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The exponentiated discrete Weibull distribution

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Abstract

In this paper, the exponentiated discrete Weibull distribution is introduced. This new generalization of the discrete Weibull distribution can also be considered as a discrete analog of the exponentiated Weibull distribution. A special case of this exponentiated discrete Weibull distribution defines a new generalization of the discrete Rayleigh distribution for the first time in the literature. In addition, discrete generalized exponential and geometric distributions are some special sub-models of the new distribution. Here, some basic distributional properties, moments, and order statistics of this new discrete distribution are studied. We will see that the hazard rate function can be increasing, decreasing, bathtub, and upside-down bathtub shaped. Estimation of the parameters is illustrated using the maximum likelihood method. The model with a real data set is also examined.

MSC: 60E05, 62E10

Keywords: Discrete generalized exponential distribution, exponentiated discrete Weibull distribution, exponentiated Weibull distribution, geometric distribution, infinite divisibility, order statistics, resilience parameter family, stress-strength parameter.

1. Introduction

It is sometimes impossible or inconvenient to measure the life length of a device on a continuous scale. In practice, we come across situations where lifetimes are recorded on a discrete scale. For example, on/off switching devices, bulb of photocopier machine, to and fro motion of spring devices, etc. (cf. Krishna and Singh, 2009) are some typical situations.

The failure rate function of an object, when the failures are reported on a discrete scale, may be bathtub-shaped or unimodal. Jiang (2010) investigated some discrete dis-

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tributions and used the exponentiated Poisson distribution and the two-fold competing risk model exhibiting bathtub-shaped or increasing failure rate functions to introduce a model for bus-motor failure data. Nooghabi et al. (2011) introduced the discrete modified Weibull distribution with increasing and bathtub-shaped failure rate function. However, in application areas, the absence of a suitable discrete model whose hazard rate function covers and contains different possible shapes, i.e., bathtub-shaped, upside-down bathtub and monotonically increasing and decreasing, is perceived. On the other hand, the traditional discrete distributions have limited applicability as models for reliability, failure times, counts, etc.

In the last two decades some papers dealing with discrete distributions obtained by discretizing a continuous distribution have appeared in the literature. Lisman and van Zuylen (1972) proposed and Kemp (1997) studied the discrete normal distribution which is characterized by maximum entropy for specified mean and variance; see also Dasgupta (1993) and Szablowski (2001). Roy (2003) introduced another discrete analog of normal distribution. Kemp (2008) also considered the discrete half-normal distribution as a maximum entropy distribution for given mean and variance. Inusah and Kozubowski (2006) and Kozubowski and Inusah (2006) introduced Laplace and skew-Laplace distributions on the lattice of integers, respectively. Barbiero (2014) considered an alternative discrete skew Laplace distribution. Krishna and Pundir (2007) introduced the discrete Maxwell distribution. Krishna and Pundir (2009) introduced the discrete Burr distribution and studied a special case of the distribution which led to perform the discrete Pareto distribution. Jazi et al. (2010) studied the discrete inverse Weibull distribution and proposed some important properties of their discrete model. Gómez-Déniz (2010) obtained a generalization of the geometric distribution from a member of the Marshall and Olkin (1997) family of distributions. Gómez-Déniz and Calderin-Ojeda (2011) considered the discrete Lindley distribution and investigated some properties and applications of the model. Chakraborty and Chakravarty (2012) studied discrete gamma distributions and discussed estimation of the parameters. In addition, Chakraborty (2013) introduced a new discrete distribution related to generalized gamma distribution. Chakraborty and Chakravarty (2013) introduced a new discrete probability distribution on the lattice of integers. Nekoukhou et al. (2013a) studied the discrete beta exponential (DBE) distribution and illustrated that the hazard rate function of this discrete analogue of the beta exponential distribution of Nadarajah and Kotz (2006) is monotone. Moreover, Hussain and Ahmad (2014) and Chakraborty and Chakravarty (2014) introduced the discrete inverse Rayleigh and discrete Gumbel distributions, respectively.

Recently, Nekoukhou et al. (2012) and (2013b) introduced two different discrete counterparts of the well-known two-parameter generalized exponential (GE) distribution of Gupta and Kundu (1999, 2001 and 2007). The probability mass functions (pmfs) of these distributions are

$$p_x = f(x; p, \gamma) = cp^{x-1}(1-p^x)^{\gamma-1}, \quad x \in \mathbb{N} = \{1, 2, 3, \dots\}, \tag{1}$$

where c is the norming constant, and

$$p_x = f(x;p,\gamma) = (1-p^{x+1})^\gamma - (1-p^x)^\gamma, \quad x \in \mathbb{N}_0 = \{0,1,2,\dots\}, \tag{2}$$

respectively. Nekoukhou et al. (2012) and (2013b) introduced these discrete analogues by using relations

$$p_x = \frac{f(x)}{\sum_{x=1}^{\infty} f(x)}, \quad x \in \mathbb{N} \tag{3}$$

and

$$p_x = S(x) - S(x+1), \quad x \in \mathbb{N}_0, \tag{4}$$

respectively, where $f(.)$ and $S(.)$ are the probability density function (pdf) and survival function of the GE distribution. Discrete generalized exponential (DGE) distribution and discrete generalized exponential distribution of a second type (DGE_2) are introduced in the literature via Eq.'s (1) and (2), respectively. The last authors denoted these two-parameter discrete distributions by $DGE(\gamma, p)$ and $DGE_2(\gamma, p)$. Eq. (2) yields that the cumulative distribution function (cdf) of the $DGE_2(\gamma, p)$ distribution is given by

$$F(x;p,\gamma) = (1-p^{[x]+1})^\gamma, \quad x \geq 0. \tag{5}$$

It is interesting to note that the above cdf coincides with the exponentiated geometric distribution which was mentioned in Jiang (2010), and investigated by Chakraborty and Gupta (2012).

