For each natural number k, $d(k)$ means the rounding of the daily parameter to k sexagesimal fractions. I also call $m(k)$, $y(k)$ and $p(k)$ to the corresponding roundings of the parameters for the months, years and periods. An asterisk after them will mean the corresponding truncation and the capital letters indicate the values used by Muḥyī al-Dīn.

Now, I am going to see how each table was calculated: we have tables for days, months, single years and periods of 90 years. In each case, the first column is the kind of table studied, the second column is the parameter explicitly given by Muḥyī al-Dīn, the third column is the value used in the table, the fourth column is the procedure of computation of the parameter with respect to Muḥyī al-Dīn's value and the last column indicates the number of sexagesimal fractions used in the computation of the table. The chosen parameter is the one which gives least differences with the table of the *Tāj al-azyāj*. There are a lot of cases in which the value obtained by rounding is the same that the value obtained by truncation: in these cases I consider rounding as the method used.

1. Longitude of the Sun

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexages. fractions
Days	$D = 0;59,\ 8,20,8,4,37$	$d = 0;59,8,20$	Rounding: $d = D(3)$	3
Months	$M = 29;34,10,4,2,18,30$	$m = 29;34,10,4$ $m' = 28;35,1,44$	Rounding: $m = M(3)$ $m' = m - d$	3 3
Years	$Y = 348;55,10,47,41,30,0$	$y = 348;55,10,47,42$ $y' = 349;54,19,\ 7,50$	Rounding: $y = Y(4)$ $y' = y + D(4)$	4 4
Periods 90 years	$Q = 38;25,55,32,13,55,0$	$p = 115;17,46,37$	$p = 3 \cdot Q(3)$	3

2. Longitude of the Moon

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexages. fractions
Days	$D = 13;10,35,1,36,32,17$	$d = 13;10,35$	Rounding: $d = D(2)$	2
Months	$M = 35;17,30,48,16,8,30$	$m = 35;17,30,48$ $m' = 22;6,55,46$	Rounding: $m = M(3)$ $m' = m - D(3)$	3 3
Years	$Y = 344;26,39,29,34,28,18$	$y = 344;26,39,29$ $y' = 357;37,14,30$	Truncation $y' = y + D(3)$*	3 3
Periods 90 years	$Q = 38;16,10,4,56,4,7$	$p = 115;17,46,37$	$p = 3 \cdot Q\ (3)$	3

3. Anomaly of the Moon

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexag. fractions
Days	$D = 13;3,53,56,9,27,7$	$d = 13;3,53,56,9$	Rounding: $d = D(4)$	4
Months	$M = 31;56,58,4,43,33,30$	$m = 31;56,58,4,44$ $m' = 18;53,4,8,35$	Rounding: $m = M(4)$ $m' = m - d$	4 4
Years	$Y = 305;0,13,19,45,59,18$	$y = 305;0,13,20$ $y' = 318;4,7,16$	Rounding: $y = Y(3)$ $y' = y + D(3)$	3 3
Periods 90 years	$Q = 293;49,33,10,43,37,17$	$p = 161;28,39,32$	$p = (3 \cdot Q)(3)$	3

4. Nodes of the Moon[44]

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexag. fractions
Days	$D = 0;3,10,38,58,42,48$	$d = 0;3,10,39$	Rounding: $d = D(3)$	3
Months	$M = 1;35,19,29,21,24,0$	$m = 1;35,39,30$ $m' = 1;32,8,51$	$m = 30 \cdot d(3)$ $m' = m - d(3) =$ $= 29 \cdot d(3)$	3 3
Years	$Y = 18;44,49,58,24,31,12$	$y = 18;44,49,58$ $y' = 18;48,0,37$	Rounding: $y = Y(3)$ $y' = y + D(3)$	3 3
Periods 90 years	$Q = 202;59,56,18,6,24,0$	$p = 111;0,11$	$p = 360° - 3 \cdot Q(2)$	2

