Pub. Mat. UAB

Vol. 31 nº 1 Abril-1987

TOPICS IN ANALYSIS

Daniel B. Henry

The following brief investigations in analysis were discussed in seminars (or less formally) during my visit to the Universitat Autónoma de Barcelona and the Centre de Recerca Matemàtica, Institut d'Estudis Catalans. Some (2,3,4) were initiated here; others were initiated in Sao Paulo; and still others were initiated here but are not yet complete. I am grateful to Professors Carles Perelló, Xavier Mora and Joan Solà-Morales for making my stay both pleasant and instructive.

1. Examples on propagation of singularities in the wave equation.

2. Non-decay of thermoelastic vibrations in dimension $\geqslant 3$.

3. On some non-linear integral inequalities of Kielhöfer and Caffarelli.

4. An example in the spectral theory of semigroups.

5. A property of the exponential function.

6. Asymptotic behavior of some scalar ODEs and an elementary example of non-minimal ω-limit sets.

1. Examples on propagation of singularities in the wave equation

1.0. Introduction

The following study arose from an attempt, in an introductory graduate course in PDEs, to describe (without proof) the general propagation of singularities theorem of Lax and Hörmander. The customary catch-phrase

"singularities propagate along bicharacteristics"

is, at best, misleading - not all bicharacteristics carry singularities. The more exact version

"micro-local singularities propagate along the corresponding bicharacteristics"

requires substantial explanation, but this effort is repaid by a more complete understanding as shown in the examples below.

We first describe our examples. Then we define characteristics and the wave-front set, state the propagation of singularities theorem, and use it to interpret the examples. Finally we give details of the calculation of the wave-front set for the example of Fritz John [2] (ex. 2). Taylor's article [5] describes also reflection and diffraction of singularities in boundary value problems. Mathematical details of the theorem (and generalizations) may be found in, for example, the books of Taylor [6] or Hörmander [4].

1.1. The examples

Example 1 Consider tempered distributions E, E_+, E_- on $\mathbb{R}_t \times \mathbb{R}^n_x$ defined by $\hat{E}(t,\xi) = |\xi|^{-1}\sin t|\xi|$ (Fourier transformation in x),

$$\hat{E}_+(t,\xi) = \begin{cases} \hat{E}(t,\xi), & t > 0 \\ 0, & t < 0 \end{cases}, \quad \hat{E}_-(t,\xi) = \begin{cases} 0, & t > 0 \\ -\hat{E}(t,\xi), & t < 0. \end{cases}$$

Then as distributions on $\mathbb{R} \times \mathbb{R}^n$,

$$(\partial^2/\partial t^2 - \Delta_x)E(t,x) = 0$$

$$(\partial^2/\partial t^2 - \Delta_x)E_\pm(t,x) = \delta_0(t,x)$$

and the singular support (outside which they are C^∞) is shown in Fig. 1

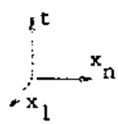
u	sing. supp. u  t, x_n, x_1	sing. supp. $(\partial_t^2 - \Delta)u$
$u = E$	 $\lvert x\rvert^2 = t^2$	(empty)
$u = E_+$	 0 $\lvert x\rvert^2 = t^2$, $t \geqslant 0$	$\cdot 0$
$u = E_-$	 0 $\lvert x\rvert^2 = t^2$, $t \leqslant 0$	$\cdot 0$

Fig. 1

Consider a point (t_0, x_0) of $|x|^2 = t^2$, $t_0 \neq 0$; only one of the bicharacteristic rays through this point carries singularities of E. No ray through (t_1, x_1), $t_1^2 \neq |x_1|^2$,

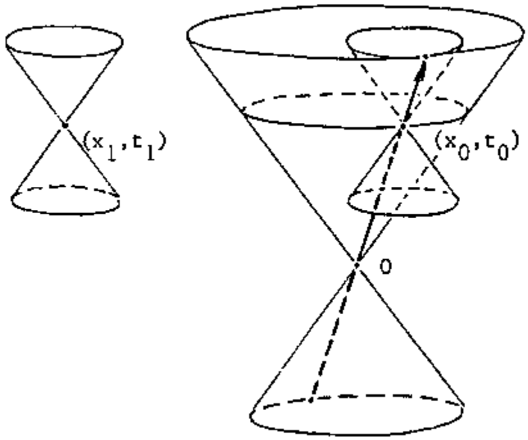


Fig. 2

carries singularities of E; though they all pass through the singular support of E. (See Fig. 2).

Example 2 (F. John [2], p. 572-574)

Imagine a vast and troubled sea; and in the midst of the sea, a circle; and inside the circle, everything is calm. There are no special forces acting. The circle is not a physical barrier. But the region of quiet continues, neither expanding nor contracting, while the storm rages without.

Sound impossible?

For any positive integer k, if J_k is the $k^{\underline{th}}$ order Bessel function; $J_k(kr)\, e^{ik(t+\theta)}$ is a smooth solution (using plane polar coordinates) of the wave equation $u_{tt} = u_{xx} + u_{yy}$ on $\mathbb{R} \times \mathbb{R}^2$ so

$$u = \sum_{k = 2^{\text{int.}} \geq 2} k^{-\lambda} J_k(kr)\, e^{ik(t+\theta)} \quad \text{[sum over powers of 2]}$$

is also a distribution solution when $\lambda > 0$. Taking $\lambda = m + 3/5$, m = integer $\geqslant 0$, u is a C^m function on $\mathbb{R}_t \times \mathbb{R}^2_{(x,y)}$, which is analytic inside the cylinder $\{x^2 + y^2 < 1, t \text{ arbitrary}\}$, C^{m+1} in the exterior $\{x^2 + y^2 > 1\}$, not C^{m+2} on any open set which meets $\{x^2 + y^2 > 1\}$ and not C^{m+1} on any open set which meets $\{x^2 + y^2 = 1\}$. If $m \geqslant 2$, u is a classical solution.

The singular support of u is $\{x^2 + y^2 \geqslant 1,\ t$ arbitrary$\}$ but the boundary $\{x^2 + y^2 = 1\}$ is nowhere characteristic. There are many bicharacteristic rays linking the regular $\{x^2 + y^2 < 1\}$ and singular $\{x^2 + y^2 \geqslant 1\}$ regions, but these do not carry singularities from one region to the other (See Fig 3).

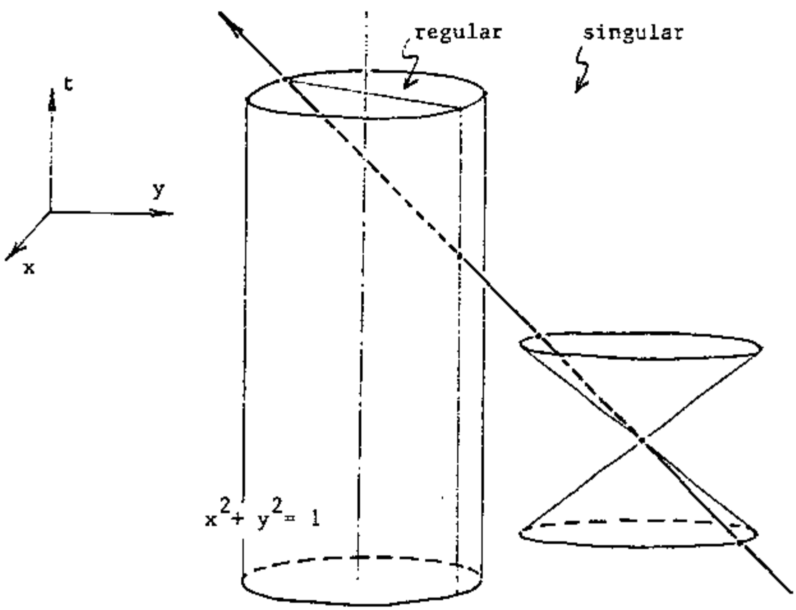


Fig. 3

Not all rays carry singularities. (We show later, for this example, that at most 4 bicharacteristic rays, through any given point outside the cylinder, can carry singularities — they are the rays tangent to the cylinder).

1.2. Characteristics and wave-front sets

Let $P(x,\xi) = \sum_{|\alpha| \leq m} p_\alpha(x)\xi^\alpha$ be a polynomial in $\xi \in \mathbb{R}^n$ with C^∞ coefficients, and suppose the principal part $P_m(x,\xi) = \sum_{|\alpha| = m} p_\alpha(x)\xi^\alpha$ is real-valued and the real characteristics are simple: $\xi \in \mathbb{R}^n$, $P_m(x,\xi) = 0$, $\frac{\partial}{\partial \xi_j} P_m(x,\xi) = 0$ for $j = 1,\dots,n$ imply $\xi = 0$.

As is well-known, if u is a C^r solution of $P(x,\frac{\partial}{\partial x})u=0$ $(r \geq m)$ which is C^{r+1} on either side of a smooth hypersurface S, but is not C^{r+1} everywhere since the $(r+1)$-order derivatives have a jump discontinuity across S, then S must be a characteristic surface: $P_m(x,N(x)) = 0$ when $x \in S$, $N(x)$ is normal to S at x, and u is not C^{r+1} on any neighborhood of x. Such a surface S may be represented as $\{x : \varphi(x) = 0\}$ where φ is a smooth function with $\nabla\varphi(x) \neq 0$ (in the region of interest) and $P_m(x,\nabla\varphi(x)) = 0$ where $\varphi(x) = 0$. We may solve (locally) the characteristic equation $P_m(x,\nabla\varphi(x)) \equiv 0$ by solving the Hamiltonian system

$$dx/d\lambda = \partial P_m/\partial\xi\ (x,\xi), \quad \partial\xi/\partial\lambda = -\partial P_m/\partial x(x,\xi)$$

for appropriate initial values with $P_m(x,\xi) \equiv 0$, and then $\varphi(x(\lambda)) = \text{constant}$, $\nabla\varphi(x(\lambda)) = \xi(\lambda)$ along such integral curves. (This construction is classical, and more details may be found (for example) in Courant — Hilbert [1], vol. 2). The resulting curves $\lambda \longrightarrow (x(\lambda), \xi(\lambda))$ are termed bicharacteristic strips and their projections $\lambda \longrightarrow x(\lambda)$ bicharacteristic curves or rays. The characteristic surface $\{\varphi = 0\}$ is then fibered by such rays, and Lax[3] proved the magnitude of the jump in the $(r+1)$-derivative across S satisfies a first-order linear ODE along such a ray, so it is either zero everywhere or non-zero at every point of such a ray. In this sense, such "jump" singularities propagate along the bicharacteristic rays. Taking this as representative of other singularities, the solutions of the wave equation in example 1 behave more-or-less as expected, though it is not clear why some rays through a singular point do not "propagate" the singularity. But F. John's example (ex.2) remains completely mysterions - the boundary of the regular region is a cylinder $\{x^2 + y^2 = 1,\ t \text{ arbitrary}\}$ which is nowhere characteristic. Of course, there is no contradiction - we are not dealing with a jump discontinuity - but it shows that propagation of singularities is a more complicated phenomenon than the classical treatment (or Lax's theorem) suggests.

For a more precise formulation, we must go beyond local analysis ("near a point") to micro-local analysis ("near a point, looking only in certain directions").

<u>Definition</u> Let $A \subset \mathbb{R}^n$ be an open set and u a distribution on A $(u \in \mathcal{D}'(A))$. The <u>wave-front set</u> of u, $WF(u)$, is a subset of $A \times (\mathbb{R}^n - \{0\})$, which we define by exclusion: $(x_0, \xi_0) \in A \times (\mathbb{R}^n - 0)$ is outside $WF(u)$, if and only if there exist $\varphi \in C_c^\infty(A)$, $\varphi(x_0) \neq 0$, and an open cone $K \subset \mathbb{R}^n - 0$ containing ξ_0, such that the Fourier transform

$$(\varphi . u)^\wedge(\xi) = O(|\xi|^{-N}) \quad \text{as} \quad \xi \longrightarrow \infty \text{ in } K$$

for every $N = 1,2,\dots$.

