Figurate Numbers in the Mathematical Tradition of al-Andalus and the Maghrib

A. Djebbar

A. Introduction

After Ḥabīb b. Baḥrīz and Thābit b. Qurra translated into Arabic the *Arithmetical Introduction* of Nicomachus, the chapters on figurate numbers, based on the properties illustrated in the table below, were further developed and were introduced in the teaching of mathematics along with other problems of Euclidian number theory.[1] For the 10th century, this interest in the figurate numbers is confirmed by the mathematical writings of al-Anṭākī[2] and Ibn Sīnā (Ibn Sīnā 1975, 53-61; 1976, 3) and by the encyclopedia of Abū ᶜAbd Allāh al-Khwārizmī (al-Khwārizmī 1968, 188-91). But the contents of these treatises do not give us much information on early developments in this subject. Also we know that in the central part of the Muslim Empire this tradition was continued for several centuries, as evidenced by the writings of ᶜAbd al-Qāhir b.

[1] The translation of Ḥabīb b. Baḥrīz was made (at the end of the 8th or the begining of the 9th century) from a Syriac version (F. Sezgin 1974, 164-66). For the translation of Thābit b. Qurra, cf. Kutsch 1958, 67-82.

[2] Abū'l-Qāsīm al-Anṭākī (cf. F. Sezgin 1974, 30) wrote the *Kitāb tafsīr al-arithmāṭīqī* which is really a commentary on the *Arithmetical Introduction* of Nicomachus. Only the third part has come down to us (Ms. Ankara Saip 5311, ff. 1b-36a).

Ṭāhir in the 11th century (Saidan 1978, 155-58), of al-Tannūkhī (Ms. Ankara Saip 5311, ff. 50a-52b; Vatican Or.317/1, ff. 76b-77b), Ibn al-Hā'im in the 14th century (Ms. Tunis 8918, ff. 19a-21b), and of Ibn al-Majdī in the 15th century (Ms. Brit. Mus. Add. 7469, ff. 21b-29b). The last two of these were particularly influential in teaching circles.

As far as the Muslim West is concerned, we have, in a previous study, drawn attention to some aspects of this tradition in the *Rafᶜ al-ḥijāb* of Ibn al-Bannā' (1256-1321) (Djebbar 1980, 76-89), without, however, being able to establish the relationship between the statements he made and previous writings on the subject, the oldest known in al-Andalus being the book of Rabīᶜ ibn Yaḥyā al-Usquf (10th century).[3]

The object of this brief paper is to present some new data about the history of this theme in al-Andalus and the Maghrib. These data come from Ibn Munᶜim (d. 1228) in his *Fiqh al-ḥisāb* (which is also important for the history of Combinatorial Analysis, as we have shown elswhere [Djebbar 1985]). Prior to the chapter labelled "*the ninth species on numerical figures*" summarized below, Ibn Munᶜim gives us very useful information on number theory in the Andalusian tradition of the 11th-12th centuries, as well as on his own personal contribution (Ms. Rabat B.G. 416 Q, pp. 297-315). It is clear that figurate numbers were extensively studied by arithmeticians before him and that Andalusians produced some original writings on them.[4]

In this chapter of the *Fiqh al-ḥisāb* we also learn about two authors : Ibn Sayyid "*the geometer*", of the 11th century (Djebbar 1993, 79-91) and ᶜAbd al-Ḥaqq b. Ṭāhir (12th century).[5] We also learn that a mistake

[3] According to Steischneider (1964, 125), this book is a paraphrase of the *Arithmetical Introduction* of Nicomachus. It was translated into Hebrew by Kalonymos b. Kalonymos, in 1316-17.

[4] These are the exact words of Ibn Munᶜim : "When I was willing to write, for my book, this part on numerical figures, I studied the works of the arithmeticians and I saw that their books were dealing deeply with this subject". (Ms. Rabat B.G. 416 Q, 297-298).

