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ON THE ENDOMORPHISM RING OF A FREE MODULE

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Throughout, let R be an (associative) ring (with 1). Let F be the free right R-module, over an infinite set C, with endomorphism ring H.

In this note we first study those rings R such that H is left coherent. By comparison with Lenzing's characterization of those rings R such that H is right coherent [8, Satz 4], we obtain a large class of rings H which are right but not left coherent.

Also we are concerned with the rings R such that H is either right (left) IF-ring or else right (left) self-FP-injective. In particular we prove that H is right self-FP-injective if and only if R is quasi-Frobenius (QF) (this is an slight generalization of results of Faith and Walker [3] which assure that R must be QF whenever H is right self-injective) moreover, this occurs if and only if H is a left IF-ring. On the other hand we shall see that if R is pseudo-Frobenius (PF), that is R is an injective cogenerator in Mod-R, then H is left self-FP-injective. Hence any PF-ring, R, that is not QF is such that H is left but not right self FP-injective.

A left R-module M is said to be FP-*injective* if every R-homomorphism $N \to M$, where N is a finitely generated submodule of a free module F, may be extended to F. In other words M is FP-injective if and only if $\mathrm{Ext}^1(K,M)=0$ for every finitely presented module K. R is said to be *left self*-FP-*injective* if R is FP-injective as left R-module. In [7, 2.3] Jain characterizes left self-FP-injective rings as those rings in which every finitely presented right R-module is torsionless. By using Morita equivalence, this is to say that for each $a \in R_n$ (where R_n denotes the ring of all n by n matrices) the right ideal aR_n is a right annihilator, for all $n \geq 1$.

R is said to be a *right* IF-*ring* if every right injective module is flat. Colby [1] characterizes the right IF-rings as those rings such that every finitely presented right R-module embeds in a free module, by Morita equivalence this is to say: for all $n \geq 1$ given $a \in R_n$ the right ideal aR_n is the right annihilator of a finite subset of R_n. In particular we see that a right IF-ring is left self-FP-injective.

Recall that R is said to be *right coherent* if every finitely generated right ideal is finitely presented, this is equivalent to say that the right annihilator (in R_n) of each $a \in R_n$ is a finitely generated right ideal, for all $n \geq 1$. We also use the fact, discovered by Chase, cf [11, p.43], that R is right coherent if and only if the direct product of any family of copies of R is flat as left R-module.

If S is a subset of R we denote by $r(S)$ and $l(S)$ its right and left annihilator, respectively.

Because F is a free module of infinite rank we see that $F \cong F^n$ all $n \geq 1$. It follows that $H \cong H^n$ as right (or left) H-modules. So H is isomorphic (as ring) to H_n, for all $n \geq 1$. Further every finitely generated right (or left) H-module is cyclic.

From the above remark we see that H is right (left) coherent if and only if the right (left) annihilator of every element of H is finitely generated.

A right R-module is said to be *torsionless* if it is contained in a direct product of copies of R. If M is a right R-module, we denote by $\bar{M}$ the torsionless module associated to M, that is $\bar{M} = M/N$, where $N = \bigcap_{t \in \mathrm{Hom}(M,R)} \mathrm{Ker}\, t$

Proposition 1. H *is left coherent if and only if for each right* R-*module,* M, *generated by a set of cardinality* $\leq |C|$ *and defined by a set of relations of cardinality* $\leq |C|$ *there exists a monomorphism* $\varepsilon : \bar{M} \to F$ *such that every* R-*homomorphism* $\varepsilon(\bar{M}) \to F$ *may be extended to* F.

Proof. Suppose H is left coherent. Let M be a right R-module generated by a set of cardinality $\leq |C|$ and defined by a set of relations of cardinality $\leq |C|$, then there exists $\eta \in H$ such that $F/\mathrm{Im}\ \eta \cong M$ and we may assume $M = F/\mathrm{Im}\ \eta$. Since H is left coherent there exists $\varphi \in H$ such that $H\varphi = \underline{1}(\eta)$. In particular $\varphi\eta = 0$ and so $\mathrm{Im}\ \eta \leq \mathrm{Ker}\ \varphi$. Suppose $t : F \to R$ is an R-homomorphism with $t(\mathrm{Im}\ \eta) = 0$, then $t\eta = 0$ and hence $t \in H\varphi$. So $\mathrm{Ker}\varphi \leq \mathrm{Ker}\ t$, that is $N = \mathrm{Ker}\ \varphi/\mathrm{Im}\ \eta \leq \bigcap_{t \in \mathrm{Hom}(M,R)} \mathrm{Ker} t$ and, the equality holds because $M/N = F/\mathrm{Ker}\varphi$ is torsionless. Thus we have shown that $\bar{M} = F/\mathrm{Ker}\ \varphi$. Set $\varepsilon : \bar{M} \to F$ the natural homomorphism induced by φ. Let $t : \varepsilon(\bar{M}) \to F$ be an R-homomorphism, then $t\varphi \in H$ and $t\varphi\eta = 0$ so $t\varphi = u\varphi$, for some $u \in H$. Clearly $u|\varepsilon(\bar{M}) = t$.

