SORT 38 (2) July-December 2014, 107-138

Stochastic cash flows modelled by homogeneous and non-homogeneous discrete time backward semi-Markov reward processes

Fulvio Gismondi[1], Jacques Janssen[2], Raimondo Manca[3]
and Ernesto Volpe di Prignano[3]

Abstract

The main aim of this paper is to give a systematization on the stochastic cash flows evolution. The tools that are used for this purpose are discrete time semi-Markov reward processes. The paper is directed not only to semi-Markov researchers but also to a wider public, presenting a full treatment of these tools both in homogeneous and non-homogeneous environment. The main result given in the paper is the natural correspondence of the stochastic cash flows with the semi-Markov reward processes. Indeed, the semi-Markov environment gives the possibility to follow a multi-state random system in which the randomness is not only in the transition to the next state but also in the time of transition. Furthermore, rewards permit the introduction of a financial environment into the model. Considering all these properties, any stochastic cash flow can be naturally modelled by means of semi-Markov reward processes. The backward case offers the possibility of considering in a complete way the duration inside a state of the studied system and this fact can be very useful in the evaluation of insurance contracts.

MSC: 60K15, 60K20, 91B28, 91B30.

Keywords: Stochastic cash flows, insurance contracts, discrete time backward semi-Markov processes, reward processes, homogeneous and non-homogeneous processes.

1. Introduction

By stochastic cash flow (SCF) we mean a financial operation (Janssen et al., 2009) in which the flow amount is stochastic. Furthermore, it is possible that the time of payments can be stochastic.

[1] "Guglielmo Marconi" University.

[2] Solvay Business School, Université Libre de Bruxelles, Belgium. janssenwets.jacques@gmail.com

[3] Sapienza Università di Roma. Dipartimento di Metodi e Modelli per l'Economia, il Territorio e la Finanza. raimondo.manca@uniroma1.it; ernesto.volpe@gmail.com

Received: August 2012
Accepted: April 2014

More precisely, the discrete flow value in both cases can fluctuate within a time interval that we suppose being a subset of $\mathbb{N}$. Indeed in real life problems, the money amount is a discrete variable and the time of payments is a discrete variable. Under these assumptions, a SCF can be seen as an example of a bivariate discrete time stochastic process where the first variable is the time and the second the money amount.

The study of SCF has been particularly developed in a financial environment and in various aspects of insurance. Many papers were written on the evaluation of annuities with stochastic interest rates and/or on stochastic cash flow evaluation. In this environment, we recall the following papers: Artikis and Malliaris (1990), Artikis and Voudori (2000), Beekman and Fuelling (1991), Browne (1995), De Schepper and Goovaerts (1992), De Schepper et al. (1994), Donati-Martin et al. (2000), Duffie et al. (2000), Dufresne (2001), Halliwell (2003), Harrison (1977), Milevsky (1997), Milevsky and Posner (1998), Parker (1994), Perry and Stadje (2000, 2001), Pliska (1986), Sato and Yor (1998), Vanneste et al. (1994, 1997) and the books of Wolthuis (2003) and Yor (2001). All these papers and books face the problem of using the stochastic calculus tools under Markovian hypotheses with continuous states and continuous times. These hypotheses implies that the waiting time distribution functions (WTDF) are exponential, and furthermore that the duration inside the state cannot be considered in a Markov environment. In real life these hypotheses in the most cases are not verified.

Another way to evaluate a SCF was given by Wilkie (1986) and subsequently improved in Wilkie (1995). This model uses traditional time series tools. In this case, the model pointed to the study of future asset returns of an insurance company. This model cannot be considered a general model for the study of evolution of SCF and also in this case the duration inside the states cannot be considered.

As is well known, (see Janssen et al., 2009), financial operation is a set of financial supply (T_k, S_k) where S_k represents an amount and T_k the time of the payment of S_k. We can suppose that:

1. S_k depends on the state of a system and, usually, also on the time of the payment;
2. the change of state happens at a random time.

The introduction of this second random variable, as specified in Estes et al. (1989) complicates the calculation of the cash flow value at a given date.

If we suppose that, in a discrete time environment, the bivariate stochastic process future depends only on the present and not on the past history and that the WTDF between two transitions can be of any type, then we are in a semi-Markov process (SMP) environment (see Janssen and Manca, 2006 and 2007 and their references).

It is already known that the applications of the SMP to finance and insurance problems assume great relevance and in the literature these applications were given, for example, in Janssen (1966), Hoem (1972), CMIR (1991), Carravetta et al. (1981), Balcer and Sahin (1979, 1986), Swishchuck (1995), De Medici et al. (1995), Janssen and Manca (1997, 1998), Janssen et al. (2004) and in the book Janssen and Manca (2007).

In the study of cash flow evolution and the related evaluation, the present value and the “accumulated value” assume great relevance. The association of an amount of money to a state of the system and/or to a state transition can be done by attaching a reward structure to the process. This structure can be thought of as a random variable associated with the state occupancies and/or the transitions, see Howard (1971) in which is given a nice presentation of Homogeneous Semi-Markov ReWard Processes (HSMPRWP).

Non-homogeneous semi-Markov reward processes (NHSMRWP) were defined in De Dominicis and Manca (1986), and we recall more recent papers (Papadopoulou et al., 2012 and Papadopoulou, 2013) that applied this tool in other fields.

The rewards can be of different kinds but, in a financial environment, it makes sense to consider only amounts of money as rewards. These amounts can be positive for the system if they can be seen as a benefit and negative if they can be considered as a cost.

SMRWP is a very important tool for applications, and most relevant when it is necessary to consider the random evolution of a multi-state system in which amounts of money are involved in this evolution. Despite these considerations, these kinds of processes, as far as the authors know, have not yet been fully dealt with in the financial and actuarial literature.

The main aim of this paper is showing how natural is modelling SCF by means of SMRWP. For these reasons, the paper is directed not only to semi-Markov experts but, in general, to financial practitioners and in particular to actuaries. Furthermore in the paper, new evolution equations are presented that give more application possibilities to this powerful tool.

A good description of HSMRWP appeared in Howard (1971), although not all the relevant aspects that could have been dealt with were outlined.

A complete description of the homogeneous and non-homogeneous SMRWP and a general presentation of SMP in homogeneous and non-homogeneous cases can be found in Janssen and Manca (2006, 2007).

In the semi-Markov environment, a backward time represents the time spent in a state before the valuation point of the system. In the first book on semi-Markov (Silvestrov, 1980) the SMP evolution equations were presented only taking into account backward times. In this case, the transition probabilities are also conditioned by the time of entrance into a given state. Instead in a semi-Markov process, when backward times are not considered then we are in the hypothesis that the entrance time was just at the valuation point. Backwards times give the possibility to take into account in a complete way the duration inside a state. In a Markov environment, backward time cannot be considered and also in Wilkie’s models this aspect was not considered.

A detailed description of homogeneous backward semi-Markov processes is reported in Limnios and Oprişan (2001). The discrete time non-homogeneous semi-Markov reward processes with backward recurrence time were described in Stenberg et al. (2007).

In the second section of the paper, the reward notations will be introduced. In Sections 3 and 4 the Discrete Time Homogeneous and Non-Homogeneous semi-Markov processes (DTHSMRP, DTNHSMRP) will be introduced before in the simplest way and after with the initial backward recurrence time. After, the Discrete Time Homogeneous and Non-Homogeneous Semi-Markov ReWard Process (DTHSMRWP, DTNHSMRWP) relations will be given. In addition, the related backward semi-Markov Reward Process will be presented. Section 5 will give matrix notations for the DTHSMRWP, DTNHSM-RWP and the second moments of these processes as defined in Stenberg et al. (2006, 2007). Section 6 will present how it is possible to follow and to evaluate, in a natural way, the evolution of SCFs by means of the presented stochastic processes. Furthermore, in this section generalizations of the models given in Stenberg et al. (2006, 2007) will also be presented. Section 7 presents a real data application of these processes that describes the construction of an insurance disability model. The last section will highlight the main results presented in the paper and the addresses of future works.

2. Rewards notation

The association of a sum of money to a state of the system and to a state transition assumes great relevance in the study of financial phenomena. This can be done by attaching a reward structure to a stochastic process. This structure can be thought as a random variable associated with state occupancies and transitions (Howard, 1971).

In the homogeneous case, the time evolution of the system is function of the duration that the system is in a state after a transition.

In the non-homogeneous environment the time evolution of the system is a function of two times: the arriving time in a state and the time of the subsequent transition. For these reasons, the rewards can be a function of the duration time in the homogeneous case, whereas, in the non-homogeneous envirnment they can also be function of both the starting time and the ending time. In the two different environments, the same differences can also hold for the financial variables.

There are two kinds of rewards; one that is paid or received because of remaining in a state (permanence rewards), the other that is paid or received because of a transition (transition rewards). In the literature, *permanence reward* and *transition reward* are also called respectively *rate reward* and *impulse reward* (see Qureshi and Sanders, 1994).

In the non-homogeneous process, it is possible to take into account the fact that the interest rates can change as a function of the starting time of the financial operation.

For this reason, the variable interest rate could change as a function of the duration of the financial operation (homogeneous variable interest rate) and/or because of the start and the end times of the financial operation (non-homogeneous variable interest rate). It may also be possible to consider a stochastic interest rate (see Janssen Manca, 2002).

In NHSMRWP, non-homogeneity makes it possible to take into account the rewards that change because of both the times s and t.

$$\psi_{i,j},\ \psi_{i,j}(t),\ \psi_{i,j}(s,t)$$

denote the reward that is given for the transition from the i^{th} state:

- the first the cases in which the payment flow in the state i is constant in time, changing only in function of the state,
- the second when the payment is a function of the state and of the time,
- the third when the rate rewards change because of starting and arriving times, in this case we say that there is a non-homogenous payment (the cash flow is function of the state, the time of entrance into the state and the time of payment).

For each state, usually, there is a different reward and $\boldsymbol{\psi}, \boldsymbol{\psi}(t), \boldsymbol{\psi}(s,t)$ represents the vector of these rewards respectively in case of constant rewards, rewards that change because of running time and non-homogeneous rewards. It should be mentioned that it may also be possible to consider permanence rewards that change because of the next transition (see Howard, 1971, Janssen and Manca, 2006, Papadopoulou and Tsaklidis, 2007), but in this paper we will not present the related evolution equations because, in a financial environment, they would not make sense.

Let γ_{ij}, $\gamma_{ij}(t)$, $\gamma_{ij}(s,t)$ denote the reward that is given for the transition from the i^{th} state to to the j^{th} one (impulse reward); the difference between the three symbols is the same as in the previous cases. $\bar{\mathbf{A}}$ is the matrix of the transition rewards. The different kinds of ψ rewards represent a stochastic money discrete time flow that is paid or received because of staying in a state. On the other hand, γ represents lump sums that are paid at the instant of transition.

As far as the impulse reward γ concerned, in the case of discounting, it is only necessary to compute the present value of the lump sum paid at the moment of the related transition.

