SORT 31 (1) January-June 2007, 45-54

Goodness of fit tests for the skew-Laplace distribution

Pedro Puig[1] and Michael A. Stephens[2]

[1] *Departament de Matemàtiques, Universitat Autònoma de Barcelona,*
[2]*Department of Statistics & Actuarial Science, Simon Fraser University, B.C. Canada*

Abstract

The skew-Laplace distribution is frequently used to fit the logarithm of particle sizes and it is also used in Economics, Engineering, Finance and Biology. We show the Anderson-Darling and Cramér-von Mises goodness of fit tests for this distribution.

MSC: 62G10, 62G30

Keywords: Maximum likelihood estimator; non-regular models; Anderson-Darling statistic; Cramér-von Mises statistic

1 Introduction

The three densities most commonly proposed to describe the logarithm of particle sizes are the normal, the hyperbolic and the skew-Laplace. Examples showing the use of these three distributions in this context can be found in Fieller et al. (1992). Julià and Vives-Rego (2005) uses the skew-Laplace distribution to analyze bacterial sizes in axenic cultures. In this paper we summarize the main properties of the skew-Laplace distribution and two useful goodness of fit tests are also presented.

The following argument is employed to justify the use of the normal distribution in particle size analysis:

Address for correspondence: P. Puig, Universitat Autònoma de Barcelona, Departament de Matemàtiques, Facultat de Ciències Edifici C, 08193 Cerdanyola del Vallés (Barcelona), Spain.
Received: November 2006
Accepted: February 2007

Suppose that a particle of initial size X_0 is repeatedly diminished by breaking off random proportions. The size in the step j is a random proportion of the size in the step $j-1$, that is, $X_j = \varepsilon_j X_{j-1}$, where ε_j are random variates taking values between 0 and 1. Then the size in the step n is

$$X_n = X_0 \prod_{j=1}^{n} \varepsilon_j.$$

If the variates ε_j are iid, for large n the distribution of $\log(X_n)$ can be approximated by the normal as a consequence of the central limit theorem.

Consider now that each observation $\log(x_i)$ follows a normal distribution with a different mean μ_i and variance σ_i^2. It is reasonable to assume that $\mu_i = f(\sigma_i^2)$, where $f()$ is a suitable monotonous function. This corresponds to the generally accepted idea that large scale observations have a wide dispersion. The simplest selection is a linear relationship, $f(\sigma_i^2) = a + b\sigma_i^2$.

We can also suppose that σ_i^2 are random variates following a suitable distribution defined on the positive reals. If σ_i^2 follows a distribution whose density function is $p(x;\gamma,\delta) = \frac{e^{-\gamma x-\delta/x}}{C(\gamma,\delta)}$, the resulting model is the hyperbolic distribution (Barndorff-Nielsen,1977). Correspondingly, if σ_i^2 follows an exponential distribution then the skew-Laplace distribution is obtained.

However the skew-Laplace distribution can arise as the difference of two exponentials as will be seen in Section 2 and in the example in section 4.1. Several properties, generalizations and applications of the skew-Laplace distribution have been reported in Kotz et al. (2001).

Sometimes the maximum likelihood estimators (MLE) of the parameters of this distribution have been calculated by maximizing directly its likelihood function. This can give numerical problems when iterative methods are used. A proper derivation of the MLE was done in Hinkley and Revankar (1977). These authors worked in the context of log-skew-Laplace models (see also Kotz et al., 2001). In Section 3 we study the maximum likelihood estimation and present a simple proof of the result of Hinkley and Revankar (1977).

In Section 4, we show the Anderson-Darling and Cramér-von Mises goodness of fit tests.