In this paper we will introduce the exponentiated discrete Weibull (EDW) distribution, which is really a generalization of the discrete Weibull (DW) distribution of Nakagawa and Osaki (1975) and also DGE_2 distribution, and illustrate its important features and properties. The failure rate function of the new model is found to be bathtub-shaped, unimodal and also increasing and decreasing. In the application section we will see that the new model provides a satisfactory fit and that is competitive with traditional and also newly developed discrete models. The new discrete distribution also contains a generalization of the discrete Rayleigh distribution of Roy (2004) which has not been introduced in the literature yet.

The paper is organized as follows. Section 2 introduces the three-parameter EDW distribution and discusses some of its important features and properties such as cumulative distribution and hazard rate functions, moments, infinite divisibility and the order statistics. In Section 3, the researchers will consider the maximum likelihood method to estimate the parameters of EDW distribution. In addition, in this section, estimation of the stress-strength parameter is discussed. Section 4 describes fitting of the proposed model to a real data set. Finally, in Section 5 some concluding remarks are given.

2. Three-parameter EDW distribution

When the cdf of the DW distribution, denoted by $DW(p,\alpha)$, of Nakagawa and Osaki (1975), i.e.,

$$G(x;p,\alpha) = 1 - p^{([x]+1)^{\alpha}}, \quad x \geq 0, \tag{6}$$

where $0 < p < 1$ and $\alpha > 0$ are the model parameters, is inserted into the *resilience parameter family* of distributions, the cdf of the resulting discrete distribution is given by

$$F(x;p,\alpha,\gamma) = \{1 - p^{([x]+1)^{\alpha}}\}^{\gamma}, \quad x \geq 0 \tag{7}$$

in which $\gamma > 0$ is the *resilience parameter*.

We call such a random variable X, with cdf (7), an *exponentiated discrete Weibull distribution* with parameters $0 < p < 1$, $\alpha > 0$ and $\gamma > 0$ and denote it by $EDW(p,\alpha,\gamma)$.

It is evident that when $\gamma > 0$ is an integer value, the cdf given by (7) agrees with the cdf of the maximum of γ independent and identical $DW(p,\alpha)$ random variables.

2.1. Probability mass, survival and hazard rate functions

The corresponding pmf of a random variable X following an $EDW(p,\alpha,\gamma)$ distribution for $x \in \mathbb{N}_0$ is given by

$$p_x = P(X = x) = f(x;p,\alpha,\gamma) = \{1 - p^{(x+1)^{\alpha}}\}^{\gamma} - \{1 - p^{x^{\alpha}}\}^{\gamma} \tag{8}$$

$$= \sum_{j=1}^{\infty} (-1)^{j+1} \binom{\gamma}{j} \{p^{jx^{\alpha}} - p^{j(x+1)^{\alpha}}\}, \tag{9}$$

where $\binom{\gamma}{j} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-j)j!}$. For integer $\gamma > 0$, the sum in Eq. (9) stops at γ.

Nekoukhou et al. (2013b) indicated that $\sum_{j=1}^{\infty}(-1)^{j+1}\binom{\gamma}{j} = 1$. Hence, if $0 < \gamma < 1$ the pmf (9) can be viewed as an infinite mixture of $DW(p^j,\alpha)$ distributions, $j = 1,2,\ldots$

It is interesting to note that the EDW distribution with pmf (8) or (9) may also be viewed as a discrete analog of the exponentiated Weibull (EW) distribution of Mudholkar and Srivastava (1993) via Eq. (4) and doing reparametrization $0 < e^{-\beta^{\alpha}} = p < 1$ in the structure of EW distribution.

Nassar and Eissa (2003) obtained expressions for the mode of the EW pdf. They stated that EW distribution is monotone decreasing for $\alpha\gamma \leq 1$ and for $\alpha\gamma > 1$, it is unimodal. Naturally, it follows that $EDW(p,\alpha,\gamma)$ distributions are also unimodal for all values of parameters. Figure 1 illustrates the pmf of an $EDW(p,\alpha,\gamma)$ distribution for different values of parameters.

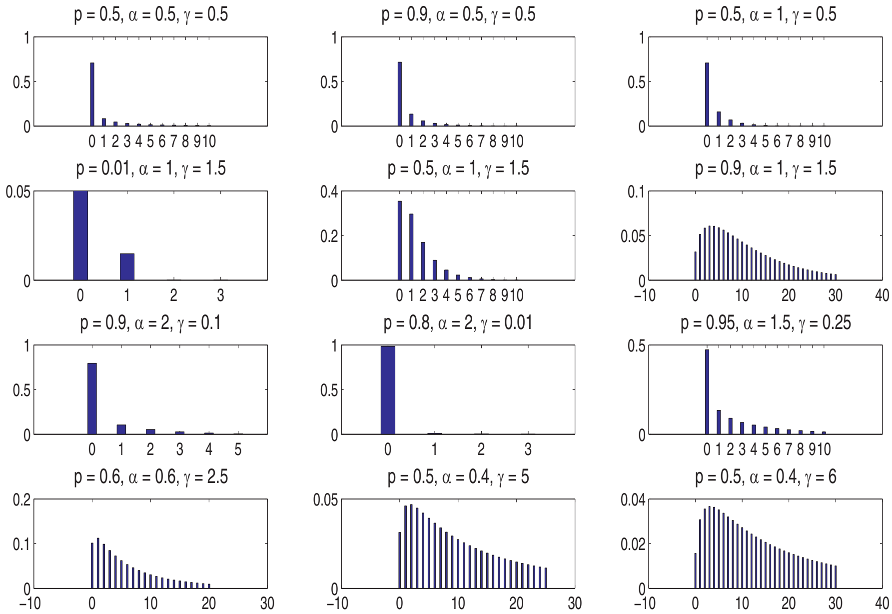

Figure 1: *Illustrations of the pmf of* $EDW(p,\alpha,\gamma)$ *for possible values of* p, α *and* γ.