Reward structure can be considered a very general structure attached to the problem being studied. This random variable evolves together with the stochastic process to which it is linked. When the studied system, that evolves dynamically in a random way, is in a state then a reward of ψ type can be paid; once there is a transition, a reward of γ type could be paid.

This behaviour is particularly efficient in the construction of models which are useful for following the dynamic evolution of insurance problems. Indeed, permanence in a state involves the payment of a premium or the receipt of a claim. In addition, the transition from one state to another can often bring about some cost or benefit.

3. DTSMP with a finite state set

In this section, DTHSMP and DTNHSMP will be described following the SMP notation given in Janssen and Manca (2006) and (2007).

Given the complete filtered probability space $(\Omega, \Im, \Im_t, P)$ in which the following two sequences of random variables (r.v.s) are

1. $J_n : \Omega \to I = \{1, 2, \ldots, m\}$, $n \in \mathbb{N}$ representing the state at the n^{th} transition.
2. $T_n : \Omega \to \mathbb{N}$ representing the time of the n^{th} transition.

We suppose that (J_n, T_n) is a homogeneous (non-homogeneous) Markov renewal process of kernel $\mathbf{Q} = [Q_{ij}(t)]$ ($\mathbf{Q} = [Q_{ij}(s,t)]$), where:

$$\begin{aligned} Q_{ij}(t) &\equiv P\left[J_{n+1} = j, T_{n+1} - T_n \le t | \sigma\left(J_a, T_a\right), 0 \le a < n, J_n = i\right] \\ &= P\left[J_{n+1} = j, T_{n+1} - T_n \le t | J_n = i\right]. \end{aligned}$$

$$\left(\begin{aligned} Q_{ij}(s,t) &\equiv P\left[J_{n+1} = j, T_{n+1} \le t | \sigma\left(J_a, T_a\right), 0 \le a < n, J_n = i, T_n = s\right] \\ &= P\left[J_{n+1} = j, T_{n+1} \le t | J_n = i, T_n = s\right]. \end{aligned} \right)$$

Furthermore we define

$$X_n = T_n - T_{n-1}.$$

X_n represents the so called *inter-arrival time*, i.e. the time spent between two subsequent transitions.

We know also that:

$$p_{ij} = P\left[J_{n+1} = j | J_n = i\right] = \lim_{t \to \infty} Q_{ij}(t); i, j \in I, t \in \mathbb{N}$$

$$p_{ij}(s) = P\left[J_{n+1} = j | J_n = i, T_n = s\right] = \lim_{t \to \infty} Q_{ij}(s,t); i, j \in I, s, t \in \mathbb{N}$$

where $\mathbf{P} = [p_{ij}]$ and $\mathbf{P}(s) = [p_{ij}(s)]$ are the transition matrices of the embedded homogeneous and non-homogeneous Markov chain respectively. It is also necessary to introduce the probability that the process will leave state i in a time t:

$$H_i(t) \equiv P\left[T_{n+1} - T_n \le t | J_n = i\right],$$

$$H_i(s,t) \equiv P\left[T_{n+1} \le t | J_n = i, T_n = s\right].$$

Obviously, it results that:

$$H_i(t) = \sum_{j=1}^{m} Q_{ij}(t),$$

$$H_i(s,t) = \sum_{j=1}^{m} Q_{ij}(s,t),$$

where $H_i(t)$ and $H_i(s,t)$ are distribution functions (d.f.); then:

$$\lim_{t\to\infty} H_i(t) = 1\,, \quad \forall i,$$

$$\lim_{t\to\infty} H_i(s,t) = 1\,, \quad \forall i.$$

Now the d.f. WTDF for each state i can be defined, given that the state successively occupied is known:

$$F_{ij}(t) = P\left[T_{n+1} - T_n \leq t | J_n = i, J_{n+1} = j\right],$$
$$F_{ij}(s,t) = P\left[T_{n+1} \leq t | J_n = i, J_{n+1} = j, T_n = s\right].$$

The difference between Markov and semi-Markov processes is mainly in these d.f. Indeed, in the discrete time Markov case, these d.f. can only be geometric distribution whereas in the semi-Markov case they could be of any type.

The related probabilities can be obtained by means of the following relations:

$$F_{ij}(t) = \begin{cases} Q_{ij}(t)/p_{ij} & \text{if} \quad p_{ij} \neq 0 \\ U_1(t) & \text{if} \quad p_{ij} = 0 \end{cases}$$

$$F_{ij}(s,t) = \begin{cases} Q_{ij}(s,t)/p_{ij}(s) & \text{if} \quad p_{ij}(s) \neq 0 \\ U_1(s,t) & \text{if} \quad p_{ij}(s) = 0 \end{cases}$$

where:

$$U_1(t) = \begin{cases} 0 & \text{if} \quad 0 > t \\ 1 & \text{if} \quad 0 \leq t. \end{cases} \quad \text{and} \quad U_1(s,t) = \begin{cases} 0 & \text{if} \quad s > t \\ 1 & \text{if} \quad s \leq t. \end{cases}$$

In a discrete time environment, it is necessary to define the following probabilities:

$$b_{ij}(t) = \mathrm{P}\left[J_{n+1} = j,\, T_{n+1} - T_n = t | J_n = i\right],$$
$$b_{ij}(s,t) = \mathrm{P}\left[J_{n+1} = j,\, T_{n+1} = t | J_n = j,\, T_n = s\right].$$

resulting in:

$$b_{ij}(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ Q_{ij}(t) - Q_{ij}(t-1) & \text{if} \quad t > 0 \end{cases}$$

$$b_{ij}(s,t) = \begin{cases} 0 & \text{if} \quad s = t \\ Q_{ij}(s,t) - Q_{ij}(s,t-1) & \text{if} \quad t > s \end{cases}$$

Fixed:

$$N(t) = \sup\{n | T_n \leq t\}, \quad \forall t \in \mathbb{N}$$

the DTHSMP and DTNHSMP $Z = (Z_t, t \in \mathbb{N})$ can be defined. where $Z(t) = J_{N(t)}$ represents, for each time t, the state occupied by the process. In the non-homogeneous case, supposing that s is a transition time, the transition probabilities are defined in the following way:

$$\phi_{ij}(t) = P[Z_t = j | Z_0 = i]$$
$$\phi_{ij}(s,t) = P[Z_t = j | Z_s = i]$$

They are obtained solving the following evolution equations:

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_{\beta=1}^{m} \sum_{\vartheta=1}^{t} b_{i\beta}(\vartheta)\phi_{\beta j}(t - \vartheta), \tag{1}$$

$$\phi_{ij}(s,t) = \delta_{ij}(1 - H_i(s,t)) + \sum_{\beta=1}^{m} \sum_{\vartheta=s+1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t), \tag{2}$$

where δ_{ij} represents the Kronecker symbol.

Both (1) and (2) can be obtained by means of a simple probabilistic argument using the regenerative property of the Markov renewal process (see Janssen and Manca, 2006 and 2007).

With the aim of clarification the meaning of the parts of (2) is given:

$$\delta_{ij}(1 - H_i(s,t))$$

represents the probability of remaining in the state i without any transition from the time s up to time t, and it only makes sense if $i = j$; this is the reason for the Kronecker delta.

$$\sum_{\vartheta=s+1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t)$$

represents the probabilities of all the possible trajectories that can be followed going from state i at time s to state j at time t.

In Figure 1 a typical trajectory of a semi-Markov process is shown.

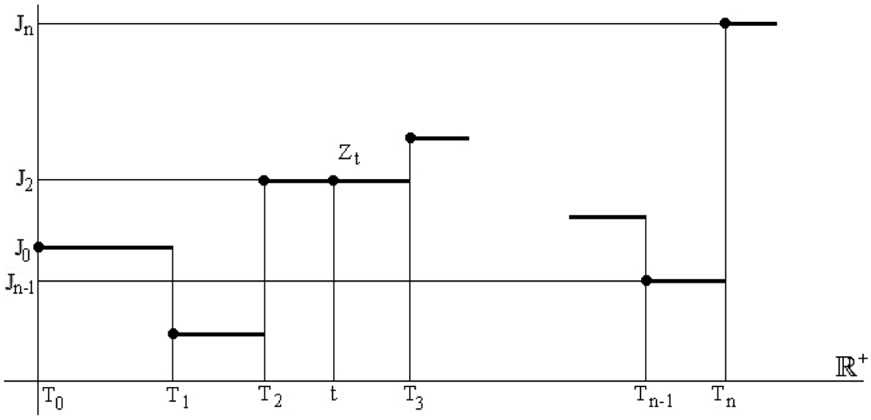

Figure 1: *Trajectory of a SMP.*

Now the backward recurrence process will be introduced. To explain the meaning of backward time, we present Figure 2 and Figure 3 in which homogeneous and non-homogeneous cases are shown.

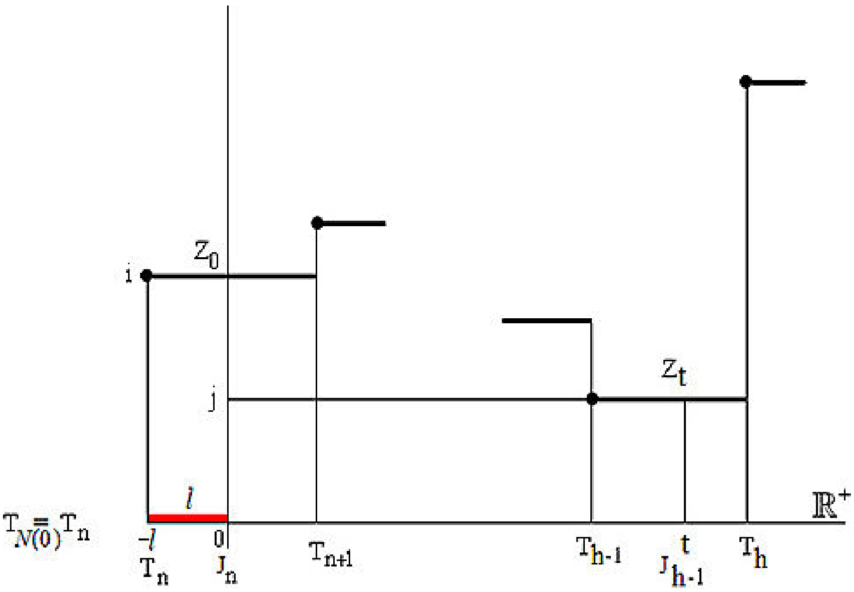

Figure 2: *Trajectory of a HSMP with recurrence backward time.*

We follow the system from time s up time t, but, this time, we consider in a non-homogeneous environment that the system entered the state i at time $s-u$ and that the system does not move from i for a period of time u that is the recurrence backward time.