[5] The only additional notes that we were able to find on Ibn Ṭāhir are :

a)- In the commentary of the *Talkhīṣ* of Ibn al-Bannā', written by Ibn Zakariyā' al-Gharnāṭī, he refers to a book of Ibn Ṭāhir and a long demonstration it contains which concerns the approximation of the cubic root of a number (Ms. Tunis, B.N. 561, f. 59a).

b)- In the commentary of Ibn Ghāzī's, *Bughyat al-ṭullāb fī sharḥ Munyat al-ḥussāb*, contained in the last proposition of Ibn Sayyid's treatise led Ibn Munᶜim to write his ninth species. In this chapter, he mainly makes Ibn Sayyid's treatise more complete by demonstrations and determines the exact expressions for the sums of some finite subsequences: of both even and odd figurate numbers.[6]

However, beside the historical aspect of this chapter of Ibn Muniᶜm's work, the mathematical aspect is also of some interest from a technical point of view. We must underline here the introduction of algebra in the resolution of an arithmetical problem which leads to a quadratic equation, the handling of arithmetical sequences of polynomials and, finally, the use of tables to write these polynomials.[7] At the same time, we notice the continued use of the original geometrical vocabulary in the definitions and in the descriptions of some properties of figurate numbers.[8]

From a methodological point of view, we notice that Ibn Munᶜim wants to assess the validity of arithmetical results by proofs and, when possible, tries to substitute a "*complete*" proof for the inductive method. In his mind this means using *analysis and synthesis* as method of demonstration and using propositions of Euclid's *Elements* or similar material from the *Kitāb*

where he is refered to with the *kunya* Abū Muḥammad (Ms. Brit. Mus. Add. 9625, f. 62b; Souissi 1983, 201).

This paper being in proof, Dr. Miquel Forcada tells me that he has identified Ibn Ṭāhir: see his forthcoming paper "Las ciencias de los antiguos en al-Andalus durante el período almohade: una aproximación biográfica", to appear in *Estudios onomásticos y biográficos en al-Andalus* (Granada). My gratitude to him.

6 "When I had just finished studying the opuscule of Ibn Sayyid, I noticed that he had inserted a problem about which he was mistaken. As a matter of fact, he indicated a method to calculate the sum of numerical figures of odd and even sides and with even or odd value, while this method only allows the calculation of the sum of the squares of successive even natural numbers or the sum of successive odd natural numbers (...). I thought about it and some excellent method appeared to me to determine, with geometrical proofs, the sums of numerical figures according to their natural divisions (...). I am going to present what Ibn Sayyid said about it and, in addition, what I have introduced" (Ms. Rabat B.G. 416 Q, p. 298).

7 Ibn Munᶜim presents seven tables giving the expressions of the sums. (Ms. Rabat B.G. 416 Q, pp. 309, 310, 312, 313).

8 Ibn Munᶜim uses the following terms : line-number, plane-number, solid-number, side, equilateral triangle, equilateral figure, truncated figure, gnomon.

al-istikmāl of al-Mu'taman (d. 1085).[9]

Moreover, his thoroughness leads him to establish, in the 8th species of his *Fiqh al-ḥisāb*, a new series of propositions relating to summation of natural numbers and their squares. (ms. Rabat B.G., 416Q, pp. 255-97).

After the 13th century, Ibn Munᶜim's book was probably still being studied, as suggested by Ibn al-Bannā'[10], by Ibn Zakariyā' al-Gharnāṭī (Ms. Tunis, B.N. 561, ff. 101b, 104b, 109b, 115a, 116a and 117a), as well as Ibn Haydūr (Ms. Vatican 1403, p. 52) and Ibn Khaldūn (1967, 897).

However, the commentators on Ibn al-Bannā' who are the major Maghribī mathematicians of 14th-15th centuries (except Ibn al-Majdī who was an Egyptian) either only paraphrased the chapter on figurate numbers

9 In the introduction of his 8th species, Ibn Munᶜim says : "If somebody wants to study this part, it is necessary to know, prior to that, the first species of the first genus of al-Mu'taman's book since Euclid's book is not sufficient for whoever wants to study this part". Concerning al-Mu'taman, cf. Djebbar 1993; Djebbar 1994, and especially Hogendijk 1986, 43-52; Hogendijk 1988, 51-66; Hogendijk 1991; Hogendijk 1994.

10 Compared to that of Ibn Munᶜim, Ibn al-Bannā''s work seems to be a kind of summary and, at the same time, to be a complementary work. Thus, in Ibn al-Bannā''s work, the *Rafᶜ al-ḥijāb*, the original part of the *Fiqh al-ḥisāb*, I mean the latest part on sub-sequences of figurate numbers, totally disappears, maybe due to its great length and its "lack of usefulness". On other occasions, Ibn al-Bannā' also thinks that theoretical problems are not useful. However, a new analysis of a part of the Nicomachus' table, made with a combinatorial point of view, is added in the *Rafᶜ al-ḥijāb*. Furthemore, in his book, Ibn al-Bannā' reestablishes the inductive method since, conversely to Ibn Munᶜim, he uses the tables properties to calculate these sums : $P_m(n)$ being the figurate number with m sides and n being the length of each side, he proves that :