2 The skew-Laplace distribution

The skew-Laplace (SKL) or skew-double exponential distribution has a density function, defined over all the reals, of the form

$$p(x;\alpha,\beta,\mu) = \begin{cases} \exp(\frac{(x-\mu)}{\alpha})/(\alpha+\beta) & x \le \mu \\ \exp(\frac{(\mu-x)}{\beta})/(\alpha+\beta) & x > \mu \end{cases} \tag{1}$$

where $\alpha, \beta > 0$ and μ can be any real number. When α or β tends to 0 then the two-parameter exponential or negative-exponential distribution is obtained. When $\alpha = \beta$ it corresponds to the classical Laplace distribution. A skew-Laplace distribution with parameters μ, α and β will be referred as SKL(μ, α, β).

The distribution function, is

$$F(x;\alpha,\beta,\mu) = \begin{cases} \alpha\exp(\frac{(x-\mu)}{\alpha})/(\alpha+\beta) & x \le \mu \\ 1-\beta\exp(\frac{(\mu-x)}{\beta})/(\alpha+\beta) & x > \mu \end{cases} \tag{2}$$

The profile of the log-density is formed by two lines of slopes $1/\alpha$ and $-1/\beta$ intersecting in $x = \mu$, the location parameter and its mode. Therefore this distribution can be easily detected empirically by plotting a log-histogram.

2.1 Moments and properties

Many of the properties described in this section can be found in Kotz et al. (2001). Given a random variable X, SKL distributed, it is easy to compute its moment generating function $\Phi(t) = E(\exp(tX))$ giving

$$\Phi(t) = \frac{\exp(\mu t)}{(1+\alpha t)(1-\beta t)} \tag{3}$$

From (3), the cumulant generating function has a very simple form, $K(t) = \log(\Phi(t)) = \mu t - \log(1+\alpha t) - \log(1-\beta t)$, and consequently the mean is $E(X) = \mu+\beta-\alpha$, the variance is $V(X) = \alpha^2+\beta^2$ and for $i > 2$ the cumulants are $k_i = (i-1)!(\beta^i + (-1)^i\alpha^i)$.

The coeficients of skewness and kurtosis are as follows:

$$\sqrt{\beta_1} = \frac{k_3}{k_2^{3/2}} = \frac{2(\beta^3-\alpha^3)}{(\alpha^2+\beta^2)^{3/2}}, \quad \beta_2 = 3 + \frac{k_4}{k_2^2} = 3 + \frac{6(\beta^4+\alpha^4)}{(\alpha^2+\beta^2)^2}$$

They can be expressed in terms of $\theta = \beta/\alpha$, giving

$$\sqrt{\beta_1} = \frac{2(\theta^3-1)}{(\theta^2+1)^{3/2}}, \quad \beta_2 = 3 + \frac{6(\theta^4+1)}{(\theta^2+1)^2} \tag{4}$$

As θ varies in $(0,\infty)$, $\sqrt{\beta_1} \in (-2,2)$ and $\beta_2 \in [6,9)$. From (4) it is evident that $\sqrt{\beta_1}$ determines β_2. Moreover $\beta_2(\sqrt{\beta_1}) = \beta_2(-\sqrt{\beta_1})$. The following table shows the relationship between both coefficients:

$\sqrt{\beta_1}$	0.0	±0.2	±0.5	±1.0	±1.5	±1.8	±2.0
β_2	6.00	6.03	6.17	6.68	7.58	8.34	9.00

The values $\sqrt{\beta_1} = 0$ and $\beta_2 = 6$ correspond to $\theta = 1$, that is, the Laplace distribution. From an empirical point of view, if the sample skewness and kurtosis coefficients do not lie near the appointed values it would be a sign that the SKL distribution is inadequate for fitting our data.

Another measure of dispersion is $E|X - \mu|$, that is, the mean deviation with respect to the location parameter. It gives, $E|X - \mu| = (\alpha^2 + \beta^2)/(\alpha + \beta) = V(X)/(\alpha + \beta)$. Then the normalizing constant in (1) can be interpreted as the quotient $E|X - \mu|/V(X)$.