Note We localize u near x_0 by multiplying by a cut-off function φ with small support including x_0; then we localize the "direction" near ξ_0 by choice of the cone K. In fact, we should consider ξ a co-vector or co-direction, the important thing being the corresponding hyperplanes $\{\xi.x = \text{const}\}$.

After multiplication by φ, we may suppose $\varphi.u$ is defined and equal to zero outside supp φ, so $(\varphi.u)\hat{}$ is well-defined.

$WF(u)$ is a conical set closed in the relative topology of $A\times(\mathbb{R}^n - 0)$.

Examples

(1) If $\varphi \in C^\infty(A)$, $WF(\varphi)$ is empty.

(2) If δ_0 is Dirac's delta, $\langle \delta_0,\varphi \rangle = \varphi(0)$ for $\varphi \in C_c^\infty(\mathbb{R}^n)$, then $WF(\delta_0) = (0,\xi)|\xi \neq 0\}$ and $WF(\partial_j\delta_0) = WF(\delta_0)$, $j = 1,2,\dots,n$.

(3) If $\Omega \subset \mathbb{R}^n$ is an open set and $\partial\Omega$ is a C^∞ hypersuface (with Ω on only one side), $\chi_\Omega = 1$ in Ω and $\chi_\Omega = 0$ outside Ω, then

$$WF(\chi_\Omega) = \{(x,\xi)|x \in \partial\Omega, \xi \neq 0 \text{ is normal to } \partial\Omega \text{ at } x\}$$

(4) Hörmander [4,v.1] shows, for each $\xi_0 \neq 0$ in $\mathbb{R}^n$, there exists $u \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$WF(u) = \{0,t\xi_0) : t > 0\}$$

Note that the direction $-\xi_0$ is not included: $WF(u)$ need not be a "double" cone.

(5) Let $g : A \longrightarrow \mathbb{R}$ be C^∞ with $\nabla g(x) \neq 0$ on A, and suppose $f \in C^0(\mathbb{R})$ is <u>not</u> C^∞; then for the composition we have

$$WF(f \circ g) = \{(x,\xi) \mid x \in A,\ f \text{ is not } C^\infty \text{ on any nbd. of } g(x),\ \xi \text{ is a non-zero multiple of } \nabla g(x)\}$$

(6) $WF(\varphi . u) \subset WF(u)$ for any C^∞ function φ, and $WF(\partial_j u) \subset WF(u)$ for $j = 1,2,\ldots,n$

(7) $WF(u + v) \subset WF(u) \cup WF(v)$

The proofs of (1), (2), (6), (7) are easily supplied, while the others are in Hörmander [4, vol. 1], for example. We only prove a special case of (5): $A = \mathbb{R}^n$, $g(x) = x_n$, $f(t) = 1$ for $t > a$, $f(t) = 0$ for $t < a$. Then if $\psi \in C_c^\infty(\mathbb{R}^n)$

$$(\psi . f \circ g)\hat{\ }(\xi) = \iint_{\{x_n > a\}} e^{-i\xi'.x' - i\xi_n x_n} \psi(x', x_n)\, dx'\, dx_n$$
$$= e^{-ia\xi_n} \int_0^\infty e^{-i\xi_n \tau} \tilde{\psi}(\xi', a+\tau)\, d\tau ,$$

where

$$\tilde{\psi}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-i\xi'.x'} \psi(x', x_n)\, dx' .$$

Integration by parts shows (for any N)

$$(\psi . f \circ g)^{\wedge}(\xi) = \frac{e^{-ia\xi_n}}{i\xi_n} \tilde{\psi}(\xi'; a) + O(\xi_n^{-2}(1+|\xi'|)^{-N})$$

$$= O(|\xi_n|^{-1}(1+|\xi'|)^{-N}) \quad \text{as} \quad |\xi| \longrightarrow \infty.$$

If $\psi(\cdot,a) \not\equiv 0$ and $\xi_n \longrightarrow \pm\infty$ with ξ' bounded, the transform does not go to zero rapidly (merely $O(|\xi|^{-1})$). But if $\xi \longrightarrow \infty$ in such a way that $|\xi'|/|\xi| \geqslant$ const. > 0, i.e. excluding some conical neighborhood of the ξ_n-axis, then $(\psi.f\circ g)^{\wedge}(\xi) =$ $= O(|\xi'|^{-N}) = O(|\xi|^{-N})$. Thus, for this example, $WF(f\circ g) =$ $= \{(x,\xi) | \ x_n = a, \ \xi = (0,\ldots,0,\xi_n) \neq 0\}$, in agreement with (5).

Now we state the general propagation of singularities theorem of Lax and Hörmander:

<u>Theorem</u>. Let $P(x,\frac{\partial}{\partial x})$ be an m-order scalar differential operator with C^∞ coefficients whose principal part P_m is real and has its real characteristics simple. If $u \in \mathcal{D}'(A)$, A open, $A \subset \mathbb{R}^n$,

$$WF(Pu) \subset WF(u) \subset WF(Pu) \cup \text{Char } P$$

$$\text{where Char } P = \{(x,\xi) | \xi \neq 0, \ P_m(x,\xi) = 0\}$$

and $WF(u) \setminus WF(Pu)$ is a subset of Char P which is invariant under the Hamiltonian flow

$$x' = \partial P_n/\partial\xi(x,\xi), \quad \xi' = -\partial P_n/\partial x(x,\xi)$$

until x reaches ∂A or (x,ξ) reaches $WF(Pu)$.

Returning to the functions E, E_+, E_- of example 1, we now see more clearly their signifance. With $\square = \partial^2/\partial t^2 - \Delta_x$, $\square E(t,x) = 0$ on all $\mathbb{R} \times \mathbb{R}^n$ so given any point $(t_0, x_0; \tau_0, \xi_0)$ of $WF(E) \subset \mathrm{Char}\square$, $\tau_0^2 = |\xi_0|^2 \neq 0$, the entire bicharacteristic through this point $(t_0 + \lambda\tau_0, x_0 - \lambda\xi_0; \tau_0, \xi_0)$, $-\infty < \lambda < \infty$, must lie in $WF(E)$. Since $E(t,x)$ is invariant with respect to rotation in the x-variables, the same is true for $WF(E)$. Analogous conclusions hold for E_+, but $WF(\square E_+) = WF(\delta_0) = \{(0,0;\tau_0,\xi_0) | (\tau_0,\xi_0) \neq (0,0)\}$, so we can only follow the bicharacteristics in $WF(E_+)$ until x reaches the origin, and can draw no conclusion about the other half-line.

We prove in the next section that, for John's example 2,

$$WF(u|_{x^2+y^2>1}) \subset Q \equiv \{(t,x,y;\ \tau,\xi,\eta) | \ x^2+y^2 > 1 \ \text{ and}$$

$$(\tau,\xi,\eta) \ \text{ is a non-zero multiple of either}$$

$$\nabla(t+\theta-\mathrm{Cos}^{-1}\frac{1}{r} - r^2) \quad \text{or} \quad \nabla(t+\theta+\mathrm{Cos}^{-1}\frac{1}{r}+r^2)\}$$

using plane polar coordinates $(x,y) = r(\cos\theta, \sin\theta)$.

The corresponding bicharacteristic curves–the only ones that can carry singularities–are

$$(T,X,Y) = (t, r\cos\theta, r\sin\theta) + \lambda(\pm 1, \sigma\frac{\sqrt{r^2-1}}{r}\cos\theta - \frac{\sin\theta}{r}, \sigma\frac{\sqrt{r^2-1}}{r}\sin\theta + \frac{\cos\theta}{r}) \ ,$$

$$-\infty < \lambda < \infty, \quad \text{where} \quad \sigma = \pm 1, \quad \text{so}$$

$$X^2 + Y^2 = r^2 + 2\lambda\sigma\sqrt{r^2-1} + \lambda^2 = 1 + (\lambda + \sigma\sqrt{r^2-1})^2$$

and such rays are tangent to the unit cylinder. No ray carrying singularities passes from the regular to the singular region. The cylinder $\{x^2 + y^2 = 1\}$ is the envelope of rays carrying singularities, which may (plausibly) be related to the additional "roughness" on the cylinder, compared to the exterior.

1.3. The wave-front set for F. John's example.

Let Q be the closed conic set defined at the end of the last section. We show, for every positive integer N,

$$u = u_N + R_N \quad \text{with} \quad WF(u_N|_{x^2+y^2>1}) \subset Q$$

and $R_N|_{x^2+y^2>1}$ is of class $C^{p(N)}$, $p(N) \longrightarrow \infty$ as $N \longrightarrow \infty$. Given a point of $\{x^2+y^2>1\} \times (\mathbb{R}^3-0)$ outside Q and any positive integer N_0, choose N so large that $p(N) \geqslant N_0$ and then (after micro-localization) the Fourier transform of (cut-off).u is $O(|\xi|^{-N_0})$ in an appropriate cone; thus

$$WF(u|_{x^2+y^2>1}) \subset Q.$$

By the method of stationary phase,

$$J_k(k \sec\rho) = \sum_{j=0}^{N} k^{-j-1/2} \left(a_j^+(\rho)e^{ikg(\rho)} + a_j^-(\rho)e^{-ikg(\rho)}\right)$$

$$+ R_N(k,\rho)$$

where $g(\rho) = \tan\rho - \rho$, $a_j^{\pm}(\rho)$ and $R_N(k,\rho)$ are analytic in ρ on $0 < \rho < \pi/2$ (so $\sec\rho > 1$) and

$$\left(\frac{\partial}{\partial\rho}\right)^s R_N(k,\rho) = O(k^{-N-3/2+s}) \quad \text{as} \quad k \longrightarrow +\infty,$$

uniformly for ρ in compact sets of $(0, \pi/2)$. For example (with $N=0$)

$$J_k(k \sec\rho) = \sqrt{\frac{2}{k \tan\rho}} \cos(kg(\rho)-\pi/4) + O(k^{-3/2}),$$

as proved in Courant-Hilbert [1, vol. 1], and more details may be found in Watson [7].

Now defining

$$f_j(s) = \sum_{k=2}^{\text{int.} \geqslant 2} k^{-\lambda-j-1/2} e^{iks} \qquad (\lambda = m+3/5)$$

we see f_j is 2π-periodic and C^{m+j+1} but the $(m+j+2)$-order distributional derivative is nowhere locally integrable. We have

$$u = \sum_{j=0}^{N} \{a_j^+(\rho)\ f_j(g(\rho)+t+\theta) + a_j^-(\rho)\ f_j(-g(\rho)+t+\theta)\} + R_N(t,\rho,\theta)$$

$$= U_N + R_N \quad \text{at} \quad (t,x,y) = (t, \sec\rho\cos\theta, \sec\rho\sin\theta)$$

where $R_N(t,\rho,\theta) = \sum_{k=2}^{\text{int.} \geqslant 2} k^{-\lambda} R_N(k,\rho) e^{ik(t+\theta)}$ is of class C^{N+m+2}, and (by examples (5), (6) and (7) of wave-front sets above)

$$WF(u_N|_{x^2+y^2>1}) \subset Q \quad \text{for each} \quad N,$$

so $WF(u|_{x^2+y^2>1}) \subset Q$.

References

1. R. Courant and D. Hilbert: Methods of Mathematical Physics, vol. 1 (1953) and vol. 2 (1960), Interscience (J. Wiley) N.Y.
2. F. John: Continuous dependence on data for solutions of PDEs with a prescribed bound, Comm. Pure and Appl. Math. 13(1960), p. 551-585 esp. p. 572-574.
3. P. Lax; Asymptotic solutions of oscillatory initial-value problems, Duke Math. J. 24 (1957), p. 627-646.
4. L. Hörmander: Analysis of Linear Partial Differential Operators, vol. 1 and 3, Springer Verlag (Grundlehren), 1984.
5. M. Taylor: Propagation, reflection and diffraction of singularities of solutions to wave equations, Bull. Am. Math. Soc. 84 (1978), p. 589-611.
6. M. Taylor, Pseudo-differential operators, Princeton (1981).
7. G.N. Watson, A Treatise on the theory of Bessel Functions, Cambridge University Press, 1958.