$$1^2 + 3^2 + \dots + (2n-1)^2 = P_3(1) + P_3(2) + \dots + P_3(2n-1)$$

$$2^2 + 4^2 + \dots + (2n)^2 = P_3(1) + P_3(2) + \dots + P_3(2n)$$

$$1^2 + 2^2 + \dots + n^2 = P_4(1) + P_4(2) + \dots + P_4(n)$$

$$1^3 + 3^3 + \dots + (2n-1)^3 = [P_3(n)]^2$$

We also find, in the *Rafᶜ al-ḥijāb*, investigation on sequences $(P_m(n)/n)_m$, whose common difference is (n-1)/2 and $(P_m(n)/n)_n$, whose common difference is (m-1)/2 (Djebbar 1980, 90-93).

of the *Rafᶜ al-ḥijāb* or simply ignored it.[11]

In conclusion, we should mention that the contributions on figurate numbers which we have cited represent only a part of all the additional work done after the *Arithmetical Introduction* was translated. This follows clearly from statements by Ibn Munᶜim in the introductory part of his 9th species. Therefore, it seems reasonable to assume that the Andalusian tradition in this area of research included results of other scholars, such as those transmitted by Ibn Sīnā[12] and, later, al-Umawī[13].

As far as works of a pedagogic nature were concerned, it seems that it was not the Archimedian tradition (in the determination of the areas of figures) but the Nichomachean tradition which occasioned the introduction of such chapters on finite power series as we describe in this paper. The introduction of this material in manuals may have occured as early as the

[11] Ibn Haydūr deals extensively with the matter in the *Tuḥfat aṭ-ṭullāb* (Ms. Vatican Or.1403, ff. 38b-48a) and in *al-Tamḥīṣ* (Ms. Rabat Ḥasaniyya 252, pp. 99-107). Ibn al-Majdī does the same thing in his *Ḥāwī l-lubāb* (Ms. Brit. Mus. Add. 7469, ff. 21a-28a). On the other hand, Ibn Ghāzī, Ibn Zakariyā', Ibn Qunfudh, al-Huwarī and al-ᶜUqbānī do not say a word about it.

[12] Especially, the search for figurate numbers that have the following property :

$$P_m(n) = P_{m'}(n') \; ; \; m \neq m' \text{ and } n \neq n'$$

and the relation between perfect numbers and figurate numbers (Ibn Sīnā 1975, 56-57).

[13] We find there the expression of the following sums :

$$1.2 + \dots + (n-1).n \; ; \; 2.4. + \dots + 2(n-1).2n \; ; \; 1.3. + \dots + (2n-3)(2n-1)$$

We also find (if we note : $S_3(n) = P_3(1) + \dots + P_3(n)$), the following relations :

$$P_3(1) + \dots + P_3(n) = [(n/3) + (2/3)]P_3(n) \text{ and } S_3(1) + \dots + S_3(n) = [(n/4) + (3/4)]S_3(n)$$

The first one is given by Ibn al-Bannā', the second one does not exist in any work previous to the 14th century (Saidan 1984, 34-36). We know that Fermat will state a generalisation of this expression (Tannery & Henry 1894, III, 273).

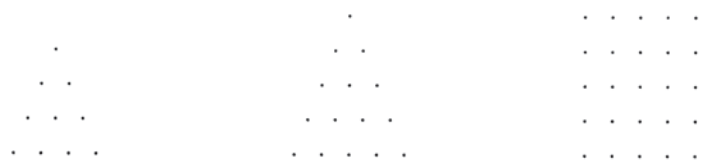
12th century in the Muslim West.[14]

B. Summary of the ninth species of the *Fiqh al-Ḥisāb*

I. Arithmetical properties of the figurate numbers table[15]

Definition:

1. Let $P_m(n)$ be the number whose geometrical figure is an *m*-sided polygon, each side consisting of *n* points. For example :



$P_3(4)$ $P_3(5)$ $P_4(5)$

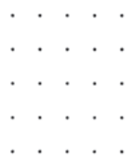
2. The following relationships exist between these figurate numbers :

$$P_3(n) = n + P_3(n-1) \quad ; \quad P_m(n) = P_3(n-1) + P_{m-1}(n)$$

($m \geq 4$ and $n \geq 2$). For example :

[14] This conjecture is based on the fact that in the Maghribī and Andalusian writings, there is no reference to the works of Thābit b. Qurra and of Ibn al-Haytham which set up und use series of powers to calculate the area of a parabola and the volume of a paraboloid, in Thābit's works, and to calculate the volume of the sphere, the paraboloid and the parabolical pirale, in Ibn al-Haytham's works.