2.2 Generation of values

Given a random variable X, two-parameter exponentially distributed with starting point x_0 and expectation $E(X) = \tau + x_0$ then the moment generating function is $\Phi_X(t) = \exp(x_0 t)/(1 - \tau t)$. Hence, $\Phi_{-X}(t) = \exp(-x_0 t)/(1 + \tau t)$ and from (3) it can be readily deduced that the difference of two-parameters exponential independent random variables follows a SKL distribution. Now the parameters of the SKL have a new meaning, that is, α and β are the means of each exponential after subtracting its starting points and μ measures the distance between these starting points.

This result leads to a first approach to simulate a SKL(μ, α, β), by subtracting two independent exponentials starting at 0 with means α and β respectively and adding the constant μ. It can be summarized in the following formula:

$$X = \alpha \log(z_1) - \beta \log(z_2) + \mu = \log(z_1^{\alpha}/z_2^{\beta}) + \mu$$

where z_1 and z_2 are two independent uniform (0, 1) variates.

A second approach comes from the mixture pattern model mentioned in Section 1. Consider that observations follow a normal distribution with mean $a + b\sigma^2$ and variance σ^2, where σ^2 is also a continuous random variable with density $g(x)$ over the positive reals. It can be easily shown that the moment generating function of the resulting distribution is $\Phi(t) = \int_0^\infty e^{(a+b\sigma^2)t+\sigma^2 t/2} g(\sigma^2) d\sigma^2$. If the mixing density is an exponential with mean τ then it gives the moment generating function in (3), and the relationship between the two parameterizations is $\mu = a$, $2\alpha\beta = \tau$ and $\beta - \alpha = b\tau$. This can be summarized in the following expression:

$$X = \mu + \frac{\beta - \alpha}{2\alpha\beta} x_1 + x_1 y_1$$

where x_1 is an exponential variate with mean $2\alpha\beta$ and y_1 is a standard normal, both being independent.

A third approach to simulate a SKL random variable is by using the classical inverse distribution method. It gives the following expression,

$$X = \begin{cases} \alpha \log(\frac{\alpha+\beta}{\alpha} z) + \mu & z \in (0, \frac{\alpha}{\alpha+\beta}) \\ \beta \log(\frac{\beta}{(1-z)(\alpha+\beta)}) + \mu & z \in (\frac{\alpha}{\alpha+\beta}, 1) \end{cases} \tag{5}$$

where z is a uniform (0, 1) variate. This method is better than the preceding ones because it only requires one uniform value for each SKL value.

3 Parameter estimation

Consider a sample $X = (x_1, \ldots, x_n)$, coming from a $SKL(\mu, \alpha, \beta)$. Our goal is to find the MLEs of the parameters. First suppose that μ is known and $x_{(r)} \leq \mu \leq x_{(r+1)}$, where $x_{(r)}$ indicates the r-th order statistic. The log-likelihood function can be written as

$$l(X; \mu, \alpha, \beta) = -\frac{1}{\alpha} L_r(\mu) - \frac{1}{\beta} U_r(\mu) - n \log(\alpha + \beta) \tag{6}$$

where $L_r(\mu) = \sum_{i=1}^{r} (\mu - x_{(i)})$ and $U_r(\mu) = \sum_{i=r+1}^{n} (x_{(i)} - \mu)$. The likelihood function is differentiable with respect to α and β in the domain of the parameters. Then, solving the likelihood equations we obtain

$$\begin{aligned} \hat{\alpha}_0(\mu) &= \frac{\sqrt{L_r(\mu)}(\sqrt{L_r(\mu)} + \sqrt{U_r(\mu)})}{n} \\ \hat{\beta}_0(\mu) &= \frac{\sqrt{U_r(\mu)}(\sqrt{L_r(\mu)} + \sqrt{U_r(\mu)})}{n} \end{aligned} \tag{7}$$

Notice that if $\mu \leq x_{(1)}$ then $L_r(\mu) = 0$ and, similarly, if $\mu \geq x_{(n)}$ then $U_r(\mu) = 0$. Consequently it can be directly shown that (7) is also valid to describe the maximum likelihood estimators of α and β in these situations.