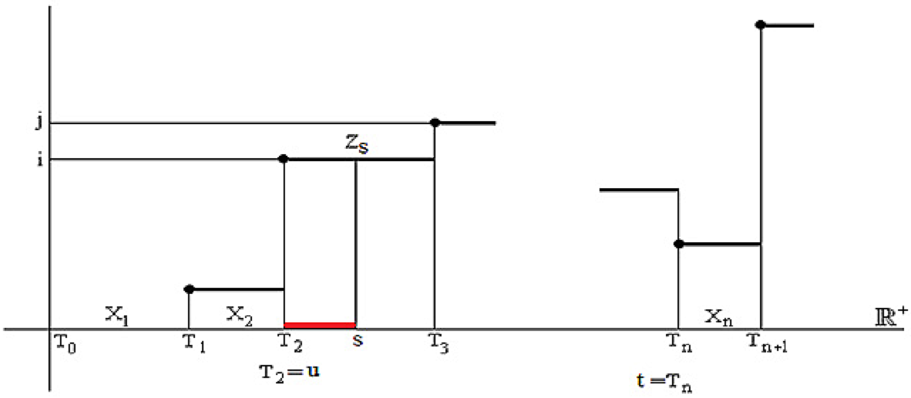

Figure 3: *Trajectory of a NHSMP with recurrence backward time.*

In order to take into account the backward time, it is necessary to condition the system to remain for a time u inside the state i.

Under the backward assumptions, the relations 1 and 2 become respectively:

$$^{b}\phi_{ij}(u;t) = \delta_{ij}\frac{(1-H_i(t+u))}{(1-H_i(u))} + \sum_{\beta=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{i\beta}(\vartheta+u)}{(1-H_i(u))}\phi_{\beta j}(0;t-\vartheta),$$

$$^{b}\phi_{ij}(u,s;t) = \delta_{ij}\frac{(1-H_i(u,t))}{(1-H_i(u,s))} + \sum_{\beta=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{i\beta}(u,\vartheta)}{(1-H_i(u,s))}\phi_{\beta j}(0,\vartheta;t),$$

where

$$^{b}\phi_{ij}(u,t) = \mathrm{P}\left[Z(t)=j|Z(0)=i, T_{N(0)}=-u\right]$$
$$^{b}\phi_{ij}(u,s;t) = \mathrm{P}\left[Z(t)=j|Z(s)=i, T_{N(s)}=u\right]$$

In the homogeneous case it is supposed that the system arrived a time u before 0 in the state i and it does not move from the arriving time up to 0. In the non-homogeneous case, it is supposed that the system arrived at time u in the state i and does not move from this state up to the time s.

4. The semi-Markov reward process with backward time

In this part, the DTHSMRWP and DTNHSMRWP will be introduced.

As in the previous section, before presenting the backward relations the SMRWP evolution equations will be given. This is to point out that the *SMRWP are a class of stochastic processes*, in the sense that, depending on the problem to be faced, a different evolution equation will be obtained. A classification of DTSMRWP was given in Janssen and Manca (2007).

In DT two general cases should be considered the DTSMRWP-immediate and the DTSMRWP-due. In the following we give two different evolution equations, one homogeneous and one non-homogeneous, the first for the due and the other for the immediate. The first case has the time-variable permanence and impulse rewards and variable rate of interest. The non-homogeneous case has non-homogeneous rate of interest and rewards.

For each case we present in (3) and (5) the reward process and in (4) and (6) the related semi-Markov reward evolution equations that are the mean of the process, as it is proved in Stenberg et al. (2006, 2007) and more recently in a more general case in D'Amico et al. (2013).

$$\begin{aligned}
\ddot{\xi}_i(t) &\equiv 1_{\left\{T_{N(0)+1>t}|J_{N(0)}=i\right\}}\left(\sum_{\tau=1}^{t}\psi_i(\tau)\,\nu(\tau-1)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}1_{\left\{J_{N(0)+1}=k,T_{N(0)+1}=\vartheta|J_{N(0)}=i\right\}}\left(\sum_{\tau=1}^{T_{N(0)+1}}\psi_i(\tau)\,\nu(\tau-1)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}1_{\left\{J_{N(0)+1}=k,T_{N(0)+1}=\vartheta|J_{N(0)}=i\right\}}\,\nu\left(T_{N(0)+1}\right)\gamma_{iJ_{N(0)+1}}\left(T_{N(0)+1}\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}1_{\left\{J_{N(0)+1}=k,T_{N(0)+1}=\vartheta|J_{N(0)}=i\right\}}\,\ddot{\xi}_{J_{N(0)+1}}\left(t-T_{N(0)+1}\right)
\end{aligned}\tag{3}$$

$$\begin{aligned}
\ddot{V}_i(t) &= (1-H_i(t))\sum_{\theta=1}^{t}\psi_i(\theta)\,\nu(\theta-1)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{\theta=1}^{\vartheta}\psi_i(\theta)\,\nu(\theta-1)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\,\nu(\vartheta)\gamma_{ik}(\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\,\nu(\vartheta)\ddot{V}_k(t-\vartheta),
\end{aligned}\tag{4}$$

$$\begin{aligned}
\xi_i(s,t) &\equiv 1_{\left\{T_{N(s)+1>t}|J_{N(s)}=i,T_{N(s)}=s\right\}}\left(\sum_{\tau=s+1}^{t}\psi_i(s,\tau)\,\nu(s,\tau)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}1_{\left\{J_{N(s)+1}=k,T_{N(s)+1}=\vartheta|J_{N(s)}=i,T_{N(s)}=s\right\}}\left(\sum_{\tau=s+1}^{T_{N(s)+1}}\psi_i(s,\tau)\,\nu(s,\tau)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}1_{\left\{J_{N(s)+1}=k,T_{N(s)+1}=\vartheta|J_{N(s)}=i,T_{N(s)}=s\right\}}\,\nu\left(s,T_{N(s)+1}\right)\gamma_{iJ_{N(s)+1}}\left(s,T_{N(s)+1}\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}1_{\left\{J_{N(s)+1}=k,T_{N(s)+1}=\vartheta|J_{N(s)}=i,T_{N(s)}=s\right\}}\,\nu\left(s,T_{N(s)+1}\right)\xi_{J_{N(s)+1}}\left(T_{N(s)+1};t\right)
\end{aligned}\tag{5}$$

$$V_i(s,t) = (1-H_i(s,t)) \sum_{\theta=s+1}^{t} \psi_i(s,\theta)\,\nu(s,\theta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\theta=s+1}^{\vartheta} \psi_i(s,\theta)\,\nu(s,\theta)$$
$$+ \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\,\nu(s,\vartheta)\gamma_{ik}(s,\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\,\nu(s,\vartheta)V_k(\vartheta,t) \qquad (6)$$

where:

$$\nu(t) = \begin{cases} 1 & \text{if} \quad t=0 \\ \prod_{\tau=1}^{t} (1+r(\tau))^{-1} & \text{if} \quad t>0 \end{cases};$$

$$\nu(s,t) = \begin{cases} 1 & \text{if} \quad t=s \\ \prod_{\tau=s+1}^{t} (1+r(s,\tau))^{-1} & \text{if} \quad t>s \end{cases}$$

$$1_{\{T_{N(0)+1>t}|J_{N(0)}=i\}} := \begin{cases} 1 & if\, T_{N(0)+1}(\omega) > t \quad \omega \in \Omega(i,0) \\ 0 & \text{otherwise,} \end{cases}$$
$$\Omega(i,0) = \left\{\omega \in \Omega : J_{N(s)}(\omega) = i, T_{N(s)}(\omega) = 0\right\}.$$

For a deeper understanding, the interested reader can refer to D'Amico et al. (2013)

- both permanence (or rate) and transition (or impulse) rewards are considered,
- the rewards in (3) and (4) are homogeneous in time and (5) and (6) non-homogeneous,
- the interest rates are in (3) fixed, in (4) and (6) variable in time and in (5) non-homogeneous.

Remark 4.1. *The introduction of stochastic interest rate does not present any difficulties (see Janssen et al., 2002).*

The interested reader can find other cases in the Janssen and Manca (2006, 2007) books.

Now we present a homogenous and a non-homogenous case with backward times. The homogeneous is in an immediate environment, the non-homogeneous in a due. As before we present before the reward processes and after the related evolution equations. The first time index gives the backward time. The hypotheses on interest rates and on rewards are the same of the relations given in (3) and (5).

In the homogeneous case the backward time is negative because it is supposed to begin following the system, after each transition, at time 0. The non-homogeneous backward relations, in which the first index time gives the backward, the second the starting horizon time and the third the ending time are the following:

$$
\begin{aligned}
&{}^{b}\ddot{\xi}_i(u,s;t) \equiv 1_{\left\{T_{N(s)+1>t}|J_{N(s)}=i,T_{N(s)}=u,T_{N(s)+1>s}\right\}}\left(\sum_{\tau=s+1}^{t}\psi_i(s,\tau)\,\nu(s,\tau-1)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}1_{\left\{J_{N(s)+1}=k,T_{N(s)+1}=\vartheta|J_{N(s)}=i,T_{N(s)}=u,T_{N(s)+1>s}\right\}}\left(\sum_{\tau=s+1}^{T_{N(s)+1}}\psi_i(s,\tau)\,\nu(s,\tau-1)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}1_{\left\{J_{N(s)+1}=k,T_{N(s)+1}=\vartheta|J_{N(s)}=i,T_{N(s)}=u,T_{N(s)+1>s}\right\}}\,\nu\left(s,T_{N(s)+1}\right)\gamma_{iJ_{N(s)+1}}\left(s,T_{N(s)+1}\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}1_{\left\{J_{N(s)+1}=k,T_{N(s)+1}=\vartheta|J_{N(s)}=i,T_{N(s)}=u,T_{N(s)+1>s}\right\}}\,\nu\left(s,T_{N(s)+1}\right)\ddot{\xi}_{J_{N(s)+1}}\left(T_{N(s)+1},T_{N(s)+1};t\right)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\ddot{V}_i(u,s;t) &= \frac{(1-H_i(u,t))}{(1-H_i(u,s))}\sum_{\theta=s+1}^{t}\psi_i(s,\theta)\nu(s,\theta)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\sum_{\theta=s+1}^{\vartheta}\psi_i(s,\theta)\nu(s,\theta)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\gamma_{ik}(s,\vartheta)\nu(s,\vartheta)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\ddot{V}_k(0,\vartheta;t)\nu(s,\vartheta).
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&{}^{b}\xi_i(-u;t) \equiv 1_{\left\{T_{N(0)+1>t}|J_{N(0)}=i,T_{N(0)}=-u,T_{N(0)+1>0}\right\}}\left(\sum_{\tau=1}^{t}\psi_i(\tau)\,\nu(\tau)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}1_{\left\{J_{N(0)+1}=k,T_{N(0)+1}=\vartheta|J_{N(0)}=i,T_{N(0)}=-u,T_{N(0)+1>0}\right\}}\left(\sum_{\tau=s+1}^{T_{N(s)+1}}\psi_i(\tau)\,\nu(\tau)\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}1_{\left\{J_{N(0)+1}=k,T_{N(0)+1}=\vartheta|J_{N(0)}=i,T_{N(0)}=-u,T_{N(0)+1>0}\right\}}\,\nu\left(T_{N(0)+1}\right)\gamma_{iJ_{N(s)+1}}\left(T_{N(0)+1}\right)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}1_{\left\{J_{N(0)+1}=k,T_{N(0)+1}=\vartheta|J_{N(0)}=i,T_{N(0)}=-u,,T_{N(0)+1>0}\right\}}\,\nu\left(T_{N(0)+1}\right)\xi_{J_{N(0)+1}}\left(0;t-T_{N(0)+1}\right)
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
V_i(u;t) &= \frac{(1-H_i(t+u))}{(1-H_i(u))}\sum_{\theta=1}^{t}\psi_i(\theta)\,\nu(\theta)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\sum_{\theta=1}^{\vartheta}\psi_i(\theta)\,\nu(\theta)\\
&+\sum_{m}k=1^m\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\nu(\vartheta)\gamma_{ik}(\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\nu(\theta)V_k(0;t-\vartheta).
\end{aligned}
\tag{10}
$$