2. Non-decay of thermoelastic vibrations in dimension $\geqslant 3$

Dafermos [1] showed the equations of linear thermoelasticity

$$\begin{cases} \rho\ddot{u} = \alpha\,\Delta\,u + (\alpha+\beta)\,\nabla\,(\operatorname{div} u) - \mu\,\nabla\,\theta \\ \rho C_D\,\overset{\circ}{\theta} + \mu\,\operatorname{div} u = \kappa\,\Delta\,\theta \end{cases} \qquad \text{in} \quad \Omega \subset \mathbb{R}^n$$

for an isotropic homogeneous material (ρ, α, $2\alpha+\beta, C_D$, κ positive constants, and μ constant $\neq 0$), with boundary condition

$$\begin{cases} u = 0 \quad \text{or stress} = 0 \\ \theta = 0 \quad \text{or } \kappa\,\partial\theta/\partial N + \beta\theta = 0 \end{cases} \qquad \text{on} \quad \partial\Omega,$$

define a semigroup of contractions in an appropriate Hilbert space X with norm

$$\|(u,\overset{\circ}{u},\theta)\|_X^2 = \int_\Omega \{\,\alpha \sum_{i,j} (\partial u_i/\partial x_j)^2 + (\alpha+\beta)(\operatorname{div} u)^2 + \rho|\overset{\circ}{u}|^2 + \rho C_D\,\theta^2\,\}$$

(In fact, he treats a much more general situation.) He also showed (under plausible hypotheses on Ω) that every solution tends to zero in X as $t \longrightarrow \infty$.

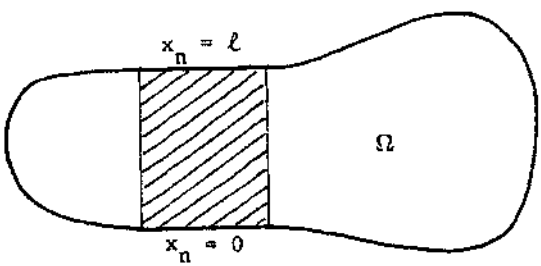
More recently, Orlando Lopes and Anizio Perissinotto, Jr. (Univ. Estadual de Campinas, S.P.,Brasil) showed, for the case $n = 1$, that the solutions thend to zero exponentially.

It is easy to show that, if $n \geqslant 2$ and we use spatial periodicity in place of the boundary conditions, there are solutions which do not decay -in fact $\theta \equiv 0$, $\rho \ddot{u} = \alpha \Delta u$, $\operatorname{div} u \equiv 0$, but $u \not\equiv 0$. Dafermos showed there are no such solutions satisfying the boundary conditions, for most regions Ω. But if the boundary has opposing "flat spots" (described below) and $n \geqslant 3$, we show there are solutions almost of this kind, so there are solutions which dacay arbitrarily slowly. In fact if $\{e^{At}, t \geqslant 0\}$ is the semigroup, we have

$$\| e^{At} \|_{\mathcal{L}(X)} = r(e^{At}) = r_{ess}(e^{At}) = 1$$

for all $t \geqslant 0$, where $r(.)$, $r_{ess}(.)$ denote the spectral radius and essential spectral radius.

Specifically assume (after appropriate rigid motions) there is a cylinder $\{(\hat{x}, x_n) \mid |\hat{x}| < \delta, 0 < x_n < \ell\}$ in Ω whose ends $(x_n = 0, \ell)$ are in $\partial\Omega$ (see figure). Then given any



$\epsilon > 0$, there is a C^∞ function $v : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ with $\operatorname{div} v \equiv 0$, $|\rho v_{tt} - \alpha \Delta v| \leqslant \epsilon$ satisfying the boundary conditions at the ends of the cylinder and vanishing whenever $|\hat{x}| \geqslant \delta$. Further $\|(v, v_t, 0)\|_X \equiv 1$.

If $T > 0$ is given with $\sqrt{\alpha/\rho}\, T/\ell$ rational, we may choose v to be T-periodic in time. If follows that the solution u, θ of the thermolasticity equations with initial values $u = v$, $\mathring{u} = v_t$, $\theta = 0$ (at $t = 0$) satisfy

$$\| (u-v,\ \mathring{u}-v_t,\ \theta) \|_X = O(\epsilon)$$

uniformly on $0 \leqslant t \leqslant T$. Thus

$$\| (e^{AT}-I)(v,\mathring{v},0)_{t=0} \|_X = O(\epsilon), \quad \| (v,\mathring{v},0) \|_X = 1,$$

so $1 \in \sigma(e^{AT})$. In fact $(v,\mathring{v},0)|_{t=0}$ tends weakly to zero as $\epsilon \longrightarrow 0$, so it cannot have a strongly convergent subsequence, hence $1 \in \sigma_{ess}(e^{AT})$.

We already know $1 \geqslant \| e^{AT} \|_{\mathcal{L}(X)} \geqslant r(e^{AT}) \geqslant r_{ess}(e^{AT})$, so in fact we have equality.

It only remains to construct v, which is embarrassingly easy. Let $c = \sqrt{\alpha/\rho}$ (the speed of transverse shear waves). Choose $\varphi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ of class C^∞, supported in $|\hat{x}| < \delta$, with div $\varphi \equiv 0$; for example

$$\begin{aligned}
\varphi_1(x) &= \alpha(x_1)\beta'(x_2)\gamma(x_3 \ldots x_{n-1}) \\
\varphi_2(x) &= -\alpha'(x_1)\beta(x_2)\gamma(x_3,,,x_{n-1}) \\
\varphi_j(x) &= 0 \text{ for } 2 < j \leqslant n-1
\end{aligned}$$

where α, β, γ are C^∞ real-valued functions with small support. (if $n = 3$, $\gamma \equiv 1$). Let $v(t,x) = e^{i\omega x_n} \frac{\cos\omega ct}{\omega}(\varphi(\hat{x}),0)$, $\omega > 0$, where we use Rev if the boundary condition at $x_n = 0$ (in the cylinder) is stress = 0, and Im v if the condition is $u = 0$. It follows easily that div $v \equiv 0$ and

$$\rho v_{tt} - \alpha \Delta v = O(\omega^{-1}) \text{ as } \omega \longrightarrow +\infty,$$

uniformly in (t,x). Since v is $2\pi/\omega$-periodic in x_n, we may choose arbitrarily large ω such that $\mathrm{Re}\,v$ or $\mathrm{Im}\,v$ also satisfies the appropriate boundary condition on $\{x_n = \ell\}$ in the cylinder. Also note (if v is either the real or imaginary part of the v above)

$$\|(v,v_t,0)\|_X^2 = \int_\Omega \rho|v_t|^2 + \alpha|\nabla|^2$$

$$= \alpha\frac{1}{2}\int_{|\hat{x}|<\delta} |\varphi|^2 + O(\omega^{-1}) \quad \text{as} \quad \omega \longrightarrow \infty.$$

Multiplication by an appropriate constant gives a solution with norm 1.

Finally if cT/ℓ is rational, we may choose arbitrarily large ω so, not only is the boundary condition at $\{x_n = \ell\}$ satisfied, but v is T-periodic (since $\omega cT/2\pi$ is an integer).

Two obvious questions:

1) Does the boundary really need to be flat?

2) What happens for $n = 2$?

I don't know the answer to either, but will say the little I know or speculate.

The construction is, of course, modeled on geometric optics which does not require flat boundaries (for example, Ralston's "solutions with localized energy"); but which becomes much more complicated when the boundary is curved. We want a solutions with $\mathrm{div}\, u \equiv 0$ (or $\mathrm{div}\, u$, $\partial_t \mathrm{div}\, u$, $\partial_x \mathrm{div}\, u$

uniformly small), which is still true after many reflections at the boundary. It may be possible to achieve this when the "flat spots" reduce to points where the tangent planes are parallel. This last condition is easily satisfied in a smooth bounded convex domain Ω: maximize $|p - q|$ with p,q in $\partial\Omega$. The problem does not look impossible - merely difficult.

When $n = 2$ the above construction fails (div $\varphi \equiv 0$ implies $\varphi \equiv$ constant). There are analogous solutions between infinite parallel planes $\{x_2 = 0\}$ and $\{x_2 = 1\}$, but I am not able to localize these. Consideration of non-normal reflections does not appear promising. Reflection at a plane boundary always generates dissipative "waves", except at normal incidence. I incline (weakly) to the view that $n = 2$ will be like the case $n \geqslant 3$, rather than $n = 1$.

Reference

C. Dafermos, On the existence and asymptotic stability of solutions to the equations of linear thermoelasticity. Arch. Rational Mech. Anal. 29(1968) pp. 241-271.

3. On some non-linear integral inequalities of Kielhöfer and Caffarelli.

Certain integral inequalities from Kielhöfer's article [1] have proved to be useful in the study of parabolic partial differential equations. One of these (lemma A.1 from the appendix to [1], p. 218), sometimes cited as "Kielhöfer's lemma", though Kielhöfer attributes the argument to L. Caffarelli, appears to be incorrect -at least the proof contains a grave error. (I suspect the estimate itself is wrong, but have no counter-example).

We will correct the proof and generalize the results. Aside from this correction, our arguments are only mild variants of those of Kielhöfer and Caffarelli. The resulting inequality (Theorem 2 below) is significantly weaker than lemma A.1[1], for application to uniform (in time) estimation of solutions, and we may hope Theorem 2 is not the best possible result.

Our first result is a generalization of [1], lemma 1.2 (p. 205 and 218).

<u>Theorem 1</u> Let α, p, q be positive constants with $1 \leq p \leq 1 + \alpha q$. Suppose A, B, C are non-negative constants, $0 < T < \infty$ and $\varphi : [0,T) \longrightarrow \mathbb{R}_+$ is continuous with

$$\int_0^T \varphi^q < \infty$$

and $0 \leq \varphi(t) \leq A\,\varphi(t_0) + B + C\int_{t_0}^{t} (t-s)^{\alpha-1}\,\varphi(s)^p ds$ for all $0 \leq t_0 < t < T$.

Then φ is bounded on $[0,T)$ and there exists t_1 in $(0,T)$ such that

$$\varphi(t) \leqslant \mathrm{Max}(1,\ 4B,\ 2A\varphi(t_1)) \text{ on } t_1 \leqslant t < T.$$

Proof $\int_0^T \varphi^q < \infty$ implies $\liminf_{t \to T-} (T-t)\varphi(t)^q = 0$. Thus for any positive ϵ_1, ϵ_2 there exists t_1 in $T-\epsilon_2 \leqslant t_1 < T$ such that $(T-t_1)\varphi(t_1)^q \leqslant \epsilon_2$.

Assuming $C > 0$, to avoid trivialities, we first suppose $B \leqslant 1/4$. Choose $\epsilon_2 > 0$ so small that $\frac{1}{\alpha}C\epsilon_2^\alpha < \frac{1}{4}$, and ϵ_1, ϵ_2 so small that

$$\frac{1}{\alpha}\, C(2A)^{p-1}\, \epsilon_1^{\frac{p-1}{q}}\ \epsilon_2^{\alpha - \frac{p-1}{q}} < \frac{1}{4}\ .$$

Then choose t_1 in $[T-\epsilon_2, T)$ as above and apply the inequality on the largest interval $[t_1, t_2) \subset [t_1, T)$ where $\varphi(t) < L = \max\{1, 2A\varphi(t_1)\}$.

If $t_2 < T$ then

$$\varphi(t_2) < A\varphi(t_1) + B + CL^p \int_{t_1}^{t} (t-s)^{\alpha-1}\, ds$$

$$= A\varphi(t_1) + B + \frac{1}{\alpha}\, C\, L^p (T-t_1)^\alpha.$$

If $2A\varphi(t_1) \leqslant 1$, then $L = 1$ and

$$\varphi(t_2) < \frac{1}{2} + \frac{1}{4} + \frac{1}{\alpha}\, C\epsilon_2^\alpha < 1 = L.$$

Otherwise $L = 2A\varphi(t_1) > 1$ and

$$\frac{\varphi(t_2)}{L} < \frac{1}{2} + \frac{1}{4} + CL^{p-1}(T-t_1)^{\alpha}$$

$$\leqslant \frac{3}{4} + \frac{1}{\alpha} C(2A)^{p-1} \epsilon_1^{(p-1)/q} \epsilon_2^{\alpha-(p-1)/q} < 1.$$

In either case $\varphi(t_2) < L$, so t_2 cannot be maximal unless $t_2 = T$.