Taking into account that $(L_r(\mu) + U_r(\mu))/n = \sum_{i=1}^{n} |x_i - \mu|/n = \Delta(\mu)$ and $(L_r(\mu) - U_r(\mu))/n = \mu - \bar{x}$, (7) can be written in a more suitable form, independent of where μ is located:

$$\begin{aligned} \hat{\alpha}_0(\mu) &= \tfrac{1}{2}(\Delta(\mu) - \bar{x} + \mu + \sqrt{\Delta^2(\mu) - (\bar{x} - \mu)^2}) \\ \hat{\beta}_0(\mu) &= \tfrac{1}{2}(\Delta(\mu) + \bar{x} - \mu + \sqrt{\Delta^2(\mu) - (\bar{x} - \mu)^2}) \end{aligned} \tag{8}$$

Notice that if $\mu \leq x_{(1)}$ then $\Delta(\mu) = \bar{x} - \mu$ and $\hat{\alpha}_0(\mu) = 0$. Similarly, if $\mu \geq x_{(n)}$ then

$\hat{\beta}_0(\mu) = 0$. For these situations the MLEs are degenerate in the sense that the estimations lie outside the domain of the parameters.

Now, by substituing (8) in (6), the maximum of the likelihood function is

$$l_M(X;\mu) = -n(\log(\Delta(\mu) + \sqrt{\Delta^2(\mu) - (\overline{x} - \mu)^2}) + 1) \tag{9}$$

Therefore, if all the parameters are unknown, the MLE of μ can be found by maximizing (9) or, equivalently, by minimizing $\psi(\mu) = \Delta(\mu) + \sqrt{\Delta^2(\mu) - (\overline{x} - \mu)^2}$. Observe that $\psi(\mu) = \overline{x} - \mu$ for $\mu < x_{(1)}$, and $\psi(\mu) = -\overline{x} + \mu$ for $\mu > x_{(n)}$. Then it is obvious that the minimum must be located in the interval $[x_{(1)}, x_{(n)}]$.

The function $\psi(\mu)$ is not differentiable at the points $\mu = x_i$, but the derivative can be computed at all other points. Then, for $x_{(r)} < \mu < x_{(r+1)}$, $r = 1, \ldots, n-1$, $\Delta(\mu) = \frac{2r-n}{n}\mu - 2\sum_{i=1}^{r} x_{(i)} + \overline{x}$ and

$$\psi'(\mu) = \frac{2r-n}{n} + \frac{\Delta(\mu)(2r-n)/n + \overline{x} - \mu}{\sqrt{\Delta^2(\mu) - (\overline{x} - \mu)^2}}$$

Straightforward calculations show that the unique solution of $\psi'(\mu) = 0$ is $\mu_0 = (r^2\overline{x} + (n-2r)\sum_{i=1}^{r} x_{(i)})/(r(n-r))$. If μ_0 is not in $(x_{(r)}, x_{(r+1)})$, then $\psi(\mu)$ is monotone in this interval. Otherwise, further calculations show that

$$\psi''(\mu_0) = -\frac{n}{2r(\overline{x} - \sum_{i=1}^{r} x_{(i)}/r)}$$

Notice that $\overline{x} - \sum_{i=1}^{r} x_{(i)}/r \geq 0$, and equality occurs with probability 0. Consequently in μ_0 we have a local maximum. Due to the continuity of $\psi(\mu)$, if the minimum is not attained inside the intervals $(x_{(r)}, x_{(r+1)})$, $r = 1, \ldots, n-1$, it must be attained at the borders, that is, in one (or several) of the sample values $x_{(i)}$), $i = 1, \ldots, n$. Now, we have proved the following theorem:

Theorem 1 (Hinkley and Revankar, 1977) *Let $x_1, \ldots, x_n$ be a sample coming from a SKL(μ, α, β). The MLEs of the parameters are given by,*

$$\begin{aligned}\hat{\mu} &= x_j\\ \hat{\alpha} &= \tfrac{1}{2}(\Delta(\hat{\mu}) - \overline{x} + \hat{\mu} + \sqrt{\Delta^2(\hat{\mu}) - (\overline{x} - \hat{\mu})^2})\\ \hat{\beta} &= \tfrac{1}{2}(\Delta(\hat{\mu}) + \overline{x} - \hat{\mu} + \sqrt{\Delta^2(\hat{\mu}) - (\overline{x} - \hat{\mu})^2})\end{aligned}$$

where $\Delta(\mu) = \sum_{i=1}^{n} |x_i - \mu|/n$ and x_j is any sample value where the function $\psi(\mu) = \Delta(\mu) + \sqrt{\Delta^2(\mu) - (\overline{x} - \mu)^2}$ attains its unique minimum. Moreover the maximum of the log-likelihood function is

$$l_M(X) = -n(\log(\psi(\hat{\mu})) + 1)$$

Remark 1 Observe that the calculation of $\hat{\mu}$ is very simple because we only need to evaluate the function $\psi(\mu)$ at a finite number of points, that is, the sample values.

The MLEs are not necessarily unique but the function $\psi(\mu)$ has a unique absolute minimum. The points where $\psi(\mu)$ attains its minimum are not necessarily consecutive as happens with the Laplace distribution ($\alpha = \beta$). For example for the sample, −1.085, 0.043, 3.326, 3.954, 5.967, the maximum likelihood estimates of the location parameter are $\hat{\mu}_1 = -1.085$ and $\hat{\mu}_2 = 5.967$. We have observed this troublesome phenomenon only with small samples.

When $\hat{\mu} = x_{(1)}$ or $\hat{\mu} = x_{(n)}$ then $\hat{\alpha} = 0$ or $\hat{\beta} = 0$ and empirically, this means that data is fitted by the exponential or negative-exponential distribution. This situation can also be troublesome and unfortunately it can occur in moderately small samples with an appreciable probability. For instance, we have simulated 10000 samples of different sizes for a SKL(0,1,2). For $n = 5$ this anomaly has been observed in 96% of the samples. For $n = 10$ in 63% and for $n = 20$ in 22%. For $n = 50$ it only happened in 1% of the samples. Consequently, MLEs are not recommended for small samples.

The density of the SKL does not satisfy the standard conditions of regularity. However the consistency and the asymptotic efficiency of the MLE can be established using the very general conditions of Daniels (1961)(see also Hinkley and Revankar, 1977). The asymptotic variance V can be calculated in a standard way by inverting the Fisher information matrix. It gives the following:

$$V = \begin{pmatrix} 2\alpha\beta & \alpha\beta & -\alpha\beta \\ \alpha\beta & \alpha(\alpha+\beta) & 0 \\ -\alpha\beta & 0 & \beta(\alpha+\beta) \end{pmatrix} \tag{10}$$

The asymptotic variance of some functions of the estimates of the parameters can be calculated from here. For instance, to test symmetry it is necessary to estimate $\theta = \beta/\alpha$. It can be shown that the asymptotic variance of $\hat{\theta} = \hat{\beta}/\hat{\alpha}$ is $V(\hat{\theta}) = \theta(1+\theta)^2$.

Notice that the MLE of the expectation is $\hat{E} = \hat{\mu} + \hat{\beta} - \hat{\alpha} = \overline{x}$, that is the sample mean. Then its variance is $V(\hat{E}) = (\alpha^2 + \beta^2)/n$ and approximate confidence intervals can be calculated easily.

In practice is important to decide if the skew-Laplace distribution is a good choice to fit a data set. In the next section some goodness of fit tests are presented.

4 Goodness of fit tests

Our goodness of fit tests will be based on statistics which compare the empirical distribution function (EDF) of the sample with the hypothesised distribution $F(x)$.