Now we explain the meaning of (8), the meaning of the other relations is similar and can be easily understood. The first part of (8)

$$\frac{(1-H_i(u,t))}{(1-H_i(u,s))}\sum_{\theta=s+1}^{t}\psi_i(s,\theta)v(s,\theta) \tag{11}$$

can be seen as a discrete time annuity with non-homogeneous variable instalments $\psi_i(s,\theta)$ discounted by means of a non-homogeneous interest rate that is calculated from time s up to time $t-1$. (11) is conditioned in function of the arrival time in the state i at time u.

The second part

$$\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\sum_{\theta=s+1}^{\vartheta}\psi_i(s,\theta)v(s,\theta)$$

represents the rewards that were paid up to the moment of the first transition, still with the same conditioning.

The third and fourth parts

$$\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\ddot{V}_k(0,\vartheta;t)v(s,\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\gamma_{ik}(s,\vartheta)v(s,\vartheta)$$

explain what happened at the transition moments. Indeed, $\gamma_{ik}(s,\vartheta)$ represents the transition reward that is paid at the transition instant and $\ddot{V}_k(0,\vartheta;t)$ the mean present value of all the payments made from time ϑ up to time t. Both sums are evaluated at time ϑ so it is necessary to discount them at time s, thus the presence of the discount factor. In this case, the conditioning to arrive at time u in the state i is also considered.

The algorithm to solve the discrete time homogeneous and non-homogeneous backward case are given in Stenberg et al. (2006, 2007) respectively.

Remark 4.2. *The choice between the homogeneous and non-homogeneous cases depends on the available data. Non-homogeneity is closer to real life problems in the case of time that depends on age or seniority is certainly better to use it (insurance problems) but, in order to apply it, a great quantity of data is necessary. If the database is not huge it can be better to remain in a homogeneous environment.*

Remark 4.3. *The backward times are fundamental in the evaluation of many insurance contracts (in our opinion any insurance contract is a SCF). For example, if we have to consider the disability in an insurance contract the dead probability of a disabled person is different from the dead probability of a healthy person. But this difference decreases as a function of the time distance from the beginning of the disability time (see D'Amico et al., 2009b).*

5. The risk estimation

The evolution equation of the reward processes gives the mean present value of the stochastic financial operation. But in a stochastic environment it remains fundamental also the evaluation of the risk, i.e. the estimation of the variability. In Stenberg et al. (2006, 2007) the formulas for the higher moments of the reward processes were presented. We will report only the second order moment formula, the calculation of variance and of standard deviation.

The relation (6), in matrix form can be rewritten in the following way:

$$\mathbf{V}(u;t) = \mathbf{D}(u;t)\cdot\mathbf{A}(t)*\mathbf{1}_m + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(u;\vartheta)\cdot\tilde{\mathbf{A}}(\vartheta)\right)*\mathbf{1}_m + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(u;\vartheta)*\mathbf{V}(0;t-\vartheta)\right)\nu(\vartheta),$$

where:
$*$ represents the row-column matrix product,
$\cdot$ represents the element by element (Hadamard) matrix product,
$\mathbf{D}(u;t)$ is a diagonal matrix whose elements in the main diagonal are $\dfrac{(1-H_i(u,t))}{(1-H_i(u,s))}$
$\mathbf{A}(t)$ is a diagonal matrix whose elements in the main diagonal are $\sum_{\theta=1}^{t}\psi_i(\theta)\,\nu(\theta)$

$$\mathbf{B}(u;\vartheta) = \left[\left(\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\right)_{ik}\right] \tag{12}$$

$$\tilde{\mathbf{A}}(\vartheta) = \left[\left(\sum_{\theta=1}^{\vartheta}\psi_i(\theta)\,\nu(\theta) + \nu(\vartheta)\gamma_{ik}(\vartheta)\right)_{ik}\right] \tag{13}$$

1 is the sum vector.

Remark 5.1. *The relations (12) and (13) denote the general element of the respective matrices.*

(8) can be rewritten in the following matrix form:

$$\ddot{\mathbf{V}}(u,s;t) = \mathbf{D}(u,s;t)\cdot\ddot{\mathbf{A}}(s,t)*\mathbf{1}_m + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(u,s;\vartheta)\cdot\ddot{\tilde{\mathbf{A}}}(s,\vartheta)\right)*\mathbf{1}_m +$$
$$+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(u,s;\vartheta)*\ddot{\mathbf{V}}(0,\vartheta;t)\right)\nu(s,\vartheta),$$

where:
$\mathbf{D}(u,s;t)$ is a diagonal matrix whose elements in the main diagonal are $\dfrac{(1-H_i(u,t))}{(1-H_i(u,s))}$

$$\mathbf{B}(u,s;\vartheta)=\left[\left(\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\right)_{ik}\right]$$

$\ddot{\mathbf{A}}(s,t)$ is a diagonal matrix whose elements in the main diagonal are $\sum_{\theta=s}^{t-1}\psi_i(s,\theta)\,\nu(s,\theta)$

$$\ddot{\tilde{\mathbf{A}}}(s,\vartheta)=\left[\left(\sum_{\theta=s+1}^{\vartheta}\psi_i(s,\theta)\prod_{\tau=s}^{\theta-1}(1+r(s,\tau))^{-1}+\gamma_{ik}(s,\vartheta)\prod_{\tau=s+1}^{\vartheta}(1+r(s,\tau))^{-1}\right)_{ik}\right].$$

Now we will present the matrix form of the evolution equations of the second order moments of our two examples

1. Homogeneous immediate case:

$$\begin{aligned}\mathbf{V}^{(2)}(u;t)&=\mathbf{D}(u;t)\cdot\mathbf{A}^{(2)}(t)*\mathbf{1}_m+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(u;\vartheta)\cdot\tilde{\mathbf{A}}^{(2)}(\vartheta)\right)*\mathbf{1}_m \qquad (14)\\&+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(u;\vartheta)\cdot\tilde{\mathbf{A}}(\vartheta)\right)2\,\nu(s,\vartheta)*\mathbf{V}(0;t-\vartheta)+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(u;\vartheta)*\mathbf{V}^{(2)}(0;t-\vartheta)\right)\nu(s,\vartheta)^2,\end{aligned}$$

2. Non-homogeneous due case:

$$\begin{aligned}\ddot{\mathbf{V}}^{(2)}(u,s;t)&=\mathbf{D}(u,s;t)\cdot\ddot{\mathbf{A}}^{(2)}(s,t)*\mathbf{1}_m+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(u,s;\vartheta)\cdot\ddot{\tilde{\mathbf{A}}}^{(2)}(s,\vartheta)\right)*\mathbf{1}_m\\&+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(u,s;\vartheta)\cdot\ddot{\tilde{\mathbf{A}}}(s,\vartheta)\right)*\ddot{\mathbf{V}}(0,\vartheta;t)\,2\prod_{\tau=s+1}^{\vartheta}(1+r(s,\tau))^{-1} \qquad (15)\\&+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(u,s;\vartheta)*\ddot{\mathbf{V}}^{(2)}(0,\vartheta;t)\right)\prod_{\tau=s+1}^{\vartheta}(1+r(s,\tau))^{-2},\end{aligned}$$

where $\mathbf{V}^{(2)}(u;t)$ and $\ddot{\mathbf{V}}^{(2)}(u,s;t)$ are the second order moment of stochastic processes (7) and (9) respectively.

The proof of relations similar to (14) and (15) can be found in Stenberg et al. (2006, 2007) respectively.

Remark 5.2. *The DTHSMRWP and DTNHSMRWP presented In Stenberg et al. (2006, 2007) did not consider the differences between immediate and due models. These differences are fundamental in the model construction for applications.*

Remark 5.3. *As already outlined the reward processes are a class of stochastic processes. The consideration of backward times increases the number of the different kinds of processes.*

Once that the first and the second moments are known it is possible to calculate in a very simple way the variance and the standard deviation (see Stenberg et al., 2006, 2007).

6. SCF and semi-Markov reward processes

A deterministic cash flow is a finite or countable vector

$$\mathbf{O} = ((S_1, T_1), (S_2, T_2), \ldots)$$

whose elements are couples (S_i, T_i) where S_i is a sum and T_i is the time in which S_i is paid (negative value) or cashed (positive value). A cash flow is fair at a given time t if its value is equal 0.

A SCF is always a vector of couples but this time the first element of the couples is always a random variables and the time in some cases but not always is random. The vector could also have a finite or infinite order.

Remark 6.1. *The time of* transition rewards *is always random. Permanence rewards can be paid or cashed at each time period or randomly. For this reason the time can sometimes be considered deterministic. It is also possible that in some state the permanence rewards should not be computed. In this last case we can consider for all the period of staying in the state without permanence rewards that are equal to 0.*

Our hypotheses are:

1. $J_n \in I, I \subset \mathbb{R}$, $|I| \in \mathbb{N}$ or $|I| = |\mathbb{N}|$ where $|A|$ represents the cardinality of A.
2. $T_n \in \mathbb{N}$

3.a $\mathrm{P}\left[J_{n+1} = j,\, T_{n+1} - T_n \le t - T_{n+1} < T_{n+2} - T_{n+1} | (J_n, T_n), (J_{n-1}, T_{n-1}), \ldots, (J_0, T_0)\right]$

$$= \mathrm{P}\left[J_{n+1} = j,\, T_{n+1} - T_n \le t - T_{n+1} < T_{n+2} - T_{n+1} | J_n\right]. \tag{16}$$

in the homogeneous environment and

3.b $\mathrm{P}\left[J_{n+1} = j,\, T_{n+1} \le t < T_{n+2} | (J_n, T_n), (J_{n-1}, T_{n-1}), \ldots, (J_0, T_0)\right]$

$$= \mathrm{P}\left[J_{n+1} = j,\, T_{n+1} \le t < T_{n+2} | (J_n,\, T_n)\right] \tag{17}$$

in non-homogeneous case.