Now suppose $B > 1/4$ and let $\psi(t) = \varphi(t)/4B$. Then ψ satisfies the hypotheses of the previous case so, for some $t_1 < T$, $\psi(t) \leqslant \max(1, 2A\psi(t_1))$ on $t_1 \leqslant t < T$ hence $\varphi(t) \leqslant \max(4B, 2A\phi(t_1))$ on $[t_1, T)$:

<u>Remark</u> Kielhöfer [1] treats the case $q = 2$, $A = C = 1$, $p = 1 + \alpha q$, $0 < \alpha \leqslant \frac{1}{2}$. In place of "B", he allows a function of (t, t_0), whose important feature is that it is bounded.

<u>Example</u> If $p > 1 + \alpha q$, we show there is an <u>unbounded</u> continuous $\varphi : [0,T) \longrightarrow \mathbb{R}_+$, satisfying the other hypotheses. In fact let $\delta > 0$ be defined by $p = 1 + \alpha/\delta$, so $0 < \delta q < 1$, and let $\varphi(t) = M(T-t)^{-\delta}$, $M > 0$. Then $\int_0^T \varphi^q < \infty$ and (with $A > 1$, $C > 0$, $B = 0$ and M sufficiently large)
$1 \leqslant A(1+N)^{-\delta} + CM^{p-1} \int_0^N \sigma^{\alpha-1}(1+\sigma)^{-\alpha-\delta}\, d\sigma$ for all $N \geqslant 0$.

On change of variables $(N = (t-t_0)/(T-t)$, we see φ satisfies the inequality of Theorem 1 for all $0 \leqslant t_0 \leqslant t < T$.

<u>Theorem 2</u> Let α, p, q be positive with $p \leqslant 1 + \alpha q$, $\alpha \leqslant 1$.

There is a continuous function L on $[0,\infty)^3$, increasing in each argument (depending also on α, p, which are kept fixed), such that:

for any $0 < T < \infty$, non-negative a, b, J, K, and continuous $\varphi : [0,T) \longrightarrow \mathbb{R}_+$ with

$$\text{(i)} \quad 0 \leqslant \varphi(t) \leqslant a + b \int_0^t (t-s)^{\alpha-1} \varphi(s)^p ds \quad \text{on } 0 \leqslant t < T,$$

$$\text{(ii)} \quad \int_0^T \varphi^q \leqslant J \quad \text{and}$$

$$\int_{(0,T) \cap \{\varphi < 1\}} \varphi^p \leqslant K,$$

we have φ bounded and in fact

$$\varphi(t) \leqslant L(a, bJ^\alpha, bK^\alpha) < \infty \quad \text{on} \quad 0 \leqslant t < T.$$

If $p \geqslant q$ we may suppose $K \leqslant J$. The function L is given explicitly below (at the end of the proof).

<u>Example</u> If $p > 1 + \alpha q$ and a, b, α, q are positive, let $p = 1 + \alpha/\delta$, $\varphi(t) = M(T-t)^{-\delta}$. If M, T are sufficiently large, φ is an unbounded function satisfying (i) and (ii) of the theorem.

<u>Remark</u> The error in the argument of [1, p. 219-220]-aside from irritating misprints - is disregard of the set of t where $\varphi(t) < 1$, so there is no dependence on K. (The proof is wrong; it is not known whether the inequality claimed is false). After correcting this point, we follow fairly closely

the argument of Kielhöfer and Caffarelli.

The form given above is independent of rescaling of time (μt in place of t, for any constant $\mu > 0$). The corollary below gives a form also independent of rescaling of φ ($Q\varphi$ in place of φ, for any constant $Q > 0$).

L is also an increasing continuous function of p and $1/\alpha$.

<u>Proof</u> Define $A_k = \{t \in (0,T) \mid 2^k \leqslant \varphi(t) < 2^{k+1}\}$ for integers $k \geqslant 0$ and $A^* = \{t \in (0,T) \ \varphi(t) < 1\}$. Then $(0,T)$ is the disjoint union of A^* and the A_k, $k \geqslant 0$, and (with $|A| =$ = measure of A)

$$\int_{A^*} \varphi^q + \sum_0^\infty |A_k| \, 2^{kq} \leqslant \int_0^T \varphi^q \leqslant J.$$

For certain $\lambda > 0$ and integer $M \geqslant 1$, depending only on α, p, a, b, J, K (chosen below), we will prove $\varphi(t) < 2^{2M(1+\lambda^{-1}J)}$ on $0 \leqslant t < T$. (To avoid trivialities, we suppose a,b,J,K gre all positive; the final estimates are continuous when one or more of these tends to zero.

Let I_λ be the set of integers $k \geqslant 0$ so $2^{kq|A_k|} > \lambda$. This is a finite set and in fact

$$\# I_\lambda = \sum_{k \in I_\lambda} 1 \leqslant \sum_{k \in I_\lambda} \lambda^{-1} \, 2^{kq|A_k|} \leqslant \lambda^{-1} J.$$

For large integers $\nu \geqslant 0$, $[\nu M, (\nu+1)M]$ does not meet I_λ, and we let $\nu_0 \geqslant 0$ be the smallest such integer. For each ν

in $0 \leqslant \nu \leqslant \nu_0 - 1$, there is a corresponding point of I_λ, and allowing for possible double-counting of end points,

$$\lambda^{-1}J \geqslant \# \, I_\lambda \geqslant \tfrac{1}{2}(\nu_0 - 1)$$

so

$$0 \leqslant \nu_0 \leqslant 1 + 2\lambda^{-1} J.$$

If $2^M > a$, as we assume, then (since $\varphi(0) \leqslant a$) $\varphi(t) < 2^M \leqslant 2^{M(1+\nu_0)}$ for small positive t. Let t_1 be the largest number in $(0,T]$ such that

$$\varphi(t) < 2^{M(1+\nu_0)} \quad \text{on} \quad 0 \leqslant t < t_1.$$

We show (for appropriate choices of λ, M) that $t_1 = T$. This says $\varphi(t) < 2^{2M(1+\lambda^{-1}J)}$ on $0 \leqslant t < T$, as desired.

Suppose for contradiction $t_1 < T$, so $\varphi(t_1) = 2^{M(1+\nu_0)}$. Then

$$2^{M(1+\nu_0)} = \varphi(t_1) \leqslant a + b \int_0^{t_1} (t_1 - s)^{\alpha-1} \varphi(s)^p \, ds$$

$$\leqslant a + b \int_{(0,t)\cap A^*} (t_1 - s)^{\alpha-1} \varphi(s)^p + 2^p b \sum_{k=0}^{M(\nu_0+1)} \int_{(0,t_1)\cap A_k} (t_1 - s)^{\alpha-1} ds \; 2^{kp}.$$

Now $0 \leqslant \varphi < 1$ on A^* so for any h in $0 < h \leqslant t_1$

$$\int_{(0,t_1)\cap A^*} (t_1 - s)^{\alpha-1} \varphi(s)^p ds \leqslant h^{\alpha-1} \int_{(0,t_1-h)\cap A^*} \varphi^p + \int_{t_1-h}^{t_1} (t_1 - s)^{\alpha-1} ds.$$

Choose $h = K$ when $0 < k \leqslant t_1$, and $h = t_1$ when $K > t_1$; in either case we obtain an upper bound $(1 + \frac{1}{\alpha})K^{\alpha}$. Thus

$$2^{M(1+\nu_0)} \leqslant a + (1+\tfrac{1}{\alpha})bK^{\alpha} + \frac{2^p b}{\alpha} \sum_{k=0}^{M(\nu_0+1)} 2^{kp} |A_k|^{\alpha}.$$

Now $|A_k| \leqslant J\, 2^{-kq}$ for every $k \geqslant 0$, but we use this estimate only for $0 \leqslant k < M\nu_0$; $[M\nu_0, M(\nu_0+1)]$ is disjoint from I_λ, and for k in this interval, $|A_k| \leqslant \lambda\, 2^{-kq}$. Substituting these estimates

$$2^{M(1+\nu_0)} \leqslant a + (1+\tfrac{1}{\alpha})bK^{\alpha} + \frac{2^p bJ^{\alpha}}{\alpha} \sum_{k=0}^{M\nu_0-1} 2^{k(p-q\alpha)}$$

$$+ \frac{2^p b\lambda^{\alpha}}{\alpha} \sum_{k=M\nu_0}^{M(\nu_0+1)} 2^{k(p-q\alpha)}.$$

Recalling $p-q\alpha \leqslant 1$, we find

$$2^{M(1+\nu_0)} \leqslant a+(1+\tfrac{1}{\alpha})bK^{\alpha} + \frac{2^p bJ^{\alpha}}{\alpha}\, 2^{M\nu_0}$$

$$+ \frac{2^{p+1} b\lambda^{\alpha}}{\alpha}\, 2^{M(\nu_0+1)}.$$

This is contradictory if $2^{p+1} b\lambda^{\alpha}/\alpha = \frac{1}{4}$,

$$a + (1+\tfrac{1}{\alpha})bK^{\alpha} \leqslant \tfrac{1}{4}.2^M, \quad 2^p bJ^{\alpha}/\alpha \leqslant \tfrac{1}{4}.2^M,$$

since it says $2^{M(1+\nu_0)} \leqslant \frac{3}{4}\, 2^{M(1+\nu_0)}$. (Recall we assumed earlier that $a < 2^M$).

Thus choosing λ as above, let M be the first positive integer $\geqslant M_0$, where M_0 is defined by

$$M_0 = \max\{0, \log_2(2^{p+2}bJ^\alpha/\alpha), \log_2 (4a+4(1+\tfrac{1}{\alpha})bK^\alpha)\},$$

an increaging continuous function of a, bJ^α and bK^α. We have $M_0 \leqslant M \leqslant M_0 + 1$ and

$$\lambda^{-1} J = (2^{p+3} bJ^\alpha/\alpha)^{1/\alpha}$$

so

$$\log_2 L\ (a, bJ^\alpha, bK^\alpha) =$$

$$= 2(M_0+1)(1+(2^{p+3}bJ^\alpha/\alpha)^{1/\alpha})$$

is the desired bound:

$$\log_2 \varphi(t) < M(1+\nu_0) \leqslant 2(M_0+1)(1+\lambda^{-1}J) = \log_2 L.$$

<u>Remark</u>

In the proof of Theorem 2 we used the following simple result: If $0 < \alpha \leqslant 1$ and A is any measurable subset of $\mathbb{R}_+$,

$$\int_{(0,t_1)\cap A} (t_1-s)^{\alpha-1}\, ds \leqslant \frac{1}{\alpha}|A|^\alpha.$$

This is clear if A is a finite union of intervals in $(0,t_1)$; $s \longrightarrow (t_1-s)^{\alpha-1}$ is increasing, so by moving the interval to the right we preserve $|A|$ while increasing the integral to $\int_{t_1-|A|}^{t_1} (t_1-s)^{\alpha-1} = \frac{1}{\alpha}|A|^{\alpha}$. Any open A is a countable union of open intervals, and the result for open A follows by taking limits of incrasing unions of finite intervals. Finally, any measurable A may be approximated in measure, from the outside, by open sets, so we get the general case.

<u>Corollary</u> We use the notation and hupotheses of Theorem 2, but also, for some $Q > 0$ and $K_Q > 0$, suppose φ satisfies

$$\int_{(0,T)\cap\{\varphi<Q\}} \varphi^p \leqslant K_Q.$$

Then on $0 \leqslant t < T$,

$$\varphi(t) \leqslant Q.L(a/Q,\ bJ^{\alpha}\ Q^{p-q\alpha-1},\ bK_Q^{\alpha}\ Q^{p(1-\alpha)-1}).$$

The last argument on the right-hand side may also be written

$$bJ^{\alpha}\ Q^{p-q\alpha-1}\ (K_Q Q^{-p}/JQ^{-q})^{\alpha}.$$

Note the simplification in the extreme case $p = 1 + \alpha\, q$.