[15] I have used the symbols *n*, *m* and *q* only in cases where the author states or proves a general result.



$$P_3(4) + P_3(5) = P_4(5)$$

3. Therefore we can draw the following table where $P_m(n)$ is the element of the $(m\text{-}1)^{th}$ row and of the n^{th} column :

Table of figurate numbers										
Sides	1	2	3	4	5	6	7	8	9	10
Triangles	1	3	6	10	15	21	28	36	45	55
Rectangles	1	4	9	16	25	36	49	64	81	100
Pentagons	1	5	12	22	35	51	70	92	117	145
Hexagons	1	6	15	28	45	66	91	120	153	190
Heptagons	1	7	18	34	55	81	112	148	189	235
Octagons	1	8	21	40	65	96	133	176	225	280
Enneagons	1	9	24	46	75	111	154	204	261	325
Decagons	1	10	27	52	85	126	175	232	297	370

4. We can see from this table that, for a fixed value of n, the n^{th} column is an arithmetical series whose first term is n and whose common difference is $P_3(n\text{-}1)$.

The author demonstrates this for $n = 3$ and $n = 4$ and then he just states that it is the same for all n.

Application:

a.- To find $P_m(n)$ when m and n are known (with the example : $m = 8$

and $n = 4$) :

$$P_8(4) = P_3(4) + (8-3)P_3(3) = 40$$

b.- To find n when $P_m(n)$ and m are known (with the example : $P_8(n) = 40$) :

The author gives us two different methods, an algebraical one and a geometrical one. With the algebraical method, he suggests to "*suppose the unknown* [number, i.e. n] *equal to the thing* [i.e. x]" and to solve the problem as follows : the first term of the arithmetic series :

$$P_3(x) = 1+2+ \dots +x = x^2/2 + x/2$$

The number of terms between $P_3(x)$ and $P_8(x)$ (except $P_3(x)$) is: $8-3 = 5$. Therefore:

$$40 = P_8(x) = P_3(x) + 5(x^2/2 - x/2) = 3x^2 - 2x$$

The algebraical resolution of the equation gives : $x = 4$ (Ms. Rabat B.G. 416Q, p. 301).

With the second method, the author follows the same approach, but he employs the word "*octagon's side*" instead of "*something*". He finally comes to the same equation that he solves "*geometrically*", that time using, without mentioning it, the 6th proposition of book II of Euclid's *Elements* (Heath 1956, I, 385; Vitrac 1990, 335).

5.- For a fixed value of m, the differences between two successive figurate numbers constitute an arithmetic series of common difference (m-2) .

The author draws this from the table.

6.- For a fixed value of m ($m \geq 3$), n being variable, the sequence having as its general term :

$$[P_m(1)+\dots+P_m(n)] / (1+\dots+n)$$

is an arithmetical series whose common difference is $(m-2) / 3$.

7.- For a fixed value of n ($n \geq 2$), m being variable, the sequence having the previously described general term is an arithmetical series whose common difference is $(n-1) / 3$.

The author states these two results for any m or n. Then he deduces them from the relationship that exists between the two sums: $[P_m(1)+\ldots+P_m(n)]$ and $(1+\ldots+n)$. This relationship is established from the following properties of the table : the sums of the first n elements of each line constitute an arithmetical series whose common difference, r, and first term, a, are :

$$r = P_3(1)+\ldots+P_3(n-1) \quad ; \quad a = 1+\ldots+n \qquad (1)$$

since the q^{th} column ($2 \leq q \leq n$) is an arithmetical series whose first term is q and whose common difference is $P_3(q-1)$.[16]

II. Expression of the sums of some sequences of figurate numbers according to the largest side

1. In this part of the 9th species of the *Fiqh al-ḥisāb*, Ibn Munᶜim first gives the sum of the first n figurate numbers of any line with the following method :

For a fixed value of n, $[P_m(1)+\ldots+P_m(n)]$ is the last term of the arithmetical series whose common difference is r and first term is a (defined by (1)). Consequently :

$$[P_m(1)+\ldots+P_m(n)] = a + (m-2)r = (n^2/2 + n/2) + (m-2)(n^3/6 - n/6)$$

$$= (m-2)/(n^3/6) + (1/2)n^2 + (5-m)(n/6)$$

since from the results established by Ibn Munᶜim in the 8th species of his book :

[16] Ibn Munᶜim gives us a "complete" arithmetical proof, pointing out: "After having considered these properties, we thought that to give it without proof would be a deficiency since one could think that it is included in these propositions which can be established only with the inductive method" (Ms. Rabat B.G., 416Q, p. 304).