The survival and hazard rate functions of $EDW(p,\alpha,\gamma)$ distribution are given by

$$S(x;p,\alpha,\gamma) = 1-\{1-p^{([x]+1)^\alpha}\}^\gamma, \quad x \geq 0 \tag{10}$$

and

$$h(x;p,\alpha,\gamma) = \frac{\{1-p^{(x+1)^\alpha}\}^\gamma - \{1-p^{x^\alpha}\}^\gamma}{1-\{1-p^{(x+1)^\alpha}\}^\gamma}, \quad x \in \mathbb{N}_0, \tag{11}$$

respectively.

Discrete hazard rates arise in several common situations in reliability theory where clock time is not the best scale on which to describe lifetime. For example, in weapons reliability, the number of rounds fired until failure is more important than age in failure. This is the case also when a piece of equipment operates in cycles and the observation is the number of cycles successfully completed prior to failure. In other situations a device is monitored only once per time period and the observation then is the number of time periods successfully completed prior to the failure of the device (cf. Shaked et al., 1995).

Figure 2 illustrates the hazard rate function of $EDW(p,\alpha,\gamma)$ distribution for different values of p, α and γ. As we see from the figure, a characteristic of the EDW distribution is that its hazard rate function can be decreasing, increasing, bathtub-shaped, and upside-down bathtub depending on its parameters values. Hence, EDW distributions are more flexible than other discrete distributions such as the geometric, DGE, DGE_2 and DBE distributions, whose hazard rate functions are constant and monotone.

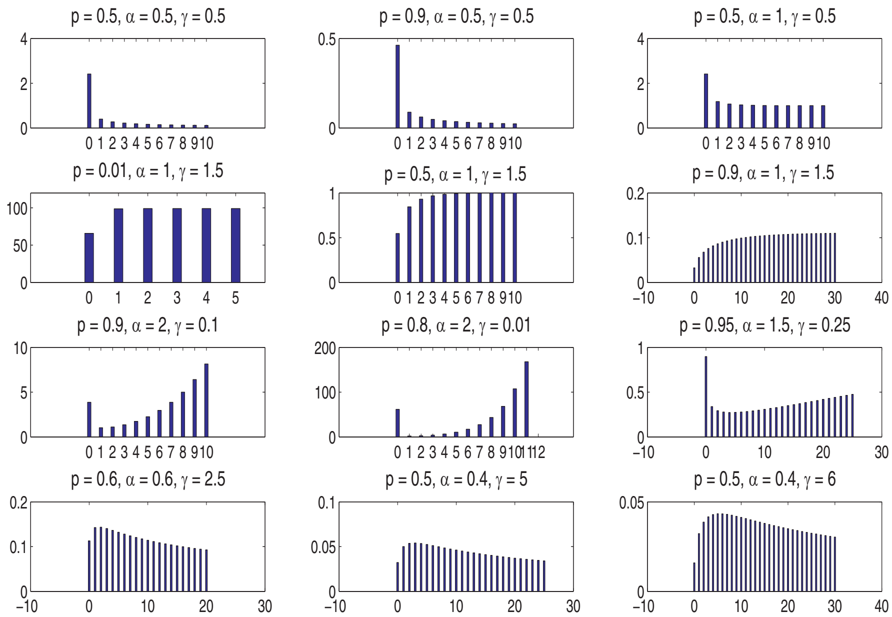

Figure 2: *Illustrations of the hazard rate function of* $EDW(p,\alpha,\gamma)$ *for possible values of p,* α *and* γ.

2.2. Special sub-models

Some special discrete distributions are achieved from EDW distribution as follows:

(1) Discrete Weibull distribution of Nakagawa and Osaki (1975), with pmf

$$p_x = (1 - p^{(x+1)^\alpha}) - (1 - p^{x^\alpha}), \tag{12}$$

is obtained when $\gamma = 1$. If, in addition, $\alpha = 1$, the geometric distribution is achieved. The discrete Weibull distribution is used for estimation of replicative senescence via population dynamics models (Wein and Wu, 2001), stress-strength reliability (Roy, 2002), evaluation of reliability of complex systems (Roy, 2002), wafer probe operation in semiconductor manufacturing (e.g., Wang, 2009), minimal availability variation design of repairable systems (e.g., Wang et al., 2010) and microbial counts in water (Englehardt and Li, 2011). Since the EDW distribution is an extension of DW distribution, one may expect from EDW model to be more flexible in such application areas.

(2) If $\alpha = 1$, then the discrete generalized exponential distribution of a second type ($DGE_2(\gamma, p)$) of Nekoukhou et al. (2013b) with pmf given by Eq. (2) is obtained. If, in addition, $\gamma = 1$, the geometric distribution will be obtained again from a different point of view.

(3) If $\alpha = 2$, then the pmf of $EDW(p, \alpha, \gamma)$ distribution reduces to

$$p_x = f(x; p, \gamma) = \{1 - p^{(x+1)^2}\}^{\gamma} - \{1 - p^{x^2}\}^{\gamma} \tag{13}$$

$$= \sum_{j=1}^{\infty} (-1)^{j+1} \binom{\gamma}{j} p^{jx^2} (1 - p^{j(2x+1)}), \tag{14}$$

which defines a generalized discrete Rayleigh distribution $GDR(\gamma, p)$ for the first time in the literature. Moreover, for $\gamma = 1$ in Eq. (13) the discrete Rayleigh (DR) distribution of Roy (2004) is obtained.