<u>Proof</u> Define $\psi : [0,T) \longrightarrow \mathbb{R}_+$ by $\psi(t) = \varphi(t)/Q$, and apply the theorem to ψ. Returning to $\varphi = Q.\psi$ gives the corollary.

Reference

1. H. Kielhöfer, Global solutions of semilinear evolution equations satisfying an energy inequality, J. Diff. Eq. 36 (1980), 188-222.

4. An example in the spectral theory of semigroups

4.1. Introduction

If $A \in \mathcal{L}(X)$ is a continuous linear operator, the spectral mapping theorem says $\sigma(e^{At}) = e^{t\sigma(A)}$. In general, the generator A of a strongly continuous semigroup $\{e^{At}, t \geqslant 0\}$ is not bounded, and the most one can assert (without further hypotheses) is that

$$\sigma(e^{At}) \supset e^{t\sigma(A)} \quad \text{for} \quad t \geqslant 0,$$

and in more detail ([3], Th.16.7.1,2,3,4)

$$P\sigma(e^{At})\setminus\{0\} = e^{tP\sigma(A)}, \; R\sigma(e^{At})\setminus\{0\} = e^{tR\sigma(A)}$$

$$\text{and} \quad C\sigma(e^{At}) \supset e^{tC\sigma(A)}$$

(note 0 does not belong to the image of the exponential function, though it may be in the spectrum of the semigroup).

A remarkable example is given in Hille and Phillips ([3], sec.23.16) of a strongly continuous group of operators $\{e^{At}, -\infty < t < \infty\}$ on the Hilbert space $L_2(0,1)$, whose <u>generator A has no spectrum</u>, while for any real $t \neq 0$

$$\sigma(e^{At}) = C\sigma(e^{At}) = \{z : e^{-|t|\pi/2} \leqslant |z| \leqslant e^{|t|\pi/2}\}.$$

The spectrum of this semigroup bears no relation to $\sigma(A)$, since $\sigma(A)$ is empty; but a recent theorem of Gearhart and Herbst [1,2] shows spectrum may also arise from lines $\operatorname{Re}\lambda$ = constant where the resolvent $(\lambda-A)^{-1}$ is unbounded. We prove

$$\|(\lambda-A)^{-1}\| \text{ is bounded on any line } \operatorname{Re}\lambda = \text{constant} \notin [-\tfrac{\pi}{2}, \tfrac{\pi}{2}],$$

$$\|(\lambda-A)^{-1}\| \text{ is bounded as } \operatorname{Im}\lambda \longrightarrow +\infty \text{ on } \operatorname{Re}\lambda = \text{constant} \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$$\text{but is unbounded as } \operatorname{Im}\lambda \longrightarrow -\infty,$$

in accordance with this theorem.

We will review the entire example, since certain details are treated differently than in [3], and other details are supplied that are omitted from [3].

4.2 The example

Given continuous $f : [0,\infty) \longrightarrow \mathbb{C}$ and $t > 0$, define

$$(1) \qquad J^t f(x) = \int_0^x \frac{(x-y)^{t-1}}{\Gamma(t)} f(y)\,dy, \; x \geqslant 0.$$

For any positive t,s, we have $J^t(J^s f) = J^{t+s} f$:

$$\begin{aligned} J^t(J^s f)(x) &= \int_0^x \frac{(x-y)^{t-1}}{\Gamma(t)}\,dy \int_0^y \frac{(y-z)^{s-1}}{\Gamma(s)} f(z)\,dz \\ &= \int_0^x \left\{\int_z^x \frac{(x-y)^{t-1}\,(y-z)^{s-1}}{\Gamma(t)\,\Gamma(s)}\,dy\right\} f(z)\,dz \\ &= \int_0^x \frac{(x-z)^{s+t-1}}{\Gamma(t+s)} f(z)\,dz\,. \end{aligned}$$

Further, if f is continuously differentiable,

$$J^t f(x) = \frac{x^t}{\Gamma(t+1)} f(x) + \int_0^x \frac{(x-y)^{t-1}}{\Gamma(t)} (f(y)-f(x))dy$$

$$= f(x) + \{\frac{x^t}{\Gamma(t+1)} - 1\}f(x) - \frac{t}{\Gamma(t+1)} \int_0^x (x-y)^t \frac{f(x)-f(y)}{x-y} dy$$

$$\longrightarrow f(x) \text{ as } t \longrightarrow 0+.$$

Defining $J^0 f = f$, we have (at least formally) a semigroup of operators, the <u>fractional integration semigroup</u>, such that

$$J^1 f(x) = \int_0^x f(y)dy \quad \text{and} \quad \frac{\partial}{\partial x}(J^t f(x)) = J^{t-1} f(x), \ t \geqslant 1.$$

(We estimate norms below to show this is strongly continuous on $L_2(0,1)$.)

In fact, the definition (1) makes sense for complex t in $\operatorname{Re} t > 0$, $t \longrightarrow J^t f$ is analytic in $\operatorname{Re} t > 0$ and $J^t(J^s f) = J^{t+s} f$ for $\operatorname{Re} t > 0$, $\operatorname{Re} s > 0$, by analytic continuation. If $f \in C_c^\infty(\mathbb{R}_+)$, $t \longrightarrow J^t f(x)$ extends to be an entire analytic function with $J^t f(x) = \frac{d^m f(x)}{dx^m}$ when $t = -m$ $(m = 0,1,2,\ldots)$, as may be seen from the formula

$$J^t f(x) = \int_0^x \frac{(x-y)^{t-1}}{\Gamma(t)} \{f(y) - \sum_{j=0}^{N} f^{(j)}(x) \ (y-x)^j/j!\} dy$$

$$+ \sum_{j=0}^{N} f^{(j)}(x)/j! \ . \ (-1)^j x^{t+j}/((t+j)\Gamma(t)).$$

Note $1/\Gamma(t)$ and $1/((t+j)\Gamma(t))$ may be considered entire analytic functions of t, for any integer $j \geq 0$.

We will work only in the half-plane $\{\mathrm{Re}\ t \geq 0\}$, and we show

(2) $\|J^t f\|_{L_2(0,1)} \leq B(t)\ \|f\|_{L_2(0,1)}$ for all $f \in C_c^\infty(0,1)$,

where $B(t) = \dfrac{1}{\mathrm{Re}t\ \Gamma(t)}$ for $\mathrm{Re}\, t > 0$, or for $0 \leq \mathrm{Re}\, t < 1/2$ we may take

$$B(t) = e^{\frac{\pi}{2}|\mathrm{Im}t|}/(1-2\,\mathrm{Re}\ t)^{1/2}.$$

Note, on $\mathrm{Re}t = \sigma > 0$, Stirling's formula gives

$$\frac{1}{\mathrm{Re}t|\Gamma(t)|} = \frac{1}{\sigma\sqrt{2\pi}}|\mathrm{Im}t|^{-\sigma+1/2}e^{\frac{\pi}{2}|\mathrm{Im}t|}\{1+O(\frac{1}{|\mathrm{Im}\ t|})\} \text{ as } \mathrm{Im}\, t \longrightarrow \pm\infty,$$

so our bounds for the norm are not too far apart. In fact, by the maximum principle, we have $\|J^t\| \leq 1.1292\ e^{\pi/2\,|\mathrm{Im}\ t|}$ for $0 \leq \mathrm{Re}\, t \leq 1/2$.

For the first estimate, define

$$K_t(u) = \begin{cases} u^{t-1}/\Gamma(t) & \text{when } u > 0 \\ 0 & \text{when } u < 0, \end{cases}$$

so on $0 < x < 1$, $\mathrm{Re}\, t > 0$, $\mathrm{supp}\ f \subset (0,1)$,

$$|J^t f(x)| = |\int_0^x K_t(x-y)f(y)dy| = |\int_0^1 K_t(x-y)\ f(y)\ dy| \leq$$

$$\leqslant \left(\int_0^1 |K_t(x-y)\ dy\right)^{1/2} \left(\int_0^1 |K_t(x-y)|\ |f(y)|^2\ dy\right)^{1/2}$$

$$\text{so } \int_0^1 |J^t f(x)|^2 dx \leqslant \int_{-1}^1 |K_t(u)| du \int_0^1 \left\{\int_0^1 |K_t(x-y)|\ dx\right\} |f(y)|^2 dy$$

$$\leqslant \left(\int_{-1}^1 |K_t(u)| du\right)^2 \|f\|^2_{L_2(0,1)} \ .$$

Thus we may take

$$B(t) = \int_{-1}^1 |K_t(u)|\, du = \int_0^1 \frac{u^{\mathrm{Re}\ t-1}}{|\Gamma(t)|}\, du = \frac{1}{\mathrm{Re}t\ |\Gamma(t)|}.$$

Before proving the other estimate, we note that this alredy shows $t \longrightarrow J^t f \in L_2(0,1)$ is continuous (and even analytic) in $\mathrm{Re}\ t > 0$, for any $f \in L_2(0,1)$, and it is also continuous as $t \longrightarrow 0$ in any sector $\{|\arg t| \leqslant \pi/2 - \epsilon < \pi/2\}$ strictly in the right half-plane.

Thus we have a strongly continuous semigroup in $\mathrm{Re}\ t > 0$, and in particular on the real axis $\{t \geqslant 0\}$.

Let A denote the generator of $\{J^t,\ t \geqslant 0\} \subset \mathcal{L}(L_2(0,1))$, so $J^t = e^{At}$, $t \geqslant 0$. Now the spectral radius

$$r(J^t) = \lim_{n\to\infty} \|J^{nt}\|^{1/n}_{\mathcal{L}(L_2)} \leqslant \lim_{n\to\infty} \frac{1}{\Gamma(nt+1)^{1/n}} = 0.$$

for any $t > 0$, so $\sigma(J^t) = \{0\}$ for $t > 0$. It is also known that $\{0\} = \sigma(e^{tA}) \supset e^{t\sigma(A)}$; but 0 is not in the image of the exponential function, and any point of $\sigma(A)$ would give a non-zero point of $\sigma(e^{tA})$ for any $t > 0$; so $\sigma(A)$ is empty. (Of course a bounded operator always has spectrum, but A is unbounded).

Now we obtain the other estimate, which shows the semigroup may be defined (and is strongly continuous) in the <u>closed</u> half-plane $\{\mathrm{Re}\,t \geqslant 0\}$. For $f \in C_c^\infty(0,1)$, $J^t f$ is well-defined for all complex t, and we consider in particular the strip $0 \leqslant \mathrm{Re}\,t < 1/2$, because then

$$|J^t f(x)| = |\int_0^1 (x-y)^{t-1} f(y) \;/\; \Gamma(t) = O(x^{\mathrm{Re}\,t-1})$$

for $x \geqslant 1$, so $\int_0^\infty |J^t f(x)|^2 dx < \infty$. Define $J^t f(x) = 0$ for $x < 0$; we compute the Fourier transform when $0 < \mathrm{Re}\,t < 1/2$

$$(J^t f)^{\widehat{}}\;(\xi) = \lim_{R\to\infty} \int_{-R}^{R} e^{-i\xi x}\, J^t f(x)\,dx$$

$$= \lim_{R\to\infty} \int_0^R e^{-i\xi x}\, J^t f(x)\,dx$$

$$= \lim_{R\to\infty} \int_0^1 dy\, \{\int_0^{R-y} u^{t-1} e^{-i\xi u} du/\Gamma(t)\}\, e^{-i\xi y} f(y).$$

Rotation of the line of integration to the negative (or positive) imaginary axis when $\xi > 0$ (or $\xi < 0$) shows

$$(J^t f)^{\widehat{}}(\xi) = |\xi|^{-t}\, e^{\mp i\pi t/2}\; \hat{f}\,(\xi), \quad (\mp = -\mathrm{sgn}\;\xi).$$

Then

$$\int_0^\infty |J^t f|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\xi|^{-2\mathrm{Re}t}\, e^{\pm\pi \mathrm{Im}t}\, |\hat{f}(\xi)|^2 \, d\xi$$

$$\leqslant \frac{1}{2\pi}\, e^{\pi|\mathrm{Im}t|} \int_{-\infty}^{\infty} |\xi|^{-2\mathrm{Re}t}\, |\hat{f}(\xi)|^2 \, d\xi$$

$$\leqslant \frac{1}{2\pi}\, e^{\pi|\mathrm{Im}t|} \left(\int_{-\infty}^{\infty} |\hat{f}|^2 + \int_{-1}^{1} (|\xi|^{-2\mathrm{Re}t} - 1)\, |\hat{f}(\xi)|^2\right).$$

For the second integral, recall supp $f \subset (0,1)$ so $|\hat{f}(\xi)| \leqslant \|f\|_{L_2(0,1)}$ for all real ξ; Thus when $0 \leqslant \mathrm{Re}t < 1/2$.

$$\|J^t f\|^2 = \int_0^1 |J^t f|^2 \leqslant \int_0^\infty |J^t f|^2 \leqslant e^{\pi|\mathrm{Im}t|}\, \|f\|^2 \left(1 + \frac{2}{\pi}\, \frac{\mathrm{Re}t}{1-2\mathrm{Re}t}\right)$$

which gives the desired estimate.