The EDF is defined by $F_n(x) = \frac{\# x_i \leq x}{n}$. The statistics considered are the Cramér-von Mises W^2 and the Anderson-Darling A^2:

$$W^2 = n \int_{-\infty}^{\infty} \{F_n(x) - F(x)\}^2 \, dF(x)$$

$$A^2 = n \int_{-\infty}^{\infty} \{F_n(x) - F(x)\}^2 \, \psi(x) dF(x)$$

where $\psi(x) = 1/[F(x)\{1 - F(x)\}]$.

Generally these tests are powerful. The percentage points for these and other EDF tests for a variety of distributions can be found in D'Agostino and Stephens (1986).

The tests procedure is as follows. Suppose the order statistics (ascending) of the sample are $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$.

a) Find the MLEs of the parameters $(\hat{\mu}, \hat{\alpha}, \hat{\beta})$ following the algorithm of Theorem 1.

b) Make the transformation $z_{(i)} = F(x_{(i)}; \hat{\mu}, \hat{\alpha}, \hat{\beta})$, for $i = 1, \ldots, n$. The $z_{(i)}$ will be in ascending order.

c) The Cramér-von Mises statistic is computed from

$$W^2 = \sum_{i=1}^{n} \{z_{(i)} - (2i - 1)/(2n)\}^2 + 1/(12n)$$

and the Anderson-Darling statistic from

$$A^2 = -n - (1/n) \sum_{i=1}^{n} (2i - 1) \left[\log(z_{(i)}) + \log(1 - z_{(n+1-i)})\right]$$

d) Estimate the coefficient of skewness by using the expression

$$\sqrt{\hat{\beta}_1} = \frac{2(\hat{\beta}^3 - \hat{\alpha}^3)}{(\hat{\alpha}^2 + \hat{\beta}^2)^{3/2}}$$

e) Look at the table 1 for the chosen statistic and interpolate to find the percentage point at a given significance level. If the value of the statistic is greater than this percentage point, then the null hypothesis is rejected at this level.

The distributions of W^2 and A^2 depend only on $|\beta/\alpha|$. Simulation studies show us that the tests performed by estimating the coefficient of skewness have a significance level slightly lower than expected.

The percentage points of the tables are those of the asymptotic distribution of W^2 and A^2 under the null hypothesis. They have been computed using the standard techniques described in Stephens (1976) (see also Puig and Stephens, 2000). For finite samples the percentage points can be calculated by using Monte Carlo methods, but they are very close to the asymptotic for samples above $n = 20$.

Table 1: *Percentage points of the asymptotic distribution of W^2 and A^2 for different values of $\sqrt{\beta_1}$ and different significance levels (in boldface).*

	W^2				A^2			
$\sqrt{\beta_1}$	**0.10**	**0.05**	**0.025**	**0.01**	**0.10**	**0.05**	**0.025**	**0.01**
0.00	.077	.091	.106	.125	.498	.582	.665	.774
±0.20	.077	.092	.107	.126	.498	.583	.666	.776
±0.40	.078	.093	.108	.129	.501	.586	.671	.784
±0.60	.079	.095	.111	.133	.504	.592	.680	.796
±0.80	.081	.098	.115	.139	.510	.601	.692	.814
±1.00	.084	.102	.121	.148	.519	.614	.710	.840
±1.20	.088	.108	.129	.158	.531	.632	.735	.875
±1.40	.093	.116	.140	.172	.550	.659	.771	.924
±1.60	.102	.127	.154	.191	.579	.700	.825	.996
±1.80	.116	.146	.177	.220	.634	.775	.922	1.121
±1.90	.128	.162	.198	.246	.692	.851	1.017	1.243
±1.95	.139	.176	.215	.268	.748	.925	1.108	1.359
±1.98	.150	.190	.233	.290	.816	1.013	1.218	1.498
±1.99	.156	.199	.243	.303	.861	1.072	1.290	1.588
±2.00	.174	.222	.271	.338	1.062	1.321	1.591	1.959