4. The horizon time is given by:

$$[0, T], T \in \mathbb{N} \text{ if the horizon is limited,}$$
$$[0, +\infty) \text{ if the horizon is infinite}$$

Remark 6.2. *(16) and (17) describe what happens to the SCF at each time of transition. The state of the system at time t is j. If there are permanence rewards they will be paid between two subsequent transitions. More precisely we know that $T_n - T_{n-1} = X_n$ and X_n rate rewards, one for each period, will be paid or received in case of deterministic permanence rewards. These rewards will be paid or cashed at the beginning of the period in the due case and at the end in the immediate case. In case of existence of transition rewards at the transition time an impulse reward will be paid or cashed.*

Supposing that:

1. at time the system start in the state i,
2. at transition time the system arrive at state j,
3. there are both permanence and transition rewards,
4. the permanence and the transition rewards change because of running time.

This financial operation in the immediate case can be described in the following way:

$$\mathbf{O} = ((\psi_i(1), 1), (\psi_i(2), 2), \ldots, (\psi_i(X_1), X_1), (\gamma_{ik_1}, T_1), (\psi_{k_1}(T_1+1), T_1+1),$$
$$(\psi_{k_1}(T_1+2), T_1+2), \ldots, (\psi_{k_1}(T_1+X_2), T_1+X_2), (\gamma_{k_1k_2}, T_2), (\psi_{k_2}(T_2+1), T_2+1),$$
$$\ldots, (\psi_{k_2}(T_2+X_3), T_2+X_3), (\gamma_{k_2k_3}, T_3), \ldots, (\gamma_{k_{n-1}k_n}, T_n), (\psi_{k_n}(T_n+1), T_n+1), \ldots,$$
$$(\psi_{k_n}(T_n+X_n), T_n+X_n), (\gamma_{k_nj}, T_{n+1}), \ldots),$$

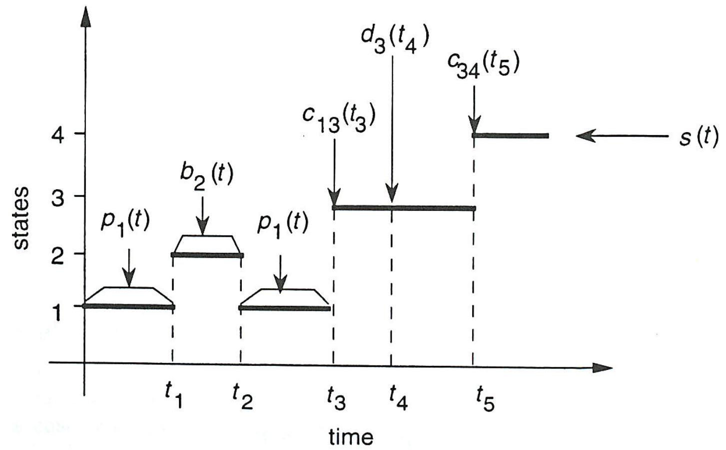

Figure 4: *Insurance contract as shown in Haberman and Pitacco (1999) [20].*

Under the 4 hypotheses it is evident that the stochastic evolution of this process can be naturally followed by means of a SMP.

The main SCF problem is the evaluation of the mean present value. Another important issue is the evaluation of the risk that, as is well-known, can be calculated knowing the second order moment and consequently the variance or the sigma square.

Remark 6.3. *Most of the insurance contracts can be modelled taking into account these four hypotheses. Indeed an insurance contract, as specified for example in Haberman and Pitacco (1999), is a stochastic cash flow. In some cases (see Janssen and Manca, 1997) it is necessary to introduce other time random variables generalizing the SMP but, de facto, also these generalized SMRWP model a more complex SCF. Figure 4 represents the trajectory of a general insurance contract as shown in Haberman and Pitacco (1999), and is the perfect reproduction of a trajectory of a SMRWP. Indeed,* $p_1(t)$ *and* $b_2(t)$ *represent respectively a premium and a benefit they are paid or received in function of the state in which the insured is. They represent permanence rewards in the SMRWP environment. The first is an entrance and the second is a cost for the insurance company* $c_{13}(t_3)$ *and* $c_{34}(t_5)$ *represent two transition rewards.* $d_3(t_4)$ *is a reward that is given in function of the staying of a time* $t_4 - t_3$ *in the state 3. Clearly, the time* $t_4 - t_3$ *is a backward time. This fact implies that we have to define also the SMRWP with backward permanence rewards.*

Remark 6.4. *In the discrete time SCF evaluation the instant of payments assumes great relevance. The distinction between immediate and due DTSMRWP is fundamental.*

6.1. The relations of SMRWP with backward time rewards

In this subsection the homogeneous and non-homogeneous SMRWP with rewards and interest rates depending on the backward time will be presented. In this paper the backward time given on the permanence rewards is pointed out. Indeed, the possibility of considering the rewards and interest rates that changes as a function of backward time is an important issue in insurance contracts. It is to outline, for example, that in a disability insurance framework it is well known that the grade of disability can decrease in function of the time spent from the disability event (see D'Amico et al., 2009b). In this case the consideration of rewards that depend on backward time can really be important in the calculation of the benefits and the premium of the insurance contract. Furthermore, in the management of a SCF with a term structure of interest rate it can happen that the interest rates of the SCF can change in function of the elapsed time between the stipulation of contract and the beginning of the financial operation. The consideration of interest rates that depends on backward time can take into consideration this other aspect of financial contracts.

We consider, in (15) and (16), at the same time both these aspects. It is clear that it is possible the writing of relations that point out only one of the two problems.

$$\begin{aligned} V_i(u;t) &= \frac{(1-H_i(t+u))}{(1-H_i(u))}\sum_{\theta=1}^{t}\psi_i(u;\theta)\,\nu(u;\theta) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\sum_{\theta=1}^{\vartheta}\psi_i(u;\theta)\,\nu(u;\theta) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\nu(u;\vartheta)\gamma_{ik}(u;\vartheta) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\nu(u;\vartheta)V_k(0;t-\vartheta). \end{aligned} \tag{15}$$

$$\begin{aligned} \ddot{V}_i(u,\,s;\,t) &= \frac{(1-H_i(u,t))}{(1-H_i(u,s))}\sum_{\theta=s+1}^{t}\psi_i(u,s;\theta)\,\nu(u,s;\theta) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\sum_{\theta=s+1}^{\vartheta}\psi_i(u,s;\theta)\,\nu(u,s;\theta) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\ddot{V}_k(0,\vartheta;t)\,\nu(u,s;\vartheta) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\gamma_{ik}(u,s;\vartheta)\,\nu(u,s;\vartheta). \end{aligned} \tag{16}$$

We would outline that it is the first time that are presented reward processes in which it is possible taking into consideration the reward and interest rates depending on backward times.

Remark 6.5. *It is clear that it is possible to write the matrix relations as in the previous case. A studious reader can try to write them.*

6.2. A mixed due and immediate SMRWP with backward recurrence times

There are insurance contracts in which the premium are paid at beginning of the period and the insured benefit at the end of the period. In these cases, it is necessary to define a SMRWP that can take into account this *mixed* situation.

(17) and (18) give the homogeneous and non-homogeneous SMRWP evolution equations in the case of rewards paid or cashed at the beginning or at the end of the period.

$$
\begin{aligned}
\bar{V}_i(u;t) &= \frac{(1-H_i(t+u))}{(1-H_i(u))}\sum_{\theta=1}^{t}\psi_i(u;\theta)\prod_{\tau=sw(i)}^{\theta+sw(i)-1}(1+r(u;\tau))^{-1} \\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\sum_{\theta=1}^{\vartheta}\psi_i(u;\theta)\prod_{\tau=sw(i)}^{\theta+sw(i)-1}(1+r(u;\tau))^{-1} \\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\prod_{\tau=1}^{\vartheta}(1+r(u;\tau))^{-1}\gamma_{ik}(\vartheta) \\
&+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\prod_{\tau=1}^{\vartheta}(1+r(u;\tau))^{-1}\bar{V}_k(0;t-\vartheta).
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\bar{V}_i(u,s;t) &= \frac{(1-H_i(u,t))}{(1-H_i(u,s))}\sum_{\theta=s}^{t}\psi_i(u,s;\theta)\prod_{\tau=s+sw(i)}^{\theta+sw(i)-1}(1+r(u,s;\tau))^{-1} \\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\sum_{\theta=s}^{\vartheta}\psi_i(u,s;\theta)\prod_{\tau=s+sw(i)}^{\theta+sw(i)-1}(1+r(u,s;\tau))^{-1} \\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}\gamma_{ik}(s,\vartheta)\prod_{\tau=s+1}^{\vartheta}(1+r(u,s;\tau))^{-1} \\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\frac{b_{ik}(u,\vartheta)}{(1-H_i(u,s))}V_k(0,\vartheta;t)\prod_{\tau=s+1}^{\vartheta}(1+r(u,s;\tau))^{-1}.
\end{aligned}
\tag{18}
$$

Remark 6.6. *The trick in the relations (17) and (18) consists in the usage of a vector of binary variable* ***sw*** *of order m that will assume at the ith element value 1 if the permanence reward is paid at the end of the period and 0 if it is paid at the beginning of the period.*

Remark 6.7. *The two relations (17) and (18) are the most general that can be given respectively in the homogeneous and non-homogeneous SMRWP for the calculation of the mean present value of a SCF.*

7. A quasi real data example

In this part we present results both for both homogeneous and non-homogeneous environments. The raw data are the same as were used in Stenberg et al. (2006, 2007) papers but the models are different. We speak of quasi real data because we truncated the Waiting Time Distribution Function (WTDF) at 10. Indeed interest rates and benefits that change because of backward recurrence times were used. Furthermore, in the model the benefits of the first state are paid at beginning of the period, instead of the benefits of the other states at the end of the period. It is the first time that a model of a SCF considers this possibility. Having data of only disabled people all the considered rewards are paid by the INAIL (Istituto Nazionale Assistenza Infortuni sul Lavoro), which is a public assistance institute and pays for the working accidents.

The historical data give the disability history of 840 people that had silicosis problems and lived in Campania, an Italian region. Each individual with silicosis will be examined by a doctor. The doctor will determine the degree of disability in the form of a percentage for each patient, ranging from 0% to 100%. Depending on the degree of disability the policy maker has determined five possible states, categorized in Table 1.

Table 1: *Disability states.*

1	[0 %,	10%)
2	[10 %,	30%)
3	[30 %,	50%)
4	[50 %,	70%)
5	[70 %,	100%)

This subdivision is the same as the ones used in Yntema (1962), Janssen (1966) that were the first to apply respectively Markov and semi-Markov environment to disability problems.

Transition between states occurs after a visit to the doctor that can be seen as the check to decide in which state the disabled person is in. This leads naturally to an example where virtual transitions are possible, i.e., the individual has become neither sufficiently better nor worse to change state. In the table we have introduced 5 different states, $E = 1, 2, 3, 4, 5$, one for each reward policy.