Now we have a semigroup in $(\mathrm{Re}t \geqslant 0)$, and we will examine in particular the behavior on the imaginary axis. First, a technical point: determining the domain of the generator.

<u>Lemma</u> ([3]), Th.23.16.1)

Let $f \in L_2((0,1), \mathbb{C})$ and define

$$F(x) = \int_0^x \log(x-y)\, f(y)\, dy, \quad 0 \leqslant x \leqslant 1 ;$$

the integral converges absolutely (Cauchy inequality) and $F(x) \to 0$ as $x \to 0$.

(i) if $\omega \neq 0$ is any complex number with $\mathrm{Re}\omega \geqslant 0$, and if

$\lim\limits_{t\to 0} \frac{1}{t} (J^{\omega t}f-f) = g$ exists as a limit in $L_2(0,1)$, then F is absolutely continuous with derivate F' in L_2(i.e. $F \in H^1(0,1)$) and $g = \omega(F' + \gamma f)$, where $\gamma = -\Gamma'(1) = \lim\limits_{N\to\infty} \left(\sum\limits_1^N \frac{1}{k} - \log N\right) =$ = Euler's Constant = 0,5772... .

(ii) if $F \in H^1(0,1)$ then as $t \to 0$ in $\operatorname{Re} t \geqslant 0$,

$$\frac{1}{t} (J^t f-f) \to F' + \gamma f \text{ in } L_2(0,1).$$

Thus wheter we consider $(t \to J^t)$ as a semigroup in $(\operatorname{Re} t \geqslant 0)$ or in $\{t \geqslant 0, \operatorname{Im} t = 0\}$, or on any ray or sector in the closed right half-plane, we always obtain the same generator A:

$$D(A) = \{f \in L_2 \mid F \in H^1\} \text{ and}$$

$$\text{for } f \in D(A),\ Af = F' + \gamma f.$$

<u>Proof</u>: It is convenient to do our calculations on $J^t f$ when $\operatorname{Re} t > 0$, and $\operatorname{Re} t$ may be chosen to be large to improve smoothness. Then we extend results by analytic continuation to the open half-plane $\operatorname{Re} t > 0$, and to $\operatorname{Re} t \geqslant 0$ by continuity.

(i) For $\operatorname{Re} \zeta \geqslant 0$, $J^{\zeta+1}f = J^1(J^\zeta f)$ or $J^{\zeta+1} f(x) = \int_0^x J^\zeta f(y)dy =$

$$= \frac{1}{\Gamma(\zeta+1)} \int_0^x (x-y)^\zeta f(y)dy.$$

Taking $\zeta = \omega t$ with $t > 0$ ($\operatorname{Re}\omega \geqslant 0$, $\omega \neq 0$) we find

$$\int_0^x \frac{1}{\omega t} \quad (J^{\omega t} f(y)-f(y))dy = \int_0^x \frac{1}{\omega t} \{ \frac{(x-y)^{\omega t}}{\Gamma(1+\omega t)} - 1\} f(y)dy.$$

By our hypothesis, the limit $t \to 0+$ is

$$\frac{1}{\omega} \int_0^x g(y)dy = \int_0^x \{\log(x-y) - \frac{\Gamma'(1)}{\Gamma(1)^2}\} f(y)dy$$

$$= F(x) + \gamma \int_0^x f,$$

so F is absolutely continuous with derivate $F' = \frac{1}{\omega} g - \gamma f$ in $L_2(0,1)$, as claimed.

(ii) Now assume $F \in H^1$, with $F(0) = 0$ as noted above, so $F = J^1 F'$ and $J^{\zeta-1}(F) = J^\zeta F'$ for $\mathrm{Re}\zeta > 1$. We show $J^\zeta(F') =$
$= \frac{\partial}{\partial\zeta} (J^\zeta f) - \gamma J^\zeta f.$

In fact, for $\mathrm{Re}\zeta > 1$,

$$J^\zeta(F') \ (x) = J^{\zeta-1}(F) \ (x) = \int_0^x \{ \int_z^x \frac{(x-y)^{\zeta-2}}{\Gamma(\zeta-1)} \log(y-z)dy\} f(z)dz$$

$$= \int_0^x \frac{(x-z)^{\zeta-1}}{\Gamma(\zeta)} \quad \{\log \ (x-z) \ \frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\zeta)}{\Gamma(\zeta)} \} f(z)dz.$$

(The inner-integral was evaluated as $\partial/\partial\epsilon \mid_{\epsilon = 0}$ of the identy

$$\int_z^x \frac{(x-y)^{\zeta-2}}{\Gamma(\zeta-1)} (y-z)^\epsilon \ dy = \frac{(x-z)^{\zeta-1+\epsilon} \ (1+\epsilon)}{\Gamma(\zeta+\epsilon)} \ ).$$

This for $\mathrm{Re}\alpha > 1$, $\mathrm{Re}\beta > 1$,

$$J^{\beta} f - J^{\alpha} f = \int_{\alpha}^{\beta} \frac{\partial}{\partial \zeta}(J^{\zeta} f)\, d\zeta = \int_{\alpha}^{\beta} J^{\zeta}(F' + \gamma f)\, d\zeta$$

But the final equation holds (by continuity and analytic continuation) in $\mathrm{Re}\alpha \geqslant 0$, $\mathrm{Re}\beta \geqslant 0$. Allowing $\alpha \to 0$,

$$\frac{1}{\beta}(J^{\beta} f - f) = \frac{1}{\beta} \int_{0}^{\beta} J^{\zeta}(F' + \gamma f)\, d\zeta, \quad \mathrm{Re}\beta \geqslant 0,$$

and then we see this converges to $F' + \gamma f$ (in L_2) as $\beta \to 0$ with $\mathrm{Re}\beta \geqslant 0$, completing the proof.

Now we are ready to close the trap!

The strongly-continuous group $\{J^{it} \mid -\infty < t < \infty\}$ of operators on $L_2(0,1)$ satisfies

$$\| J^{it} \|_{\mathcal{L}(L_2)} \leqslant e^{|t| \pi/2}$$

Now J^{it} has spectrum -as does any bounded linear operator- and the spectral radius

$$r(J^{it}) \leqslant \| J^{it} \|_{\mathcal{L}(L_2)} \leqslant e^{|t| \pi/2}$$

while the inverse of J^{it} is J^{-it}, so

$$\sigma(J^{it}) \subset \{z : e^{|t| \pi/2} \leqslant |z| \leqslant e^{|t| \pi/2}\}$$

(We see below that these sets are equal whenever $t \neq 0$. But the generator of this group is iA, where A is the generator of $\{J^t, t \geqslant 0\}$ (see the lemma if you don't believe me); and $\sigma(A)$ is empty, so $\sigma(iA)$ is empty.

Just to make the situation definite, we show every z in $e^{-|t|\pi/2} < |z| < e^{|t|\pi/2}$ is in $\sigma(J^{it})$. Since the spectrum is always a closed set, this holds egnalty for the closed annulus, when $t \neq 0$. (Hille and Phillips [3,23,16] give as an open problem whether the interior of the annulus is in the spectrum; it is, as we show).

For any complex p with Rep > 0 define

$$g_p(x) = x^p \sqrt{2\text{Re}p + 1}\ ,\ 0 \leqslant x \leqslant 1,$$

so $\|g_p\|_{L_2(0,1)} = 1$. By the definition of J^{it}, we may compute $J^{it}g_p$ to find

$$J^{it}g_p - \frac{\Gamma(p+1)}{\Gamma(p+1+it)}\, g_p = \frac{\Gamma(p+1)}{\Gamma(p+1+it)}\ (x^{it}-1)\ g_p(x).$$

Now $\int_0^1 |x^{it}-1|^2\, |g_p(x)|^2 dx \to 0$ as $\text{Re}p \to +\infty$, for any real t; and we choose p with arg p fixed in $(-\pi/2, \pi/2)$ and $|p| \to \infty$, say

$$p = R_n\, e^{i\theta}$$

$(-\pi/2 < \theta < \pi/2,\ R_n = \exp\ (\alpha+2\pi n/|t|\ )\ \alpha$ and θ fixed)

Then $\Gamma(p+1)/\Gamma(p+1+it) = p^{-it}\,(1+O(|p|^{-1}))$

$$= R_n^{-it}\, e^{-t\theta}(1+O(R_n^{-1}))$$

$$= e^{-t\theta - it\alpha}(1+O(R_n^{-1}))$$

so with $z = \exp\,(-\,(t\theta + it\alpha))$,

$$\| J^{it} g_p - z g_p \|_{L_2} \to 0, \quad \| g_p \|_{L_2} = 1,$$

Any such z is in $\sigma(J^{it})$, and (θ, α) may be chosen freely in $(-\pi/2, \pi/2) \times \mathbb{R}$, so the interior of the annulus above is in the spectrum, as claimed.

4.3. Interpretation

From 1948 (in the first edition of [3]) until 1978, this example was complately mysterious and outside the theoretical structure of spectral theory. Even after 1978, the theorem of Gearhart [1] -extended and simplified by Herbst [2] in 1983- was not applied to this example. We will apply it, and then see it more as an example than a counter-example.

The theorem of Gearhart and Herbst says: if $\{ e^{tA}, t \geq 0\}$ is any strongly continuous semigroup of linear operators on a Hilbert space, then for any $t > 0$, $z \in \mathbb{C}$,

$e^{zt} \in \sigma(e^{At})$ if and only if either $z \in \sigma(A)$, mod $\frac{2\pi i}{t}$,

or $z + \frac{2\pi i n}{t} \notin \sigma(A)$ for all integers n but the resolvent

$(z + \frac{2\pi i n}{t} - A)^{-1}$ is not uniformly bounded as $n \to \pm\infty$.

(No comparable result is known for Banach space semigroups).

We compute and estimate the resolvent of the generator iA of $\{J^{it},\ t \geq 0\}$ on lines Re z = constant, or -what is the same thing- consider $(z-A)^{-1}$ on lines Imz = constant.

First recall

$$(z-A)^{-1} = \int_0^\infty e^{-zt}\, J^t\, dt$$

for Rez large -and in fact, for all complex z, since both sides are entire. Thus

$$(z-A)^{-1}f(x) = \int_0^\infty e^{-zt}dt \int_0^x \frac{(x-y)^{t-1}}{\Gamma(t)}\, f(y)dy$$

$$= \int_0^x E(x-y;z)\, f(y)dy, \qquad 0 \leq x \leq 1,$$

where $$E(u;z) = \int_0^\infty dt\, e^{-zt}\, u^{t-1}/\Gamma(t) \quad \text{for } u > 0.$$

As before

$$\|(z-A)^{-1}\|_{L_2(0,1)} \leq \int_0^1 |E(u;z)|\,du\, \|f\|_{L_2(0,1)}$$

so $$\|(z-A)^{-1}\|_{\mathcal{L}(L_2(0,1))} \leq \int_0^1 |E(u;z)|\,du.$$

There is no difficulty when Rez $\geq$ Const.$> -\infty$.