4.1 An example

Bain and Engelhardt (1973) consider the following data set, consisting of 33 differences in flood levels between stations on a river:

1.96, 1.97, 3.60, 3.80, 4.79, 5.66, 5.76, 5.78, 6.27, 6.30, 6.76, 7.65, 7.84, 7.99, 8.51, 9.18, 10.13, 10.24, 10.25, 10.43, 11.45, 11.48, 11.75, 11.81, 12.34, 12.78, 13.06, 13.29, 13.98, 14.18, 14.40, 16.22, 17.06

They fit the data by using the Laplace distribution arguing that the observations could occur as the difference of two exponential distributions with the same mean. However, the fit does not work well as can be seen in Puig and Stephens (2000) who perform EDF tests for the Laplace distribution. Possibly the two exponentials do not have the same mean and consequently a reasonable alternative is the skew-Laplace distribution.

By using theorem 1, the MLEs are $\hat{\mu} = 11.75$, $\hat{\alpha} = 4.4654$ and $\hat{\beta} = 2.0691$. The estimated coefficient of skewness is $\sqrt{\hat{\beta}_1} = -1.345$ and the EDF test statistics are $W^2 = .097$ and $A^2 = .568$. From Table 1 the skew-Laplace assumption is not rejected for the Cramér-von Mises statistic even at a significance level of 0.10 and for the Anderson-Darling statistic at a level of 0.05.

Given the above, an approximate 95% confidence interval for the mean can be calculated from the expression

$$\bar{x} \pm 1.96\sqrt{\frac{\hat{\alpha}^2 + \hat{\beta}^2}{n}} = 9.354 \pm 1.679$$

We then test the Laplace assumption against the skew-Laplace for this example. It is equivalent to consider $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$. As has been pointed out in Section 3, an approximate 95% confidence interval for θ can be calculated from

$$\hat{\theta} \pm 1.96\sqrt{\frac{\hat{\theta}}{n}(1 + \hat{\theta})} = 0.463 \pm 0.340$$

Consequently the Laplace assumption must be rejected in favour of the general skew-Laplace.

Acknowledgements

This research was partially supported by grant MTM2006-01477 from the Ministry of Education of Spain.

References

Bain, Lee J. and Englehardt, Max (1973). Interval estimation for the two-parameter double exponential distribution. *Technometrics*, 15, 875-887.

Barndorff-Nielsen, O. E. (1977). Exponentially decreasing distributions for the logarithm of a particle size. *Proceedings of the Royal Society (London), Ser. A*, 353, 401-419.

D'Agostino and Stephens, M. A. (1986). *Goodness of Fit Techniques*. Marcel Decker, New York.

Daniels, H. E. (1961). The asymptotic efficiency of a maximum likelihood estimator. *Proc. 4th Berkeley Sympos. Math. Statist. and Prob.*, Vol. I, 151-163 Univ. California Press, Berkeley, Calif.

Fieller, N. R. J., Flenley, E. C. and Olbricht, W. (1992). Statistics of particle size data. *Applied Statistics*, 41, 127-146.

Hinkley, D. V. and Revankar, N. S. (1977). On the estimation of the Pareto law from underreported data. A further analysis. *Journal of Econometrics*, 5, 1-11.

Julià, O. and Vives-Rego, J. (2005). Skew-Laplace distribution in Gram-negative bacterial axenic cultures: new insights into intrinsic cellular heterogeneity. *Microbiology*, 151, 749-755.

Kotz, S., Kozubowski, T. J. and Podgórski, K. (2001). *The Laplace Distribution and Generalizations*. Birkhäuser, Berlin.

Puig, P. and Stephens, M. A. (2000). Tests of fit for the Laplace distribution, with applications. *Technometrics* 42, 4, 417-424.

Stephens, M. A. (1976). Asymptotic results for goodness-of-fit statistics with unknown parameters. *Annals of Statistics*, 4, 357-369.