2.3. Quantiles, mean and variance

The m-th quantile of an EDW distribution is obtained by solving the equation

$$F(q_m; p, \alpha, \gamma) = m,$$

where $F(.)$ is the cdf of an $EDW(p, \alpha, \gamma)$ distribution and q_m denotes the corresponding quantile function which is given by

$$q_m = \left\{ \frac{\log(1 - m)^{1/\gamma}}{\log p} \right\}^{1/\alpha} - 1. \tag{15}$$

Particularly, the median is immediately achieved by setting $m = 0.5$ in the above equation.

The mean and variance of a random variable X following an $EDW(p, \alpha, \gamma)$ distribution are given, respectively, by

$$E(X) = \sum_{j=1}^{\infty} \binom{\gamma}{j} (-1)^{j+1} \frac{p^{\alpha j}}{1 - p^{\alpha j}} \tag{16}$$

and

$$Var(X) = 2 \sum_{j=1}^{\infty} \binom{\gamma}{j} (-1)^{j+1} \frac{p^{\alpha j}}{(1 - p^{\alpha j})^2} + E(X) - \{E(X)\}^2. \tag{17}$$

Remark 2.1 For an integer value of $\gamma > 0$, $\sum_{j=1}^{\infty}$ should be replaced by $\sum_{j=1}^{\gamma}$ in the above equations.

Remark 2.2 For $\alpha = 1$, Eq. (16) reduces to

$$E(X) = \sum_{j=1}^{\infty} \binom{\gamma}{j} (-1)^{j+1} \frac{p^j}{1-p^j}, \tag{18}$$

which is the mean of the $DGE_2(\gamma, p)$ distribution obtained by Nekoukhou et al. (2013b). In addition, in this case, it is easy to show that the variance of an EDW distribution reduces to the variance of a DGE_2 distribution.

The mean and variance of an $EDW(p, \alpha, \gamma)$ distribution for different values of p, α and γ, using Eq.'s (16) and (17), are calculated in Table 1 below. It appears that depending on the values of the parameters, the mean of the distribution can be smaller or greater than its variance. Hence, EDW models are appropriate for modeling both over and under dispersed data since, in these models, the variance can be larger or smaller than the mean which is not the case with some standard classical discrete distributions.

Table 1: *Mean (Variance) of $EDW(p, \alpha, \gamma)$ for different values of p, α and γ.*

	$\gamma = 0.50$		
α/p	0.25	0.5	0.75
0.50	0.3938 (2.6970)	2.0105 (46.7602)	12.7288 (1517.5424)
0.75	0.2253 (0.5441)	0.8322 (3.9644)	3.2749 (44.2522)
1.00	0.1761 (0.2573)	0.5546 (1.2777)	1.7437 (8.2081)
2.00	0.1359 (0.1213)	0.3256 (0.2856)	0.7168 (0.7357)
3.50	0.1339 (0.1160)	0.2930 (0.2075)	0.5194 (0.2885)
	$\gamma = 1.00$		
α/p	0.25	0.5	0.75
0.50	0.7598 (5.0596)	3.7882 (85.6990)	23.5837 (2743.1543)
0.75	0.4296 (0.9364)	1.5272 (6.6530)	5.8068 (71.5589)
1.00	0.3333 (0.4444)	1.0000 (1.9999)	2.9999 (11.9999)
2.00	0.2539 (0.1972)	0.5644 (0.3787)	1.1522 (0.8241)
3.50	0.2500 (0.1875)	0.5003 (0.2507)	0.7885 (0.2439)
	$\gamma = 3.00$		
α/p	0.25	0.5	0.75
0.50	2.0009 (12.1856)	9.3360 (197.0330)	56.1430 (6147.2083)
0.75	1.0828 (1.8981)	3.4610 (11.9090)	12.2932 (123.5352)
1.00	0.8158 (0.7907)	2.1428 (2.9251)	5.8725 (16.5319)
2.00	0.5898 (0.2653)	1.0569 (0.3155)	1.9058 (0.6676)
3.50	0.5781 (0.2438)	0.8761 (0.1108)	1.0957 (0.1178)

Remark 2.3 Remember that a random variable X with cdf G is stochastically smaller than Y with cdf F, denoted by $X \leq_{st} Y$, if for all x, $G(x) \geq F(x)$. This is the most basic and oldest stochastic order in Probability and Statistics. In this case, if G is simpler than F, $G(x)$ may provide a useful lower bound for $F(x)$ (see, e.g., Shaked and Shanthikumar

(2007) for more details). Now, let G and F denote the cdfs of the DW and EDW distributions which are defined via Eq.'s (6) and (7), respectively. It is obvious that for $\gamma > 1$, we have $X \leq_{st} Y$ because $[G(x)]^{\gamma} \leq G(x)$ and if $0 < \gamma < 1$, it follows that $X \geq_{st} Y$. Hence, For $\gamma \geq 1$ it follows that $E(X) \leq E(Y)$ and corresponding result holds if X is stochastically larger than Y. One can consider the results of Table 1 again.

2.4. Infinite divisibility

The researchers here make the following note in regards to the famous structural property of infinite divisibility of the distribution in question. Such a characteristic has a close relation to the Central Limit Theorem and waiting time distributions. Thus, it is a desirable question in modeling to know whether a given distribution is infinitely divisible or not. To settle this question, we recall that according to Steutel and van Harn (2004, pp. 56), if $p_x, x \in \mathbb{N}_0$, is infinitely divisible, then $p_x \leq e^{-1}$ for all $x \in \mathbb{N}$. However, e.g., in an $EDW(0.9,3,1)$ distribution we see that $p_2 = 0.372 > e^{-1} = 0.367$. Therefore, in general, $EDW(p,\alpha,\gamma)$ distributions are not infinitely divisible. In addition, since the classes of self-decomposable and stable distributions, in their discrete concepts, are subclasses of infinitely divisible distributions, we conclude that an EDW distribution can be neither self-decomposable nor stable in general.