$1+\ldots+n = n^2/2 + n/2$ and $1^2+\ldots+n^2 = (1/3)n^3 + (1/2)n^2 + (1/6)n$

and :

$$r = P_3(1)+\ldots+P_3(n-1) = [P_4(1)+\ldots+P_4(n)] - [P_3(1)+\ldots+P_3(n)]$$

$$= (1/2)(1^2+\ldots+n^2) - (1/2)(1+\ldots+n).$$

Thus, the sum of the first n elements of a fixed row is seen to be a polynomial in n, n^2 and n^3. Ibn Mun^c^im presents the coefficients of these polynomials in the following table :

Table of the sums of [numerical] figures of each species according to their longest side			
The figures	according to cube [n^3]	according to square [n^2]	according to the greatest side [n]
Sides		1/2	1/2
Triangles	1/6	1/2	1/3
Rectangles	1/3	1/2	1/6
Pentagons	1/2	1/2	
Hexagons	2/3	1/2	- 1/6
Heptagons	5/6	1/2	- 2/6
Octagons	1	1/2	- 3/6
Enneagons	1+(1/6)	1/2	- 4/6
Decagons	1+(1/3)	1/2	- 5/6

2.- In the second part of the 9th species (attributed to Ibn Sayyid although Ibn Mun^c^im claims credit for the proofs), the objective is to calculate, for each row of Nicomachus' table, the sum of the even figurate numbers and that of the odd ones. The approach is identical to the previous one : all the calculations rely on the formula for the sum of an arithmetical series. But, when determining the first term and the common difference of each series (which are polynomials with variable n), the author takes into account the distribution of the even values (respectively of the odd ones) of the figurate numbers on each even line (respectively on each odd one).

We shall explain his method through one of the cases that he studied : how to calculate the sum of the figurate numbers of odd species (triangles, pentagons, etc, i.e. m odd).

For example, the sum of the pentagons with odd values for the sides included between the first and the tenth ($n = 10$) is :

$$1 + 5 + 35 + 51 + 117 + 145$$

The respective sides are :

$$1, 2, 5, 6, 9, 10$$

As a general rule, the table shows that the sides for the odd figurate numbers of each row (m odd) are distributed like this :

odd, even-odd, odd, even-odd, ...

(an even-odd number being $2(2n+1)$).

On the row of the polygons of side m, the sum :

$$S_m = P_m(1) + P_m(2) + P_m(5) + P_m(6) + \dots + P_m(n\text{-}1) + P_m(n)$$

is, therefore, the last term of the arithmetical series whose first term is S_3 and whose common difference is :

$$r = 2(S_3 - a),$$

with :

$$a = 1+2+5+6+ \dots +(n\text{-}1)+n \qquad (2)$$

But since each column is an arithmetical series, we have :

$$P_4(n) - P_3(n) = P_3(n) - n$$

and consequently :

$$2P_3(n) = n+P_4(n)$$

Then :

$$r = S_4 - a = (1^2+2^2+5^2+6^2+ \ldots +(n-1)^2 + n^2) - a \quad (3)$$

Finally, we obtain the following results :
1.- If the last side, *n*, is even-odd :

$$S_3 = (1/2)(1/6)n^3 + (1/2)n^2 + (2/3)n$$

and :

$$r = (1/6)n^3 + (1/2)n^2 - (1/6)n - 1$$

2.- If *n* is odd :

$$S_3 = (1/2)(1/6)n^3 + (1/4)n^2 + (1/3 + (1/2)(1/6))n + 1/4$$

and :

$$r = (1/6)n^3 + (1/3)n - 1/2$$

owing to the results achieved in the 8th species of the *Fiqh al-ḥisāb*. These results give, in each case, the sums of both series (2) and (3), according to the longest side *n*.

The author proceeds with the same method to determine the sums of the other series of the figurate numbers as polynomials in n, n^2 and n^3, and, in each case, he presents a table of coefficients of these polynomials.

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