In Figure 5 the graph of the transitions is reported.

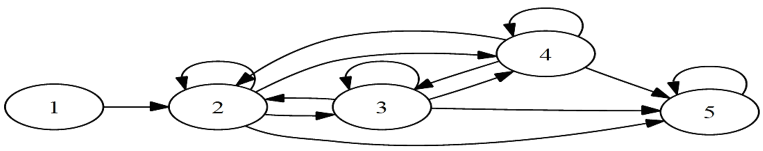

Figure 5: *Graph of Markov transition matrix.*

From the graph of Figure 5 it results:

- there is one absorbing state,
- the states 2, 3 and 4 are interconnected (they form a class of states),
- the state 5 is an absorbing state that forms the only absorbing class.

Under these conditions both homogeneous and non-homogeneous embedded Markov chains are mono-unireducible, see D'Amico et al. (2009a). These matrices were constructed by the real data.

Given the five states defined previously, we attached a reward policy to the disability degree. The reward that is given to construct the example represents the money amount that is paid for each time period to the disabled person with respect to the degree of illness. We did not have these data so we did not use real data values. The same was done for the evaluation of the interest rates. The data that we used were not sufficient for the evaluation of waiting time cumulated distribution functions. We construct these functions by means of random number extraction. Given that the applications have a horizon time of 10 years we truncate the waiting time distribution function in the way that the 10^{th} value is included between 0.9 and 1.

Transition rewards are not provided. Resuming, the hypothesis of the models that we apply are the following:

1. the evolution equations have only permanence rewards,
2. the rewards are all of the same sign (it is a disability insurance that is paid by social assistance),
3. the first state is paid at the beginning of the period the others at the end (we put this hypothesis to show the potential of our model).

Our hypothesis implies different evolution equations and we report them in (19) and (20) in homogeneous and non-homogeneous case respectively.

$$\begin{aligned}\bar{V}_i(u;t) = {} & \frac{(1-H_i(t+u))}{(1-H_i(u))}\sum_{\theta=1}^{t}\psi_i(u;\theta)\prod_{\tau=sw(i)}^{\theta+sw(i)-1}(1+r(u;\tau))^{-1} \\ & +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\sum_{\theta=1}^{\vartheta}\psi_i(u;\theta)\prod_{\tau=sw(i)}^{\theta+sw(i)-1}(1+r(u;\tau))^{-1} \\ & +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\prod_{\tau=1}^{\vartheta}(1+r(u;\tau))^{-1}\bar{V}_k(u;t-\vartheta).\end{aligned} \tag{19}$$

$$\begin{aligned}\bar{V}_i(u;t) = {} & \frac{(1-H_i(t+u))}{(1-H_i(u))}\sum_{\theta=1}^{t}\psi_i(u;\theta)\prod_{\tau=sw(i)}^{\theta+sw(i)-1}(1+r(u;\tau))^{-1} \\ & +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\sum_{\theta=1}^{\vartheta}\psi_i(u;\theta)\prod_{\tau=sw(i)}^{\theta+sw(i)-1}(1+r(u;\tau))^{-1} \\ & +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\frac{b_{ik}(\vartheta+u)}{(1-H_i(u))}\prod_{\tau=1}^{\vartheta}(1+r(u;\tau))^{-1}\bar{V}_k(u;t-\vartheta).\end{aligned} \tag{20}$$

7.1. The results

7.1.1. Homogeneous case

In this real data example we cover an horizon time of 10 years. The inputs of the program are the embedded Markov chain and the waiting time d.f.s

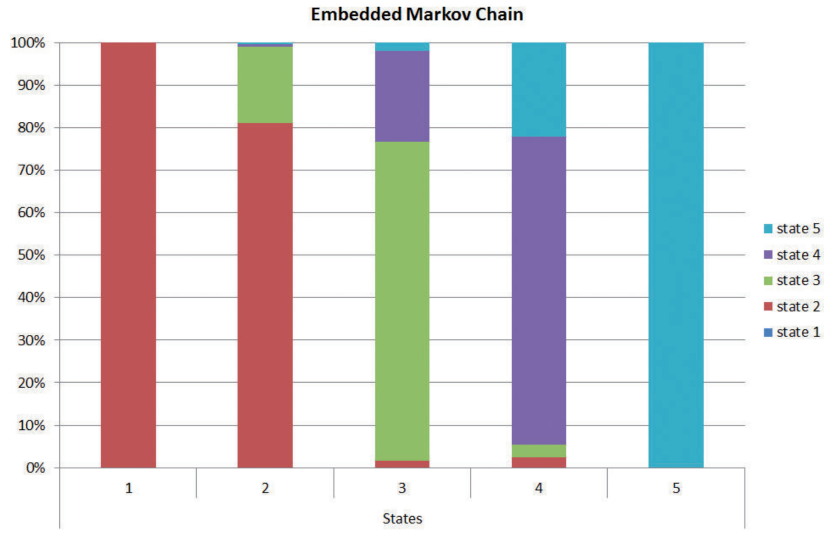

Figure 6: *The Markov chain embedded in the homogeneous process.*

In Figure 6 the matrix of the embedded Markov chain is reported. Each starting state is represented by a rectangle of the histogram. The transition probabilities are represented by different colors, for example the state 1 can go only in the state 2.

Figure 7 shows the waiting time d.f.'s. We would like to remember that a waiting time d.f. is given for each possible transition.

In Figure 8 the mean present values of rewards with no backward time, 3 years and 5 years of backward are given. We observe that with different backward and similar durations are obtained different results.

Finally Figure 9 shows the variance values. We would like to comment on the calculation of variance gives the possibility to have a measure of the risk. It is to observe that the variance has, by far, higher values of the reward values and this means a high risk in the financial operation.

7.1.2. Non-homogeneous case

In this subsection the results of the non-homogeneous case are reported.

In Figure 10 the non-homogeneous embedded Markov chains at time 1, 3 and 5 are shown. As it is possible to observe that at different times the embedded Markov chains have different transition probabilities.