2.5. Order statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. They enter the problems of estimation and hypothesis testing in a variety of ways. The aim of the present section is to establish some general relations regarding the EDW distributions. More precisely, let $F_i(x;p,\alpha,\gamma)$ and $f_i(x;p,\alpha,\gamma)$ be the cdf and pmf of the i-th order statistic of a random sample of size n from $EDW(p,\alpha,\gamma)$ distribution.

Since,

$$F_i(x;p,\alpha,\gamma) = \sum_{k=i}^{n} \binom{n}{k} [F(x;p,\alpha,\gamma)]^k [1-F(x;p,\alpha,\gamma)]^{n-k}, \tag{19}$$

using the binomial expansion for $[1-F(x;p,\alpha,\gamma)]^{n-k}$, we obtain the following result:

$$\begin{aligned} F_i(x;p,\alpha,\gamma) &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j}(-1)^j [F(x;p,\alpha,\gamma)]^{k+j} \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j}(-1)^j [\{1-p^{([x]+1)^{\alpha}}\}^{\gamma}]^{k+j} \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j}(-1)^j F_{EDW}(x;p,\alpha,\gamma(k+j)), \end{aligned} \tag{20}$$

where F_{EDW} denotes the cdf of an EDW distribution. The corresponding pmf of the i-th order statistic, $f_i(x; p, \alpha, \gamma) = F_i(x; p, \alpha, \gamma) - F_i(x-1; p, \alpha, \gamma)$ for an integer value of x, then is given by

$$f_i(x; p, \alpha, \gamma) = \sum_{k=i}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} (-1)^j f_{EDW}(x; p, \alpha, \gamma(k+j)), \tag{21}$$

where f_{EDW} denotes the pmf of an EDW distribution.

Remark 2.4 In view of the fact that $f_i(x; p, \alpha, \gamma)$ is a linear combination of a finite number of $EDW(p, \alpha, \gamma(k+j))$ distributions, we may obtain some properties of order statistics, such as their moments, from the corresponding EDW distribution. For example, the mean of the i-th order statistic is given by

$$\mu_{i:n} = \sum_{t=1}^{\infty} \sum_{k=i}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \binom{\gamma(k+j)}{t} (-1)^{j+t+1} \frac{p^{\alpha t}}{1-p^{\alpha t}}. \tag{22}$$

3. Estimation

To apply the method of maximum likelihood for estimating the parameter vector $\boldsymbol{\theta} = (p, \alpha, \gamma)^\top$ of EDW distribution, assume that $x = (x_1, x_2, ..., x_n)^\top$ is a random sample of size n from an $EDW(p, \alpha, \gamma)$ distribution. The log-likelihood function becomes

$$\ell = \sum_{i=1}^{n} \log[(1 - p^{(x_i+1)^\alpha})^\gamma - (1 - p^{x_i^\alpha})^\gamma]. \tag{23}$$

Hence, the likelihood equations are

$$\frac{\partial \ell}{\partial p} = \sum_{i=1}^{n} \frac{v_{\alpha,\gamma}(x_i+1) - v_{\alpha,\gamma}(x_i)}{m_{\alpha,\gamma}(x_i)}, \tag{24}$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{\gamma \log p [u_{\alpha,\gamma}(x_i) \log x_i - u_{\alpha,\gamma}(x_i+1) \log(x_i+1)]}{m_{\alpha,\gamma}(x_i)} \tag{25}$$

and

$$\frac{\partial \ell}{\partial \gamma} = \sum_{i=1}^{n} \frac{\gamma [u_{\alpha,\gamma}(x_i) - u_{\alpha,\gamma}(x_i+1)]}{p m_{\alpha,\gamma}(x_i)}, \tag{26}$$

where

$$m_{\alpha,\gamma}(x) = \{1 - p^{(x+1)^\alpha}\}^\gamma - \{1 - p^{x^\alpha}\}^\gamma,$$

$$v_{\alpha,\gamma}(x) = (1 - p^{x^\alpha})^\gamma \log(1 - p^{x^\alpha})$$

and

$$u_{\alpha,\gamma}(x) = (1 - p^{x^\alpha})^{\gamma-1} p^{x^\alpha} x^\alpha.$$

The solutions of likelihood equations (24)-(26) provide the maximum likelihood estimators (MLEs) of $\boldsymbol{\theta} = (p, \alpha, \gamma)^\mathsf{T}$, say $\hat{\boldsymbol{\theta}} = (\hat{p}, \hat{\alpha}, \hat{\gamma})^\mathsf{T}$, which can be obtained by a numerical method such as the three variable Newton-Raphson type procedure.

For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 3×3 observed information matrix is

$$I_n(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} -\dfrac{\partial^2 \ell}{\partial p^2} & -\dfrac{\partial^2 \ell}{\partial p \partial \alpha} & -\dfrac{\partial^2 \ell}{\partial p \partial \gamma} \\ -\dfrac{\partial^2 \ell}{\partial \alpha \partial p} & -\dfrac{\partial^2 \ell}{\partial \alpha^2} & -\dfrac{\partial^2 \ell}{\partial \alpha \partial \gamma} \\ -\dfrac{\partial^2 \ell}{\partial \gamma \partial p} & -\dfrac{\partial^2 \ell}{\partial \gamma \partial \alpha} & -\dfrac{\partial^2 \ell}{\partial \gamma^2} \end{bmatrix}, \qquad (27)$$

whose elements are given in the Appendix.

One can show that the EDW family satisfies the regularity conditions which are fulfilled for parameters in the interior of the parameter space but not on the boundary (see, e.g., Cox and Hinkley, 1974). Hence, the MLE vector $\hat{\boldsymbol{\theta}}$ is consistent and asymptotically normal. That is, $I_n^{\frac{1}{2}}(\boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ converges in distribution to trivariate normal with the (vector) mean zero and the identity covariance matrix.