5. Longitude of Saturn

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexages. fractions
Days	$D = 0;2,0,41,30,59,54$	$d = 0;2,0,42$	Rounding: $d = D(3)$	3
Months	$M = 1;0,20,45,29,57,0$	$m = 1;0,20,45,30$ $m' = 0;58,20,3,59$	Rounding: $m = M(4)$ $m' = m - D(4)$	4 4
Years	$Y = 11;52,4,56,53,24,36$	$y = 11;52,4,56,53$ $y' = 11;54,5,38,24$	Rounding: $y = Y(4)$ $y' = y + D(4)$	4 4
Periods 90 years	$Q = 356;24,36,3,23,16,54$	$p = 349;13,48,10$	$p = 3 \cdot Q(3)$	3

6. Longitude of Jupiter

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexages. fractions
Days	$D = 0;4,59,14,46,58,13$	$d = 0;4,59,15$	Rounding: $d = D(3)$	3
Months	$M = 2;29,37,23,29,6,30$	$m = 2;29,37,23,29$ $m' = 2;24,38,8,42$	Rounding: $m = M(4)$ $m' = m - D(4)$	4 4
Years	$Y = 29;25,33,13,7,28,42$	$y = 29;25,33,13$ $y' = 29;30,32,28$	Rounding: $y = Y(3)$ $y' = y + d$	3 3
Periods 90 years	$Q = 163;41,28,16,21,1,23$	$p = 131;4,24,50$	$(3 \cdot Q)(3) =$ $131;4,24,49$	3

[44] The table for the nodes of the Moon corresponding to the periods is additive while the other tables are subtractive.

7. Longitude of Mars

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexages. fractions
Days	$D = 0;31,26,38,16,2,26$	$d = 0;31,26,38$	Rounding: $d = D(3)$	3
Months	$M = 15;43,19,8,1,13,0$	$m = 15;43,19,8$ $m' = 15;11,52,30$	Rounding: $m = M(3)$ $m' = m - d$	3 3
Years	$Y = 185;31,9,49,38,21,24$	$y = 185;31,9,46,33$ $y' = 186;2,36,24,49$	Rounding: $y = Y(4)$ $y' = y + D(4)$	4 4
Periods 90 years	$Q = 171;20,46,20,7,8,46$	$p = 154;2,19$	$p = 3 \cdot Q(3)$	2

8. Anomaly of Venus

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexages. fractions
Days	$D = 0;36,59,28,56,37$	$d = 0;36,59,29$	Rounding: $d = D(3)$	3
Months	$M = 18;29,44,28,18,30$	$m = 18;29,44,28$ $m' = 17;52,44,59$	Rounding: $m = M(3)$ $m' = m - d$	3 3
Years	$Y = 218;14,56,46,2,18$	$y = 218;14,56,46$ $y' = 218;51,56,15$	Rounding: $y = Y(3)$ $y' = y + d$	3 3
Periods 90 years	$Q = 74;15,17,19,31,47$	$p = 222;45,52$	$p = (3 \cdot Q)(2)$	2

9. Anomaly of Mercury

Table	Muḥyī al-Dīn's parameter	Parameter of the table	Procedure	Sexages. fractions
Days	$D = 3;6,24,8,11,4,1$	[3;6,24,9,32,44-3;6,24,9,38,34]	¿?	3
Months	$M = 93;12,4,5,32,0,30$	$m = 93;12,4$ $m' = 90;5,40$	Rounding: $m = M(2)$ $m' = m - D(2)$	2 2
Years	$Y = 19;46,24,17,17,41,54$	$y = 19;46,24,17$ $y' = 22;52,48,25$	Rounding: $y = Y(3)$ $y' = y + D(3)$	3 3
Periods 90 years	$Q = 267;22,34,8,52,41,11$	$p = 81;7,42,27$	$p = (3 \cdot Q)(3)$	3

The table for the days is computed with a greater parameter than the one given by Muḥyī al-Dīn.