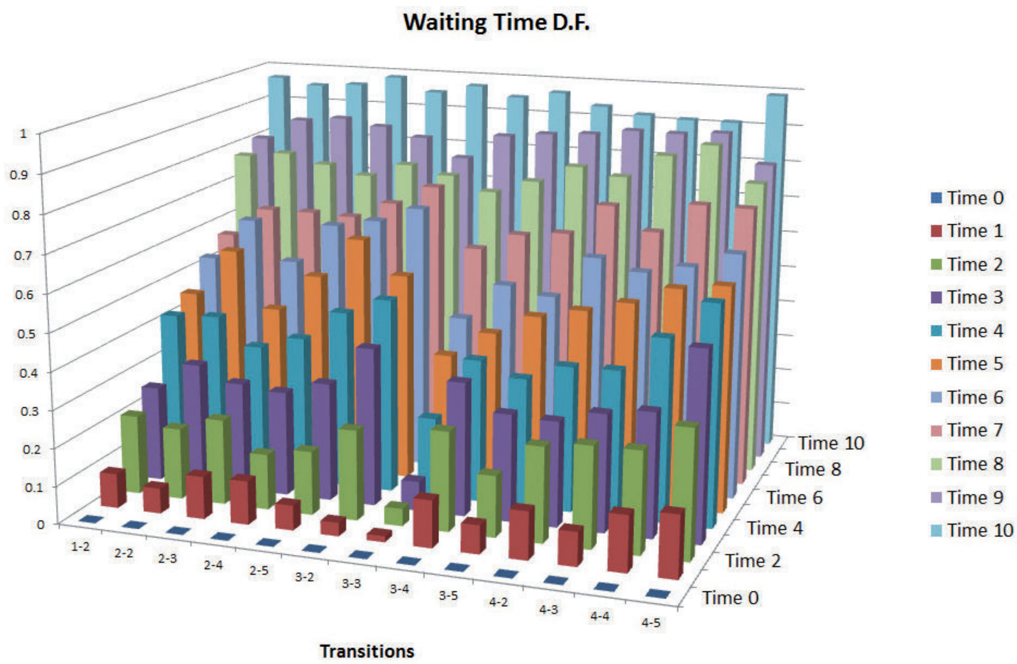

Figure 7: *The waiting time d.f.*

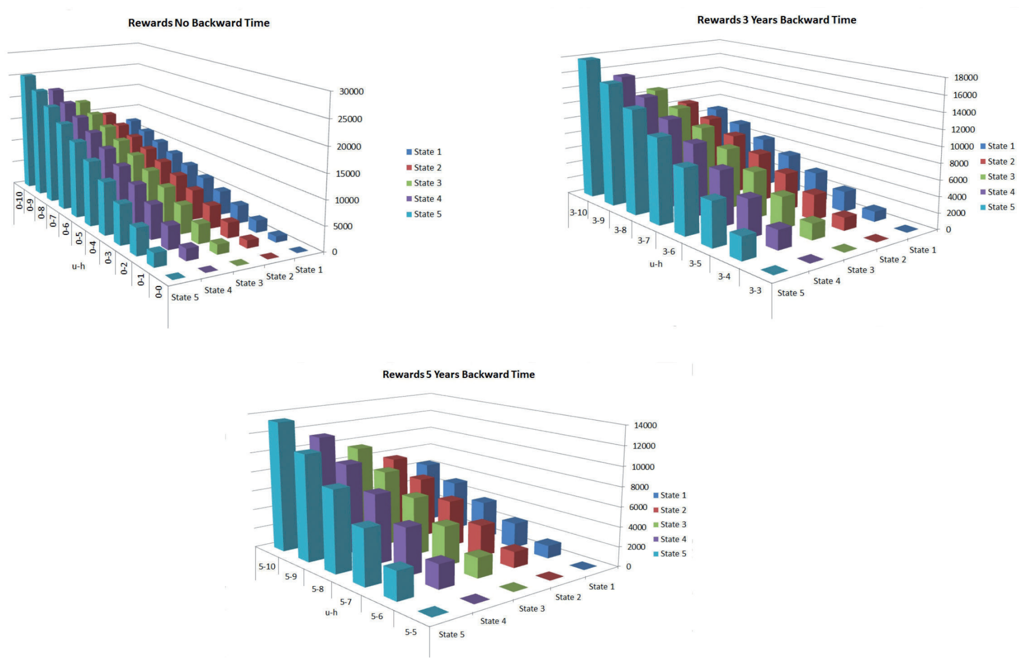

Figure 8: *Reward mean present value (backward time 0, 3, 5).*

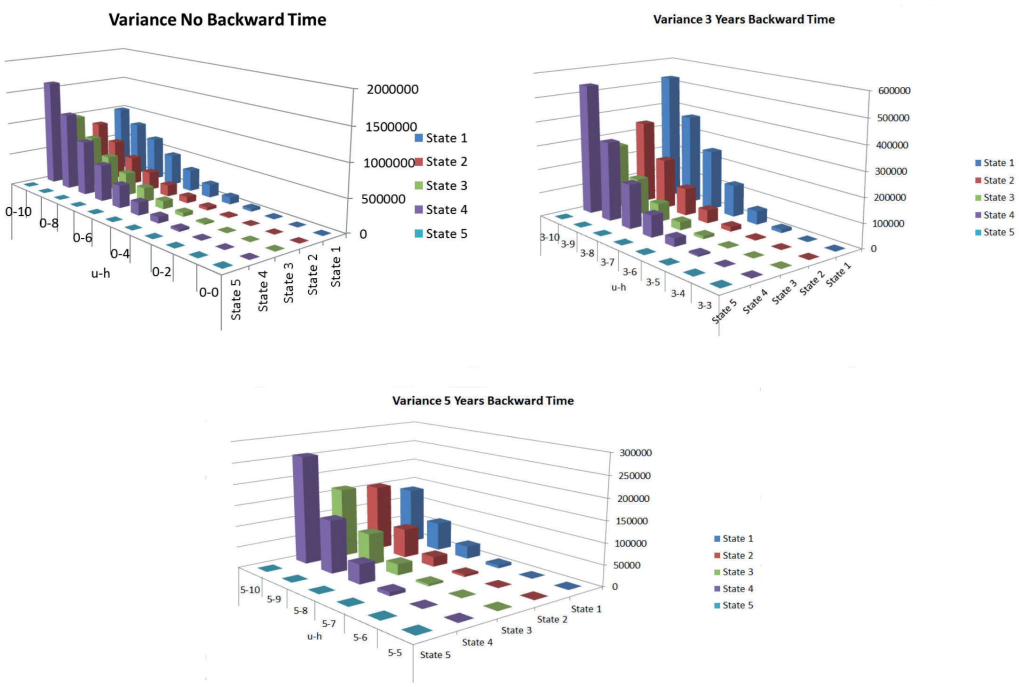

Figure 9: *Variance values (backward time 0, 3, 5).*

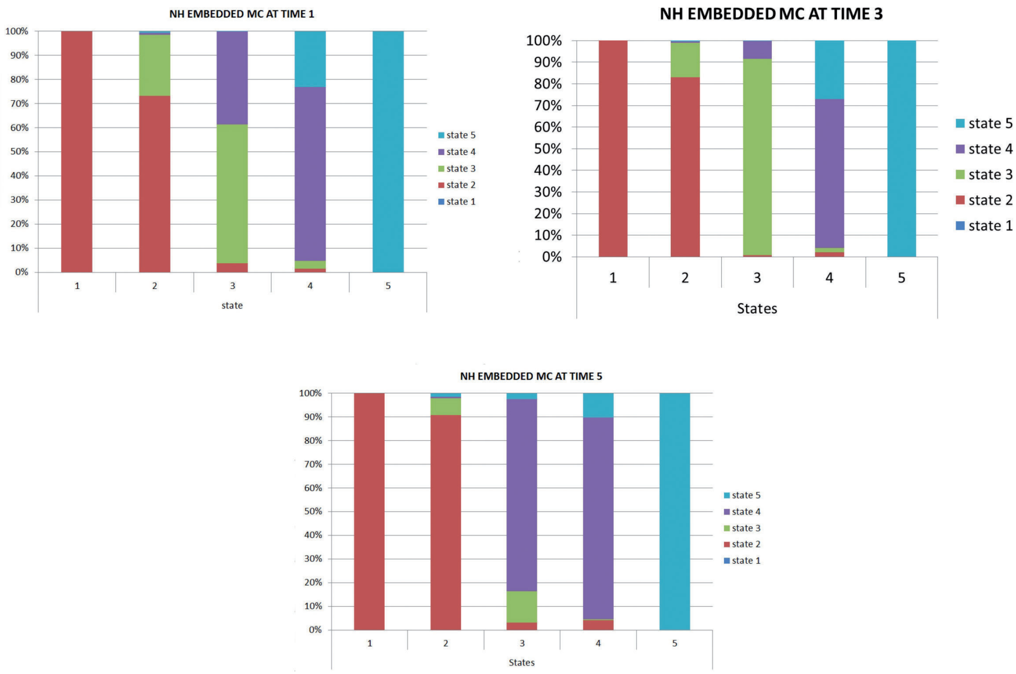

Figure 10: *Non-homogeneous embedded Markov chain at time 1, 3, 5.*

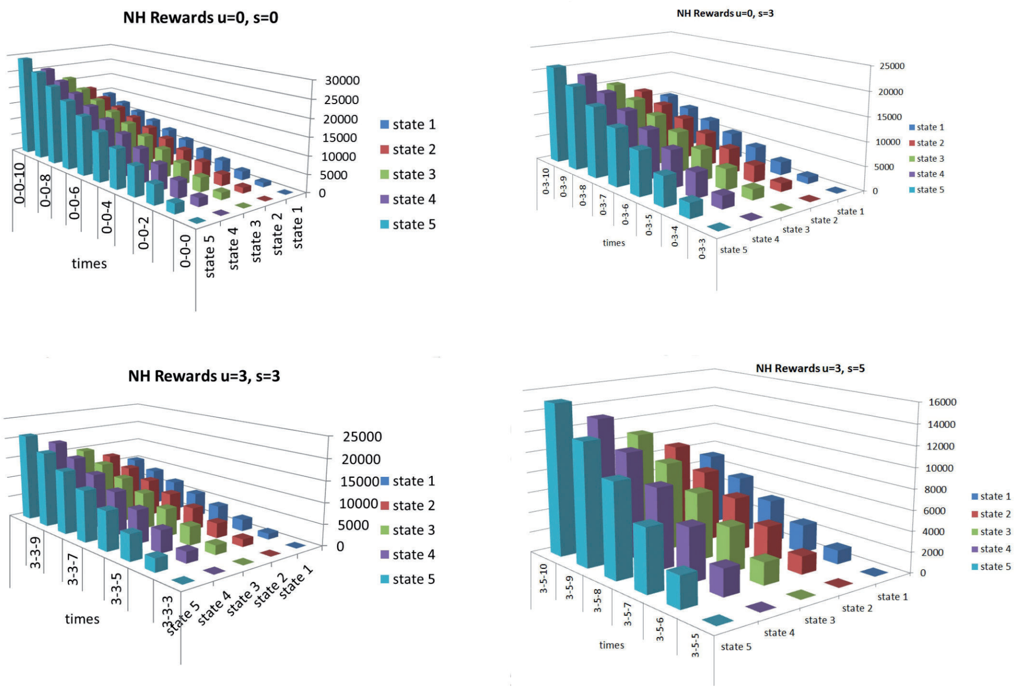

Figure 11: *Non-homogenous reward mean present values (back time 0 start time 0,3, 3; back time 3 start time 3,5).*

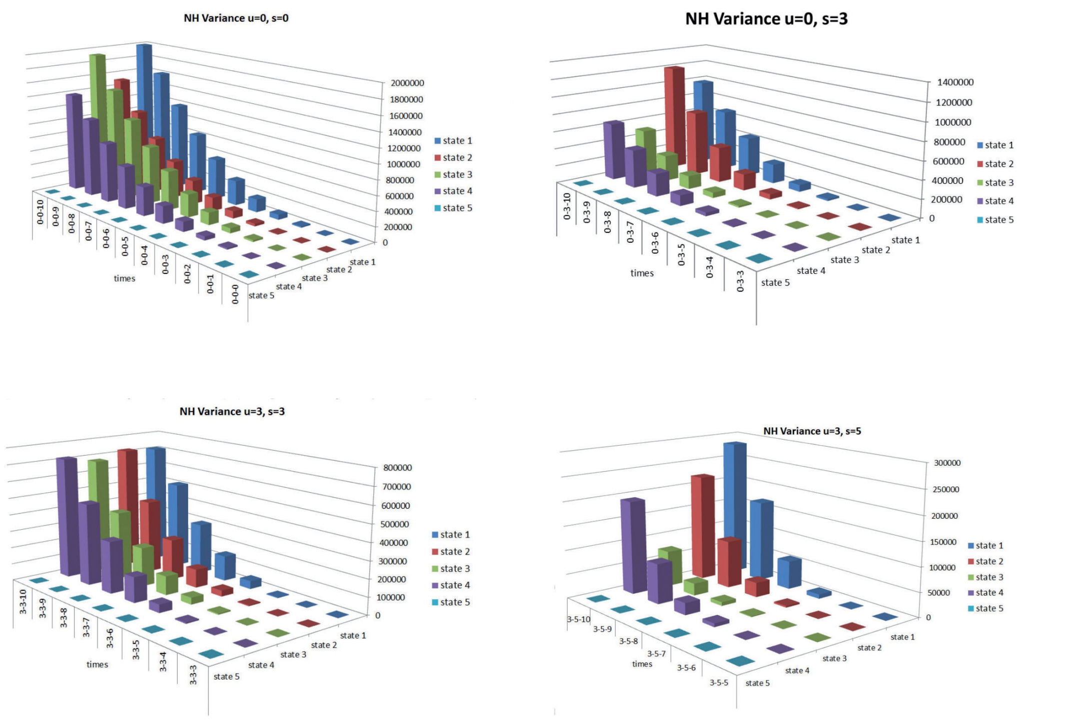

Figure 12: *Non-homogenous variance values (back time 0 start time 0,3, 3; back time 3 start time 3,5).*

In Figure 11 the non-homogeneous reward present values are given. The first two images report the values obtained without backward recurrence time with starting time equal to 0 and 3. The other two images report the results with backward recurrence time 3 and with starting time 3 and 5. It is interesting to observe that both non-homogeneity and backward time influence the result in substantial way.

Finally Figure 12 presents the variance values corresponding to the reward results that were given previously. We should point out that they are bigger by far than the mean values of our process. This means an over-dispersion of starting data and that our financial operation has a high risk.

8. Conclusions

The paper presented how to apply DTSMRWP in both the homogeneous and non-homogeneous case with backward recurrence time for calculation of the evolution and of the evaluation of SCFs. The model gives the possibility to take into account permanence rewards that are immediate, due and also a mixture of the two cases. Another important result presented is given by the calculation of the second order moment and gives the possibility to calculate the variance and consequently the sigma square, obtaining in this way an evaluation of the risk of the studied SCF.

It is the first time that the evaluation of a SCF by means of a DTSMRWP is presented and we retain that this new approach at this problem gives an important step in the study in general of SCF and more specifically of the most part of insurance contracts.

In the paper, for the first time the possibility is presented to construct a model in which some rewards are paid at the beginning of the period and others at the end of the period. This simple new approach gives the possibility to evaluate in the right way many insurance contracts, because frequently the premium of insurance contracts are paid at the beginning of the periods and the benefits at the end. Furthermore, for the first time SMRWP evolution equations with rewards and interest rates that depend on backward times have been presented.