One can use the normal distribution of $\hat{\boldsymbol{\theta}}$ to construct approximate confidence regions for some parameters. Indeed, an asymptotic $100(1-\xi)$ confidence interval for each parameter θ_i, is given by

$$(\hat{\theta}_i - z_{\xi/2}\sqrt{\hat{J}_{ii}}, \hat{\theta}_i + z_{\xi/2}\sqrt{\hat{J}_{ii}}), \qquad i = 1, 2, 3,$$

where $\hat{J}_{ii}$ denotes the (i,i) diagonal element of $I_n^{-1}(\hat{\boldsymbol{\theta}})$ and $z_{\xi/2}$ is the $(1-\xi/2)$-th quantile of the standard normal distribution.

3.1. Simulation study

Let X be a random variable that follows an EW distribution with cdf

$$F(x;\alpha,\beta,\gamma)=\{1-e^{-(\beta x)^{\alpha}}\}^{\gamma}, \quad x>0,$$

where $\alpha>0$, $\beta>0$ and $\gamma>0$ (two shapes and one scale) are the model parameters. It is easy to show that $[X]$ has an $EDW(p,\alpha,\gamma)$ distribution in which $0<p=e^{-\beta^{\alpha}}<1$. Therefore, we can simulate an EDW random variable from the corresponding continuous EW distribution. Table 2 below presents the maximum likelihood estimates of $\boldsymbol{\theta}=(p,\alpha,\gamma)^{\mathsf{T}}$ of an EDW distribution and also contains their standard errors for different values of n as a simulation study. Standard errors are attained by means of the asymptotic covariance matrix of the MLEs of EDW parameters when the Newton-Raphson procedure converges in, e.g., MATLAB software.

Table 2: *MLEs of EDW parameters for different values of n.*

n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{p}(\hat{SE}(\hat{p}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{p}(\hat{SE}(\hat{p}))$
(α,γ)		$(0.5,0.75)$			$(0.75,0.5)$	
p		0.25			0.75	
40	0.525(0.563)	1.113(2.987)	0.221(0.657)	0.822(0.427)	0.591(0.431)	0.784(0.322)
100	0.492(0.477)	0.984(2.764)	0.213(0.527)	0.812(0.396)	0.442(0.393)	0.753(0.297)
200	0.511(0.323)	0.788(1.347)	0.288(0.410)	0.730(0.268)	0.551(0.379)	0.719(0.237)
500	0.501(0.242)	0.792(1.099)	0.217(0.264)	0.745(0.158)	0.526(0.204)	0.751(0.129)
1000	0.568(0.175)	0.799(0.675)	0.257(0.185)	0.743(0.108)	0.534(0.144)	0.745(0.090)
(α,γ)		$(2,3)$			$(3,2)$	
p		0.5			0.9	
40	2.197(1.083)	2.536(2.931)	0.542(0.410)	2.857(1.453)	1.903(1.879)	0.927(0.194)
100	2.077(0.951)	2.656(2.652)	0.564(0.349)	2.912(1.197)	1.872(1.542)	0.897(0.156)
200	1.904(0.663)	2.941(2.352)	0.494(0.289)	2.937(0.818)	2.022(1.163)	0.888(0.113)
500	1.915(0.465)	3.290(1.880)	0.462(0.187)	3.153(0.605)	1.980(0.781)	0.914(0.065)
1000	2.004(0.321)	2.950(1.068)	0.511(0.124)	2.918(0.306)	1.981(0.427)	0.895(0.041)
(α,γ)		$(1,1)$			$(1.5,0.5)$	
p		0.5			0.95	
40	1.202(0.996)	1.183(1.210)	0.717(0.712)	1.139(0.611)	0.733(0.693)	0.896(0.206)
100	1.278(0.723)	0.864(1.005)	0.601(0.416)	1.257(0.436)	0.808(0.494)	0.867(0.150)
200	0.933(0.363)	0.974(0.850)	0.488(0.293)	1.443(0.393)	0.471(0.198)	0.947(0.060)
500	0.982(0.230)	1.043(0.553)	0.484(0.177)	1.521(0.233)	0.522(0.125)	0.957(0.023)
1000	1.058(0.172)	0.909(0.318)	0.542(0.122)	1.507(0.177)	0.481(0.087)	0.955(0.012)

3.2. Stress-strength parameter

The stress-strength parameter $R = P(X > Y)$ is a measure of component reliability and its estimation problem when X and Y are independent and follow a specified common distribution has been discussed widely in the literature. Suppose that the random variable X is the strength of a component which is subjected to a random stress Y. Estimation of R when X and Y are independent and identically distributed following a well-known distribution has been considered in the literature. Many applications of the stress-strength model, for its own nature, are related to engineering or military problems. There are also natural applications in Medicine or Psychology, which involve the comparison of two random variables, representing for example the effect of a specific drug or treatment administered to two groups, control and test. Almost all of these studies consider continuous distributions for X and Y, because many practical applications of the stress-strength model in engineering fields presuppose continuous quantitative data. A complete review is available in Kotz et al. (2003). However, in this regard, a relatively small amount of work is devoted to discrete or categorical data. Data may be discrete by nature. For example, the stress pattern in a step-stress accelerated life test can be treated as a discrete random variable of which the possible values can be obtained from all stress levels, and the corresponding probabilities can be obtained from the acting times of each stress levels. Moreover, the stress state of a component can be categorized based on the characteristic of external loads. For instance, the stress state of a mechanical component can be simply classified as state 1, state 2 and state 3, which correspond to low load, moderate load and heavy load, respectively. More generally, according to the change of external loads, the stress of a component can be categorized into arbitrary finite state: state 1, state 2, ..., state m.