For example, if $\operatorname{Re} z \geqslant -1$,

$$\int_0^1 du\, |E(u;z)| \leqslant \int_0^1 du \int_0^\infty dt\, e^{-t\operatorname{Re} z} u^{t-1}/\Gamma(t)$$

$$\leqslant \int_0^\infty dt\, e^{-t\operatorname{Re} z}/\Gamma(t+1) \leqslant C/(\operatorname{Re} z+2)$$

where $C = \max_{t\geqslant 0} e^{2t}/\Gamma(t+1) \simeq 238{,}83$.

In fact, by rotating the line of integration, we may estimate $E(u;z)$ also in the left-half plane, provided $|\operatorname{Im} z| > \pi/2$. Since $\overline{E(u;z)} = E(u;\bar{z})$, it suffices to treat the case $\operatorname{Im} z = \eta > \pi/2$. Stirling's approximation shows we way rotate the line of integration to the negative imaginary axis, and then.

$$E(u;\xi+i\eta) = \frac{-i}{u} \int_0^\infty e^{i\tau(\xi-\log u)} \frac{e^{-\tau\eta}}{\Gamma(-it)}\, dt$$

for $0 < u < 1$, $\eta > \pi/2$. Note $|\Gamma(-i\tau)| \simeq \sqrt{\frac{2\pi}{t}}\, e^{-\tau\pi/2}$ as $\tau \to +\infty$

but the integral converges since $\eta > \pi/2$. In fact $\tau \to e^{-\tau\eta}/\Gamma(i\tau)$, along with its derivates, tends to zero exponentially as $\tau \to +\infty$, so we may integrate by parts twice to obtain

$$E(u;\xi+i\eta) = \frac{1}{u(\xi-\log u)^2} \left\{1 + i\int_0^\infty e^{i\tau(\xi-\log u)} \frac{\partial^2}{\partial\tau^2} \left(\frac{e^{-\tau\eta}}{\Gamma(-i\tau)}\right) d\tau\right\}$$

This shows, when $|\xi-\log u| \geqslant 1$,

$$|E(u;\xi+i\eta)| \leqslant C/(u|\xi-\log u|^2)$$

for a constant C depending only on η. In case $|\xi-\log u| \leqslant 1$, we use the earlier representation:

$$|E(u;\xi+i\eta)| \leqslant \frac{1}{u} \int_0^\infty d\tau\, |e^{-\tau\eta}/\Gamma(-i\tau)| = C_1/u,$$

where C_1 depends only on η.

Now for $\eta > \pi/2$ fixed and $\xi \leqslant -1$ (we estimated $\xi \geqslant -1$ above)

$$\int_0^1 du\, |E(u;\xi+i\eta)| = \left(\int_0^{e^{\xi-1}} + \int_{e^{\xi-1}}^{e^{\xi+1}} + \int_{e^{\xi+1}}^{1}\right) |E|\, du$$

$$\leqslant C + 2C_1 + C\left(1-\frac{1}{\xi}\right) \leqslant 2(C+C_1).$$

Thus $\|(\xi+i\eta-A)^{-1}\|_{\mathcal{L}(L_2)}$ is uniformly bounded on $-\infty < \xi < \infty$ provided $\eta > \pi/2$ or $\eta < -\pi/2$. For any η, it is bounded as $\xi \to +\infty$; but we prove it is unbounded when $\xi \to -\infty$, for $-\pi/2 < \eta < \pi/2$. In fact given any $g \in L_2(0,1)$

$$\|(\xi+i\eta-A)^{-1}\|_{\mathcal{L}(L_2)} \geqslant \|g\|_{L_2} / \|(\xi+i\eta-A)g\|_{L_2}.$$

We choose, of course, $g = g_p(x) = x^p \sqrt{2\mathrm{Re}p+1}$, with $\mathrm{Re}p \to +\infty$ $\arg p =$ constant, as before. Then $\|g_p\|_{L_2}=1$ and

$$(J^t g_p)(x) = g_p(x) \frac{x^t\,\Gamma(p+1)}{\Gamma(p+1+t)}$$

so differentiating with respect to t,

$$A g_p(x) = g_p(x) \{ \log x - \frac{\Gamma'(p+1)}{\Gamma(p+1)} \}$$

But $\| g_p(x) \log x \|_{L_2(0,1)} \to 0$ as $\mathrm{Re}\, p \to +\infty$ and $\frac{\Gamma'(p+1)}{\Gamma(p+1)} =$
$= \log p + O(\frac{1}{|p|})$ so we may choose

$$p = e^{-(\xi + i\eta)} \quad ;$$

then $\arg p = -\eta$ is fixed in $(-\pi/2, \pi/2)$, $|p| = e^{-\xi} \to +\infty$ and $\mathrm{Re}\, p \to +\infty$ as $\xi \to -\infty$, and $\| A g_p - (\xi + i\eta) g_p \|_{L_2} \to 0$, $\| g_p \|_{L_2} = 1$, so $\| (\xi + i\eta - A)^{-1} \|_{\mathcal{L}(L_2)} \to +\infty$. In fact, uniformly on $|\eta| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$, $\| (\xi + i\eta - A)^{-1} \| \to +\infty$ as $\xi \to -\infty$.

References

1. L. Gearhart, Spectral theory for contraction semigroups on Hilbert space, Trans. Am.Math.Soc 236 (1978).
2. I. Herbst, The spectrum of Hilbert space semigroups, J. Operator Theory 10 (1983), 87-94.
3. E. Hille and R. Phillips, Functional Analysis and Semigroups, Amer.Math.Soc. 1957.

5. A property of the exponential function

Theorem Let Q be any subset of the complex plane, and define

$$X = \{ x \in \mathbb{R} \mid \exists \text{ sequence } z_n \in Q \text{ with } \operatorname{Re} z_n \to x,\ |\operatorname{Im} z_n| \to +\infty \}.$$

Of course, X may be empty; but if not, it is a closed subset of $\mathbb{R}$ and we set $Z = X + i\mathbb{R} = \{ z \in \mathbb{C} : \operatorname{Re} z \in X \}$. For any real t, e^{tZ} is a (relatively closed) collection of circles or annuli in $\mathbb{C} \setminus \{0\}$.

For almost all real t,

$$e^{tZ} \subset \text{Closure } e^{tQ}$$

and in fact

$$e^{tZ} = \bigcap_{N \geqslant 1} \text{Closure } e^{tQ_N} \setminus \{0\}$$

where $Q_N = \{ z \in Q : |\operatorname{Im} z| \geqslant N \}$. The excluded set of t is of measure zero and also meager (=Baire category I). If X is empty and Q is closed, then e^{tQ} is closed in $\mathbb{C} \setminus \{0\}$ for every real t.

Example Let $Q = \{ 0, \pm i, \pm 2i, \pm 3i, \dots \}$. so $X = \{0\}$ and e^{tZ} is the unit circle for $t \neq 0$. Then e^{tQ} is dense in the unit circle if and only if t/π is irrational.

Proof It is sufficient to treat the case when Q is contained in the imaginary axis and $X = \{0\}$. There is a sequence $\{i\,\omega_n\}_{n \geqslant 1}$ in Q, with $|\omega_n| \to \infty$, and for definiteness suppose $\omega_n \longrightarrow +\infty$.

For any real α and any δ in $0 < \delta < \pi$, define $S_\alpha(\delta) = \{t \in \mathbb{R} \mid$ for some integers $j,k, |\omega_j t - \alpha - 2\pi k| < \delta\}$. Note $\mathrm{dist}\,(e^{i\alpha}; e^{Qt}) < 2 \sin \delta/2$ when $t \in S_\alpha(\delta)$. Clearly $S_\alpha(\delta)$ is open; we prove its (closed) complement has measure zero, hence also has no interior. To do this, we estimate the density of $S_\alpha(\delta)$.

Recall the density of $S_\alpha(\delta)$ at $t \in \mathbb{R}$ is

$$\lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \text{ meas } \{S_\alpha(\delta) \cap (t-\epsilon, t+\epsilon)\},$$

which exists for almost every t, equals 1 a.e. in $S_\alpha(\delta)$, and equals zero a.e. outside $S_\alpha(\delta)$.

Now any real interval of length > 1 contains an integer, so for any real t and positive integer j, there is an integer k in the open interval of length 2 centered at $(\omega_j t - \alpha)/2\pi$. Let $t^* = (2\pi k + \alpha/\omega$; then $|t - t^*| < 2\pi/\omega_j$ and (since $\delta < \pi$).

$$(t^* - \frac{\delta}{\omega_j}, t^* + \frac{\delta}{\omega_j}) \subset S_\alpha(\delta) \cap (t - \frac{3\pi}{\omega_j}, t + \frac{3\pi}{\omega_j})$$

so $\frac{1}{2\omega}$ meas $\{S_\alpha(\delta) \cap (t-\epsilon, t+\epsilon)\} \geqslant \frac{\delta}{3\pi}$ when $\epsilon = \frac{3\pi}{\omega_j}$.

Thus the density of $S_\alpha(\delta)$ at t is $\geqslant \delta/3\pi > 0$, and is never zero. But it is zero a.e. outside $S_\alpha(\delta)$. We conclude that a.e. t is inside $S_\alpha(\delta)$.

Let $\{\alpha_1, \alpha_2, \ldots\}$ be a dense sequence in $[0, 2\pi]$ and let $\tilde{S} = \bigcup_{j,k=1}^{\infty} \mathbb{R}\setminus S_{\alpha_j}(\frac{1}{k})$. $\tilde{S}$ is a meager set of measure zero, since this is true for each $\mathbb{R}\setminus S'_{\alpha_j}(\frac{1}{k})$. If $t \in \mathbb{R}\setminus\tilde{S}$, then $t \in S_{\alpha_j}(\frac{1}{k})$ for all j,k, so $\{e^{i\alpha_j}, j \geq 1\}$ is in the closure of e^{tQ^j}, so the whole unit circle is in the closure of e^{tQ}.

For each $N = 1,2,\ldots$, there is a meager null-set $\tilde{S}_N \subset \mathbb{R}$ such that

$$\text{Closure } e^{tQ_N} \supset \text{unit circle}$$

when $t \in \mathbb{R}\setminus\tilde{S}_N$. Then for $t \in \mathbb{R}\setminus\tilde{S}$, $\tilde{S} = \bigcup_{N\geq 1} \tilde{S}N$,

$$\bigcap_{N \geq 1} \text{closure } e^{tQ_N} \supset \text{unit circle.}$$

When Q is an arbitrary subset of $\mathbb{C}$, we choose a countable dense sequence $\{x_n\}$ in X, and for each n there is a meager null set $\tilde{S}_n$ such that Clousure e^{tQ} contains $\{|z| = e^{tx_n}\}$ for $t \in \mathbb{R}\setminus\tilde{S}_n$. Then for $t \in \mathbb{R}\setminus\bigcup_{n\geq 1}\tilde{S}'_n$, Closure e^{tQ} contains e^{tZ}.

<u>Corollary 1</u> Let $\alpha < \beta$ and suppose $h(z)$ is an analytic function in the strip $\alpha < \mathrm{Re}z < \beta$ which is asymptotically almost-periodic, i.e.

$$h(z) - h_{\pm}(z) \to 0 \text{ as } \mathrm{Im}z \to \pm\infty \quad \text{with} \quad \alpha < \mathrm{Re}z < \beta$$

where $h_{\pm}(.)$ are analytic almost-periodic functions in the strip. Define

$$X_\sigma = \{\mathrm{Re}z \mid h_\sigma(z) = 0, \ \alpha < \mathrm{Re}z < \beta\}, \ \sigma = \pm$$

and let

$$Z_\sigma = \{ z \in \mathbb{C} \mid \mathrm{Re} z \in \text{Closure } X_\sigma \}$$

Then for a.e. real t

$$\text{Closure } \{ e^{\lambda t} \mid \alpha < \mathrm{Re}\lambda < \beta \text{ , } h(\lambda) = 0 \} \supset e^{tZ_+} \cup e^{tZ_-}.$$

Remark In applications to difference equations, functional differential equations of neutral type [1] , hyperbolic systems in one space dimension with general boundary conditions, and some other problems, we have a "characteristic equation" $(h(\lambda) = 0)$ given by an asymptotically almost-periodic analytic function $h(.)$, such that there are nontrivial solutions with exponential time-dependence $e^{t\lambda}$ if and only if $h(\lambda) = 0$. The set $\{e^{t\lambda} \mid h(\lambda) = 0\}$ is contained in the spectrum of the corresponding semigroup.