The authors believe that recurrence times processes attached to a semi-Markov environment can be of great relevance in the study of finance and insurance problems. It is well known that SCF are a fundamental tool for the evaluation of financial and insurance contract and this paper proposes a model that solves the problem.

The presented relations are in discrete time and discrete state but in a future work we will give also the continuous version of these models and the inter-correlations between the discrete and the continuous cases.

The main aim of the paper was to show that a discrete time SCF can be studied naturally by means of a semi-Markov reward process. Furthermore, by means of the semi-Markov approach, it is possible to overcome all the difficulties regarding the study of SCFs that are highlighted in the literature.

Acknowledgments

The authors would like to thank the friends and colleagues Guglielmo D'Amico and Filippo Petroni for the discussions and suggestions that they gave during the drawing up of this paper. Furthermore, they are grateful to the two anonymous referees who with their comments and suggestions improved substantially the paper.

References

Artikis, T. P. and Malliaris, A. G. (1990). Discounting certain random sums. *Scandinavian Actuarial Journal*, 1-2, 34–38.

Artikis, T. and Voudouri, A. (2000). A stochastic integral arising in discounting continuous cash flows and certain transformed characteristic functions. *Applied Mathematical Letters*, 13, 87–90.

Balcer, Y. and Sahin, I. (1979). Stochastic models for a pensionable service. *Operations Research*, 27, 888–903.

Balcer, Y. and Sahin, I. (1986). *Pension accumulation as a semi-Markov reward process, with applications to pension reform. Semi-Markov models.* Plenum N.Y.

Beekman, J. A. and Fuelling, C. P. (1991). Extra randomness in certain annuity models. *Insurance Mathematics and Economics*, 10, 275–287.

Browne, S. (1995). Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. *Mathematics of Operations Research*, 20, 937–958.

Carravetta, M., De Dominicis, R. and Manca, R. (1981). Semi-Markov stochastic processes in social security problems. *Cahiers du C.E.R.O.*, 23, 333–339.

CMIR12 (1991). (Continuous Mortality Investigation Report, number 12). *The analysis of permanent health insurance data.* The Institute of Actuaries and the Faculty of Actuaries.

D'Amico, G., Guillen, M. and Manca, R. (2009). Full backward non-homogeneous semi-Markov processes for disability insurance models: A Catalunya real data application. *Insurance: Mathematics and Economics*, 45, 173–179.

D'Amico, G., Guillen, M. and Manca, R. (2013). Semi-Markov Disability Insurance Models. *Communications in Statistics-Theory and Methods*, 42, 2172–2188.

D'Amico, G., Janssen, J. and Manca, R. (2009a). The dynamic behavior of single unireducible non-homogeneous Markov and semi-Markov chains. In *Networks: Topology and Dynamic. Lectures Notes in Economic and Mathematical Systems*, (A. Naimzada, S. Stefani and A. Torriero, eds.). Springer, 195–211.

De Dominicis, R. and Manca, R. (1986). Some new results on the transient behaviour of semi-Markov reward processes. *Methods of Operations Research*, 53, 387–397.

De Medici, G., Janssen, J. and Manca, R. (1995). Financial operation evaluation: a semi-Markov approach. *Proc. V AFIR Symposium Brussels*, vol. 3 bis. Janssen J. Ed. CESIAF, Bruxelles.

De Schepper, A. and Goovaerts, M. (1992). Some further results on annuities certain with random interest rate. *Insurance Mathematics and Economics*, 11, 283–290.

De Schepper, A., Teunen, M. and Goovaerts, M. (1994). An analytical inversion of a Laplace transform related to annuities certain. *Insurance Mathematics and Economics*, 14, 33–37.

Donati-Martin, C., Matsumoto, H. and Yor, M. (2000). On positive and negative moments of the integral of geometric Brownian motions. *Statistics and Probability Letters*, 49, 45–52.

Duffie, D., Pan, J. and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68, 1343–1376.

Dufresne, D. (2001). The integral of geometric Brownian motion. *Advances in Applied Probability*, 33, 223–241.

Estes, J. H., Moor, W. C. and Rollier, D. A. (1989). Stochastic cash flow evaluation under conditions of uncertain timing. *Engineering Costs and Production Economics*, 18, 65–70.

Haberman, S. and Pitacco, E. (1999). *Actuarial Models for Disability Insurance*, Chapman & Hall.

Halliwell, L. J. (2003). The Valuation of Stochastic Cash Flows. *CAS Forum*, (Reinsurance Call Papers, Spring 2003), 1-68, available on line at: www.casact.org/pubs/forum/03spforum/03spf001.pdf.

Harrison, M. J. (1977). Ruin problems with compounding assets. *Stochastic Processes and their Applications*, 5, 67–79.

Hoem, J. M. (1972). Inhomogeneous semi-Markov processes, select actuarial tables, and duration-dependence in demography. In *Population, Dynamics*, ed. T.N.E. Greville Academic Press, 251–296.

Howard, R. (1971). *Dynamic probabilistic systems*, vol II, Wiley.

Janssen, J. (1966). Application des processus semi-markoviens à un probléme d'invalidité. *Bulletin de l'Association Royale des Actuaries Belges*, 63, 35–52.

Janssen, J. and Manca, R. (1997). A realistic non-homogeneous stochastic pension funds model on scenario basis. *Scandinavian Actuarial Journal*, 113–137.

Janssen, J. and Manca, R. (1998). Non-homogeneous stochastic pension fund model: an algorithmic approach. *Transactions of the 26th ICA Birmingham Vol. V*, 216–238.

Janssen, J. and Manca, R. (2002). General actuarial models in a semi-Markov environment. *Proceedings of* 27^{th} *ICA 2002, Cancun*, available on line at http://www.actuaries.org/ EVENTS/Congresses/ Cancun/astin subject/astin 78 janssen manca.pdf.

Janssen, J. and Manca, R. (2006). *Applied semi-Markov Processes*, Springer.

Janssen, J. and Manca, R. (2007). *Semi-Markov Risk Models for Finance, Insurance and Reliability*, Springer.

Janssen, J. and Manca, R. and Volpe di Prignano, E.(2002). Non homogeneous interest rate structure in a semi-Markov framework. *Proceedings of* 27^{th} *ICA 2002, Cancun*, available on line at http:// www.actuaries.org/ EVENTS/Congresses/Cancun/afir subject/afir 47 janssen manca volpe.pdf.

Janssen, J. and Manca, R. and Volpe di Prignano, E. (2004). Continuous Time Non Homogeneous Semi-Markov a. Reward Processes And Multi-State Insurance Application. *Proceedings IME 2004 Roma*.

Janssen, J. and Manca, R. and Volpe di Prignano, E. (2009). *Mathematical Finance, Deterministic and Stochastic models*, ISTE Wiley.

Limnios, N. and Oprişan, G. (2001). *Semi-Markov Processes and Reliability*, Birkhäuser, Boston.

Marocco, P. and Pitacco, E. (1998). Longevity risk and life annuity reinsurance. *Transactions of the 26th ICA Birmingham, vol VI*, 453–479.

Milevsky, M. (1997). The present value of a stochastic perpetuity and the gamma distribution. *Insurance Mathematics and Economics*, 20, 243–250.

Milevsky, M. and Posner, E. (1998). Asian options, the sum of lognormals and the reciprocal gamma distribution. *Journal of Financial and Quantitative Analysis*, 33, 409–422.

Papadopoulou, A. A. (2013). Some results on modeling biological sequences and web navigation with a semi-Markov chain. *Communications in statistics-Theory and Methods*, 42, 2153–2171.

Papadopoulou, A. A. and Tsaklidis, G. M. (2007). Some reward paths in semi-Markov models with stochastic selection of the transition probabilities. *Methodology and Computing in Applied Probability*, 9, 399–411.

Papadopoulou, A. A., Tsaklidis, G., McClean, S. and Garg, L. (2012). On the moments and the distribution of the cost of a semi-Markov model for healthcare systems. *Methodology and Computing in Applied Probability*, 14 717–737.

Parker, G. (1994). Limiting distribution of the present value of a portfolio. *Astin Bulletin*, 24, 47–60.

Perry, D. and Stadje, W. (2000). Risk analysis for a stochastic cash management model with two types of customers. *Insurance Mathematics and Economics*, 26, 25–36.

Perry, D. and Stadje, W. (2001). Function space integration for annuities. *Insurance Mathematics and Economics*, 29, 73–82.

Pitacco, E. and Olivieri, A. (1997). *Introduzione alla teoria attuariale delle assicurazioni di persone*, Quaderno UMI, 42.

Pliska, S. R. (1986). A stochastic calculus model of continuous trading. *Mathematics of Operations Research*, 11, 371–382.

Qureshi, M. A. and Sanders, H. W. (1994). Reward model solution methods with impulse and rate rewards: an algorithmic and numerical results. *Performance Evaluation*, 20, 413–436.

Papadopoulou, A. A., Tsaklidis, G., McClean, S. and Garg, L. (2012). On the moments and the distribution of the cost of a semi-Markov model for healthcare systems. *Methodology and Computing in Applied Probability*, 14, 717–737.

Sato, S. and Yor, M. (1998). Computation of moments for discounted Brownian additive functional. *Journal of Mathematics of Kioto University*, 38, 475–486.

Silvestrov, D. S. (1980). *Semi-Markov Processes with a Discrete State Space*, Sovetske Radio, Moskow (Russian).

Stenberg, F., Manca, R. and Silvestrov, D. (2006). Semi-Markov reward models for insurance. *Theory of Stochastic Processes*, 12, 239–254.

Stenberg, F., Manca, R. and Silvestrov, D. (2007). An algorithmic approach to discrete time non-homogeneous backward semi-Markov reward processes with an application to disability insurance. *Methodology and Computing in Applied Probability*, 9, 497–519.

Swishchuk, A. V. (1995). Hedging of options under mean-square criterion and with semi-Markov volatility. *Ukrainian Mathematic Journal*, 47, 1119–1127.

Vanneste, M., Goovaerts, M. J., De Schepper, A. and Dhaene, J. (1997). A straightforward analytical calculation of the distribution of an annuity certain with stochastic interest rate *Insurance Mathematics and Economics*, 15, 37–48.

Vanneste, M., Goovaerts, M. J. and Labie, E. (1994). The distribution of annuities. *Insurance Mathematics and Economics*, 15, 37–48.

Wilkie, A. D. (1986). A stochastic investment model for actuarial use. *Transactions of the Faculty of Actuaries*, 39, 341–403.

Wilkie, A. D. (1995). The risk premium on ordinary shares (with discussion). *British Actuarial Journal*, 1, 251–293.

Wolthuis, H. (2003). *Life Insurance Mathematics (the Markovian Model)*, II edition, Peeters Publishers, Herent. XI, 288.

Yntema, L. (1962). A markovian treatment of silicosis. *Acta III Conferencia Int. De Actuarios y Estadísticos de la Seguridad Social*, Madrid.

Yor, M. (2001). *Exponential Functionals of Brownian Motion and Related Processes*, Springer.