The stress-strength parameter, in discrete case, is defined as

$$R = P(X > Y) = \sum_{x=0}^{\infty} f_X(x) F_Y(x),$$

where f_X and F_Y denote the pmf and cdf of the independent discrete random variables X and Y, respectively. Now, let $X \sim EDW(\boldsymbol{\theta_1})$ and $Y \sim EDW(\boldsymbol{\theta_2})$, where $\boldsymbol{\theta_1} = (p_1, \alpha_1, \gamma_1)^\mathsf{T}$ and $\boldsymbol{\theta_2} = (p_2, \alpha_2, \gamma_2)^\mathsf{T}$. Using Equations (7) and (8), we obtain

$$R = \sum_{x=0}^{\infty} [\{1 - p_1^{(x+1)^{\alpha_1}}\}^{\gamma_1} - \{1 - p_1^{x^{\alpha_1}}\}^{\gamma_1}]\{1 - p_2^{(x+1)^{\alpha_2}}\}^{\gamma_2}.$$

Using the binomial expansion, it is easy to show that

$$R = \sum_{j=1}^{\infty}\sum_{t=1}^{\infty}\sum_{x=0}^{\infty} (-1)^{j+t+1} \binom{\gamma_1}{j} \binom{\gamma_2}{t} p_2^{t(x+1)^{\alpha_2}} \{p_1^{jx^{\alpha_1}} - p_1^{j(x+1)^{\alpha_1}}\}. \tag{28}$$

Now, assume that $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ are independent observations from $X \sim EDW(\boldsymbol{\theta_1})$ and $Y \sim EDW(\boldsymbol{\theta_2})$, respectively. The total likelihood function is $\ell_R(\boldsymbol{\theta}^*) = \ell_n(\boldsymbol{\theta_1})\ell_m(\boldsymbol{\theta_2})$, where $\boldsymbol{\theta}^* = (\boldsymbol{\theta_1}, \boldsymbol{\theta_2})$. The score vector is given by

$$U_R(\boldsymbol{\theta}^*) = (\partial\ell_R/\partial p_1, \partial\ell_R/\partial\alpha_1, \partial\ell_R/\partial\gamma_1, \partial\ell_R/\partial p_2, \partial\ell_R/\partial\alpha_2, \partial\ell_R/\partial\gamma_2),$$

and the MLE of $\boldsymbol{\theta}^*$, say $\hat{\boldsymbol{\theta}}^*$, may be attained from the nonlinear equation $U_R(\hat{\boldsymbol{\theta}}^*) = \mathbf{0}$. Thus, by inserting the MLEs in equation (28) the stress-strength parameter R will be estimated.

4. Application

In this section, the EDW model will be examined for a real data set which is given by Karlis and Xekalaki (2001) on the numbers of fires in Greece for the period from 1 July 1998 to 31 August of the same year. This data set consists of 123 observation and are presented in Table 3. Only fires in forest districts are considered. Bakouch et al. (2014) considered these data to indicate the potentiality of discrete Lindley (DL) distribution in data modeling and compared it with Poisson, geometric and discrete gamma (DG) distributions. The pmf of the DG distribution, which has been used first by Yang (1994) and recently considered by Chakraborty and Chakravarty (2012), for $x \in \mathbb{N}_0$, is given by

$$p_x = \frac{\gamma(\alpha, \beta(x+1)) - \gamma(\alpha, \beta x)}{\Gamma(\alpha)}, \quad \alpha > 0,\ \beta > 0,$$

where $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ denotes the incomplete gamma function. Additionally, the pmf of the DL distribution for $x \in \mathbb{N}_0$ is given by

$$p_x = \frac{p^x}{1+\theta}\{\theta(1-2p) + (1-p)(1+\theta x)\}, \quad 0 < p < 1,\ \theta > 0.$$

Here, we compare the EDW and GDR models with these discrete distributions. In addition, because of the over dispersion phenomena in the data set, $\bar{x} = 5.3984$ and $s^2 = 30.0449$, the negative binomial (NB) distribution is also compared with the others. Maximum likelihood method is used to obtain the estimates of the parameters of the proposed new distributions (EDW and GDR). Comparing the EDW model with its rival models is performed by using the Akaike information criterion (AIC) and Kolmogrov-Smirnov (K-S) test statistic. Table 4 indicates the MLEs, AICs and the values of the K-S test statistics determined by the fitted models. The MLEs and K-S test statistic values of the DL and DG distributions, given in this table, are directly reported from Table 7 of Bakouch et al. (2014).

Table 3: *Numbers of fires in Greece.*

Numbers	0	1	2	3	4	5	6	7	8	9	10	11	12	15	16	20	43
Frequency	16	13	14	9	11	13	8	4	9	6	3	4	6	4	1	1	1

Table 4: *Summary.*

Models	MLEs	AIC	K-S statistic
EDW	$(\hat{\alpha},\hat{\gamma},\hat{p}) = (1.0809, 1.0923, 0.8599)$	685.5859	0.1254
GDR	$(\hat{\gamma},\hat{p}) = (0.3934, 0.9924)$	694.6178	0.1467
DGE_2	$(\hat{\gamma},\hat{p}) = (1.2548, 0.8225)$	683.7049	0.1301
NB	$(\hat{r},\hat{p}) = (1.3360, 0.1984)$	683.2989	0.3350
DL	$(\hat{\theta},\hat{p}) = (0.3090, 0.7343)$	685.8067	0.1122
DG	$(\hat{\alpha},\hat{\beta}) = (0.7525, 0.1543)$	749.7162	0.2683

According to the values of the K-S test statistics and AICs in Table 4, it seems that EDW model gives a satisfactory fit to this real data set.

To construct approximate confidence intervals for the parameters of EDW model and also for evaluating accuracy of the estimated parameters, we use the corresponding estimated standard errors. For instance, 95% asymptotic confidence intervals for EDW parameters are obtained as $\alpha \in (1.081 \mp 0.4531)$, $\gamma \in (1.0923 \mp 0.8554)$ and $p \in (0.8599 \mp 0.1895)$.