Proof of Cor.1 Basic properties of analytic almost-periodic functions are described in [1, lemma 3.1-3.3] . One of these [1, lemma 3.2] is:

If $h_\sigma(\lambda_1) = 0$, $\alpha < \mathrm{Re}\lambda_1 < \beta$, there exist $\lambda_2, \lambda_3, \ldots$ with $h_\sigma(\lambda_n) = 0$, $\mathrm{Re}\lambda_n \to \mathrm{Re}\lambda$, and $\sigma.\mathrm{Im}\lambda_n \to +\infty$.

Using Rouche's theorem, we find:

if $h_\sigma(\lambda_1) = 0$, $\alpha < \mathrm{Re}\lambda < \beta$, there exist $\lambda'_2, \lambda'_3, \ldots$ with $h(\lambda'_n) = 0$, $\mathrm{Re}\lambda'_n \to \mathrm{Re}\lambda_1$, and $\sigma.\mathrm{Im}\lambda'_n \to \infty$.

It follows that the set X of the theorem, corresponding to $Q = h^{-1}(0)$, is the closure of $X_+ \cup X_-$, and the Corollary is proved.

In the spectral theory of [1, th, 4.1.], the only point left open is wheter the set of circles (e^{tZ}, in our notation above) is contained in the spectrum of the semigroup. This is proved by Cor.1 for a.e. $t \geq 0$ so we have:

Corollary 2. If $\{ e^{At}, t \geq 0\}$ is the semigroup of [1] with generator A, then for a.e. $t \geq 0$

$$\sigma(e^{At}) \setminus \{0\} = \text{Closure } e^{t\sigma(A)} \setminus \{0\}$$

Remark. Corollary 2, as you may have guessed, was the original motive for this investigation. I tried to prove this about 1971, and failed. In 1981, I found approximately the above argument but concluded only meagerness, not realizing it proved measure zero until 1984. Which shows sufficient patience may compensate a lack of brilliance.

Reference

[1] D. Henry, Linear autonomous neutral FDEs, J. Diff. Eq. 15 (1974), 106-128.

6. Asymptotic behavior of some scalar ODEs and an elementary example of non-minimal ω-limit sets.

A bounded positive-semiorbit of a finite-dimensional autonomous ordinary differential equation (and some infinite -dimensional equations) has a nonempty compact connected invariant ω-limit set. "Invariance" means the ω-limit set is composed of solutions of the equation, but it need not be a single solution. Correcting an example in Hale's book [1], and also in Coleman's article [2], we show the 2-dimensional system (in polar coordinates)

$$\dot{r} = r\,(1-r^2)^3\ ,\quad \dot{\theta} = r^2\sin^2\theta + (1-r^2)^2$$

has solutions $r(t)$, $\theta(t)$ with $r(t)\to 1$ and $\theta(t)\to+\infty$ as $t\to+\infty$. Thus the ω-limit set is the unit circle, which consist of four solutions : the equilibrium points $(r,\theta) = (1,0)$ and $(1,\pi)$, and the two orbits joining these. In the version given by Hale and Coleman, the first equation is $\dot{r} = r(1-r^2)$; but this implies $r(t)\to 1$ exponentially and $\theta(t)$ has a finite limit ($\equiv 0$, mod π), so every solution approaches an equilibrium.

More generally, we study the asymptotic behavior of solutions of the scalar equation

$$(1)\qquad \dot{u} = f(t,u) = f_o(t) + f_1(t)\,u + f_2(t,u)u^2$$

where

$$f_o(t) \to 0,\ f_1(t) \to 0 \text{ and } f_2(t,0) \to a \neq 0$$

as $t \to +\infty$, and give conditions for the existence or nonexistence of solutions tending to zero as $t \to +\infty$.

<u>Theorem 1</u> Assume $f(t,u)$ and its partial derivate $f_u(t,u)$ are continuous on $\{t_0 \leqslant t \leqslant \infty, -r_0 \leqslant u \leqslant r_0\}$ and

$$\overline{\lim_{\substack{t \to \infty \\ u \to 0}}} \frac{|f(t,u)-f_o(t)-f_1(t)u|}{u^2} \leqslant |a| , \quad \overline{\lim_{\substack{t \to \infty \\ u \to 0}}} \frac{|f_u(t,u)-f_1(t)|}{|u|} \leqslant 2|a| .$$

Also assume $m_o = \overline{\lim\limits_{t \to +\infty}} t^2 |f_o(t)|$, $m_1 = \overline{\lim\limits_{t \to +\infty}} t|f_1(t)|$ are finite, $m_1 < 1$ and $m_o < (m_1-1)^2 /4|a|$.

Then there exists a solution $u(t)$ (for t sufficiently large) of $\dot{u} = f(t,u)$ which tends to zero as $t \to +\infty$; in fact,

$$|u(t)| = O(t^{-1}) \quad \text{as} \quad t \to \infty .$$

<u>Remark</u> Many variations are possible by change of variable, for example $u = v - \frac{1}{2a} f_1(t)$ or $u = v \exp(\int_o^t f_1)$. It seems desirable to reduce or eliminate f_1.

<u>Proof</u> The function $r \longrightarrow m_o + (m_1-1)r + |a| r^2$ has a minimum at $r_{min} = (1-m_1)/2|a| > 0$, where it is negative, by our assumptions. We may choose $r > 0$ slighthy less than r_{min}, so that

$$m_0 + m_1 r + |a| r^2 < r$$

and

$$m_1 + 2|a| r < 1.$$

Fix such r, and let $\epsilon > 0$ be sufficiently small that these inequalities remain true when m_0, m_1, $|a|$ are increased by ϵ. For sufficiently large $A > 0$ we have

$$|f(t,u)-f_0(t)-f_1(t)u| \leqslant (|a|+\epsilon)\, u^2$$

$$|f_u(t,u)-f_1(t)| \leqslant 2(|a|+\epsilon)\,|u|$$

$$|f_0(t)| \leqslant (m_0+\epsilon)/t^2 \;, |f_1(t)| \leqslant (m_1+\epsilon)/t$$

on the set

$$\{(t,u) : \quad t \geqslant A\,, \; |u| \leqslant r/A \}\,.$$

Now define

$$S_A \equiv \{\text{continuous } u: (A,\infty) \to \mathbb{R} \mid t|u(t)| \leqslant r \text{ for } t \geqslant A\}$$

a complete metric space with the distance d_A

$$d_A(u,\bar{u}) = \sup_{t \geqslant A} t\,|u(t) - \bar{u}(t)|.$$

Also define $\Phi : S_A \to C([A,\infty), \mathbb{R})$ by

$$\Phi(u)(t) = -\int_t^\infty f(s,u(s))ds, \; t \geqslant A,$$

It is easily verified that $\Phi(S_A) \subset S_A$ and

$$d_A(\Phi(u), \Phi(\bar{u})) \leqslant (m_1+\epsilon+2r(|a|+\epsilon))\, d_A(n,\bar{n})$$

For $u,\bar{u}$ in S_A, so there is a unique fixed point which is the desired solution.

Theorem 2 Suppose $a > 0$, $m_1 \geqslant 0$, $m_0 > (m_1 \pm 1)^2/4a$ and $|f_1(t)| \leqslant m_1/t$ for large t. Then no solution $u(.)$ of $\dot{u} \geqslant a\,u^2 + m_0/t^2 + f_1(t)u$ tends to zero as $t \to +\infty$; in fact, every solution blows up (to $+\infty$) in finite time.

Proof Let $v(t) = tu(t)$; then

$$t\dot{v} = t^2\dot{u} + tu \geqslant m_0 + v - m_1|v| + av^2$$

$$\geqslant a(v+(1\pm m_1)/2a)^2 + m_0 - (1\pm m_1)^2/4a$$

$$\geqslant \text{constant} > 0$$

so $v(t) \geqslant$ (Constant) $\log t > 0$, for large t.

Choose $C_1 > 0$ such that

$$a(v+(1\pm m_1)/2a)^2 + m_0 - (1\pm m_1)^2/2a \geqslant \frac{a}{2}v^2 + v$$

whenever $v \geqslant C_1$. Now $v \geqslant C_1$ for large t, say $t \geqslant t_1$, hence $t\dot{v} \geqslant \frac{a}{2}v^2 + v$ and so $u > 0$ and $\dot{u} \geqslant \frac{a}{2}u^2$ for $t \geqslant t_1$, so u blows up.

Returning to our example

$$\dot{r} = g(r)\ ,\ \theta = r^2 \sin^2\theta + (1-r^2)^2$$

with $g(r) = r(1-r^2)^3$ or $r(1+r^2)$ (or something similar). We study a solution $r(t) \to 1$ as $t \to +\infty$.

We apply the theorem with

$$f(t,\theta) = r(t)^2 \sin^2\theta + (1-r(t)^2)^2,$$

say $\qquad f_0(t) = (1-r(t)^2)^2,\ f_1(t) \equiv 0$, and $a = 1$.

By Theorem 1, there exists a solution $\theta(t) \to 0$ as $t \to +\infty$ provided $\overline{\lim}_{t\to\infty} (1-r(t)^2)^2\, t^2 < 1/4$, which certainly holds if $g(r) = r(1-r^2)$, since $r(t) \to 1$ exponentially. Since $\theta(t)+k\pi$ is also a solution for any integer k, it follows that every solution $\theta(t)$ is bounded. The solutions are monotonic so they have limits, necessarily $\equiv 0 \pmod{\pi}$. Thus in the examples of [1,2], the ω-limit set is always a single point.

Suppose instead that $g(r) = r(1-r^2)^3$; then every solution $r \not\equiv 0$ satisfies $-r(t)+1 \approx 1/(4\sqrt{t})$ so $t^2 f_0(t) = t^2(1-r(t)^2)^2 \approx t/4$ as $t \to +\infty$ and by Theorem 2, there is no solution $\theta(t)$ which tends to zero as $t \to +\infty$. This means there is no bounded solution. A bounded solution has a limit which (after possible shifting, $\theta \to \theta + k\pi$) we may assume to be zero; which is impossible. This is the desired example of a non-minimal ω-limit set.

A more delicate example is obtained when $g(r) = -r(1-r^2)^2$. Then $t^2(1-r(t)^2)^2 \to 1/4$ as $t \to +\infty$ (assuming $r(t) > 1$, so $r(t) \to 1$ as $t \to +\infty$). Thus if $|\lambda| < 1$, any solution (r,θ) of

$$\dot{r} = -r(1-r^2)^2 \quad , \quad \dot{\theta} = r^2 \sin^2\theta + \lambda(1-r^2)^2$$

with $r > 1$ tends to an equilibrium $[(r,\theta) = (1,0)$ or $(1,\pi)]$ as $t \to +\infty$; but for $\lambda > 1$, $r(t) \to 1$ while $\theta(t) \to +\infty$ and the ω-limit set is the whole circle $\{r=1\}$. All these examples may be written as polynomial systems in the plane: the last case, for instance, is

$$\dot{x} = -x(1-x^2-y^2)^2 - y(y^2 + \lambda(1-x^2-y^2)^2)$$

$$\dot{y} = -y(1-x^2-y^2)^2 + x(y^2 + \lambda(1-x^2-y^2)^2).$$

References

1. J.K. Hale, Ordinary Differential Equations, R.E. Krieger Publ., Malabar, Florida, 1980.
2. C. Coleman, "The five-fold way" in Modules in Applied Mathematics, vol. 1, Braun, Coleman, Drew, eds., Springer--Verlag, 1983.

Rebut el 11 d'Agost de 1986

Centre de Recerca Matemàtica

Barcelona, and

Universidade de Sao Paulo

Brazil.