5. Conclusions and comments

In this paper, a new three-parameter generalization of the discrete Weibull distribution is proposed, so-called exponentiated discrete Weibull (EDW) distribution which is, indeed, a member of resilience parameter family of distributions. The hazard rate function of the new model can be increasing, decreasing, upside-down bathtub and also bathtub-shaped and hence presents a very flexible behavior. Fitting the EDW model to a real data set indicates the flexibility and capacity of the proposed distribution in data modeling. In addition, a special sub-model of EDW distribution, i.e., the generalized discrete Rayleigh distribution is introduced for the first time in the literature.

Appendix

The elements of the 3×3 information matrix in Eq. (27) are given by

$$\frac{\partial^2 \ell}{\partial p^2} = \sum_{i=1}^{n} \{ \frac{\gamma}{p^2 m_{\alpha,\gamma}(x_i)} \{ (\gamma-1) u_{\alpha,\gamma-1}(x_i+1) p^{(x_i+1)^\alpha} (x_i+1)^\alpha$$

$$- u_{\alpha,\gamma}(x_i+1)[(x_i+1)^\alpha - 1]$$

$$- (\gamma-1) u_{\alpha,\gamma-1}(x_i) p^{x_i^\alpha} x_i^\alpha + u_{\alpha,\gamma}(x_i)(x_i^\alpha - 1) \} - \frac{\gamma^2 [u_{\alpha,\gamma}(x_i) - u_{\alpha,\gamma}(x_i+1)]^2}{p^2 m_{\alpha,\gamma}^2(x_i)} \},$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \sum_{i=1}^{n} \{ \frac{\gamma}{m_{\alpha,\gamma}(x_i)} \{ (\gamma-1) u_{\alpha,\gamma-1}(x_i+1) p^{(x_i+1)^\alpha} (x_i+1)^\alpha \log^2(x_i+1) \log^2 p$$

$$- (\gamma-1) u_{\alpha,\gamma-1}(x_i) p^{x_i^\alpha} x_i^\alpha \log^2 x_i \log^2 p + u_{\alpha,\gamma}(x_i) \log^2 x_i \log p [x_i^\alpha \log p + 1]$$

$$- u_{\alpha,\gamma}(x_i+1) \log^2(x_i+1) \log p [(x_i+1)^\alpha \log p + 1] \}$$

$$- \frac{\{\gamma \log p [u_{\alpha,\gamma}(x_i) \log x_i - u_{\alpha,\gamma}(x_i+1) \log(x_i+1)]\}^2}{m_{\alpha,\gamma}^2(x_i)} \},$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = \sum_{i=1}^{n} \{ \frac{v_{\alpha,\gamma}(x_i+1) \log(1 - p^{(x_i+1)^\alpha}) - v_{\alpha,\gamma}(x_i) \log(1 - p^{x_i^\alpha})}{m_{\alpha,\gamma}(x_i)}$$

$$- \frac{\{v_{\alpha,\gamma}(x_i+1) - v_{\alpha,\gamma}(x_i)\}^2}{m_{\alpha,\gamma}^2(x_i)} \},$$

$$\frac{\partial^2 \ell}{\partial p \partial \alpha} = \sum_{i=1}^{n} \{ \frac{\gamma}{p m_{\alpha,\gamma}(x_i)} \{ (\gamma-1) u_{\alpha,\gamma-1}(x_i+1) p^{(x_i+1)^\alpha} (x_i+1)^\alpha \log(x_i+1) \log p$$

$$- (\gamma-1) u_{\alpha,\gamma-1}(x_i) p^{x_i^\alpha} x_i^\alpha \log x_i \log p + u_{\alpha,\gamma}(x_i) \log x_i [x_i^\alpha \log p + 1]$$

$$- u_{\alpha,\gamma}(x_i+1) \log(x_i+1) [(x_i+1)^\alpha \log p + 1] \}$$

$$- \frac{\gamma^2 \log p}{p m_{\alpha,\gamma}^2(x_i)} \{ [u_{\alpha,\gamma}(x_i) \log x_i - u_{\alpha,\gamma}(x_i) \log(x_i+1)][u_{\alpha,\gamma}(x_i) - u_{\alpha,\gamma}(x_i+1)] \} \},$$

$$\frac{\partial^2 \ell}{\partial p \partial \gamma} = \sum_{i=1}^{n} \{ \frac{u_{\alpha,\gamma}(x_i)[\gamma \log(1 - p^{x_i^\alpha}) + 1] - u_{\alpha,\gamma}(x_i+1)[\gamma \log(1 - p^{(x_i+1)^\alpha}) + 1]}{p m_{\alpha,\gamma}(x_i)}$$

$$- \frac{\gamma [u_{\alpha,\gamma}(x_i) - u_{\alpha,\gamma}(x_i+1)][v_{\alpha,\gamma}(x_i+1) - v_{\alpha,\gamma}(x_i)]}{p m_{\alpha,\gamma}^2(x_i)} \}$$

and

$$\frac{\partial^2 \ell}{\partial\alpha\partial\gamma} = \sum_{i=1}^{n} \{ \frac{\log p}{m_{\alpha,\gamma}(x_i)} \{ u_{\alpha,\gamma}(x_i) \log x_i [\gamma \log(1 - p^{x_i^\alpha}) + 1]$$

$$- u_{\alpha,\gamma}(x_i+1) \log(x_i+1) [\gamma \log(1 - p^{(x_i+1)^\alpha}) + 1] \}$$

$$- \frac{\gamma \log p \{ u_{\alpha,\gamma}(x_i) \log x_i - u_{\alpha,\gamma}(x_i+1) \log(x_i+1) \} \{ v_{\alpha,\gamma}(x_i+1) - v_{\alpha,\gamma}(x_i) \}}{m^2_{\alpha,\gamma}(x_i)} \}.$$

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