Conversely, let $\eta \in H$ and set $M = F/\mathrm{Im}\ \eta$. Certainly M is generated by a set of cardinality $\leq |C|$, so let $\varepsilon : \bar{M} \to F$ satisfying the hypothesis of the proposition. Consider $\alpha_1 : F \to M$ and $\alpha_2 : M \to \bar{M}$ the natural projections. If $\beta = \varepsilon\alpha_2\alpha_1$, we claim that $H\beta = \underline{1}(\eta)$. Since $\mathrm{Im}\ \eta \leq \mathrm{Ker}\ \beta$ we have $\beta\eta = 0$. On the other hand, if $t \in H$ and $t\eta = 0$ then t induces an R-homomorphism $\bar{t} : \bar{M} \to F$ such that $\bar{t}\alpha_2\alpha_1 = t$. By hypothesis there exists $u \in H$ such that $u\varepsilon = \bar{t}$. Therefore $u\beta = u\varepsilon\alpha_2\alpha_1 = \bar{t}\alpha_2\alpha_1 = t$. This proves the claim and the result follows. □

For completeness we mention without proof the following result of Lenzing.

Theorem 2. (Lenzing [8]). H *is right coherent if and only if every finitely generated right ideal of* R *can be defined by a set of relations of cardinality* $\leq |C|$. □

By comparison of the above theorem and the following result one can obtain a large class of rings which are right but not left coherent.

Recall that a ring R is said to be *right perfect* if the following equivalent conditions hold:

(a) All flat right R-modules are projective

(b) $J(R)$, the Jacobson radical of R, is right T-nilpotent, and $R/J(R)$ is artinian.

(c) R satisfies the descending chain condition on principal left ideals.

For proofs that these are equivalent the reader is referred to [6, 5.7].

A submodule N of a right R-module M is *pure* if $M^m A \cap N^n = N^m A$ for each m x n matrix, A, of elements in R.

It is a consequence of Chase's Lemma, cf [2, 20.20, 20.21], that R is right perfect provided that any direct product of any family of copies of R is a pure submodule of a free right R-module.

Theorem 3. *The endomorphism ring of every free right* R*-module of infinite rank is left coherent if and only if* R *is right perfect and left coherent.*

Proof. Suppose that R is left coherent and right perfect. Let F be a free right R-module generated by an infinite set, say C, and set $H = \mathrm{Hom}_R(F,F)$. We have only to prove that H^I, the direct product of I-copies of H, is right H-flat, for every set I. Since R is right perfect and left coherent, F^I is projective, cf [6, 5.15], so $F^I \oplus T \cong \oplus_J F$, for some R-module T and some set J. Now, as right H-modules, we have the following isomorphisms

$$H^I \cong \mathrm{Hom}(F,F^I), \quad \mathrm{Hom}(F,F^I) \oplus \mathrm{Hom}(F,T) \cong \mathrm{Hom}(F, \oplus_J F).$$

Hence we need only to prove that $\mathrm{Hom}(F, \oplus_J F)$ is H-flat, that is the multiplication map

$$h : \mathrm{Hom}(F, \oplus_J F) \otimes_H I \to \mathrm{Hom}(F, \oplus_J F)$$

is injective, for every finitely generated left ideal I of H. Since H is left Bezout we have that $I = Hf$, for suitable $f \in H$. Suppose now that $\varphi f = 0$, where $\varphi \in \mathrm{Hom}(F, \oplus_J F)$. Let $(f_c)_{c\in C}$ and $(e_b)_{b\in B}$ be R-basis for F and $\oplus_J F$ respectively. We can write $\varphi(f_c) = \sum_{b\in B_c} e_b\, r_{cb}$, where B_c is a finite subset of B for all $c \in C$. Since C is infinite, clearly $|\bigcup_{c\in C} B_c| \leq |C|$ so that we can choose an injective map $i : \bigcup_{c\in C} B_c \to C$. Define now the right R-linear $t : F \to \oplus_J F$ by

$$t(f_c) = \begin{cases} 0 & \text{if } c \notin i(\bigcup_{c\in C} B_c) \\ e_b & \text{if } c = i(b) \end{cases}$$

If $\eta : F \to F$ is the right R-linear map given by $\eta(f_c) = \sum_{b\in B_c} f_{i(b)} r_{cb}$, then it is clear that $\varphi = t\eta$. Moreover $\mathrm{Ker}\, t \cap \mathrm{Im}\, \eta = (0)$, thus from $\varphi f = 0$ we deduce that $\eta f = 0$. Then $\varphi \otimes f = t\eta \otimes f = t \otimes \eta f = 0$. Therefore h is injective.

Conversely, assume H is left coherent for all free right R-module F of infinite rank. Let I be any infinite set, by proposition 1 there exists a monomorphism $\varepsilon : R^I \to \oplus_J R$ such that every R-homomorphism $\varepsilon(R^I) \to \oplus_J R$ can be extended to $\oplus_J R$. Now we will prove that $\varepsilon(R^I)$ is a pure submodule of $\oplus_J R$. For if suppose that $A = (a_{ij})$ is a $p \times k$ matrix over R and $(f_1,\dots,f_p)A =$ $= (\varepsilon(m_1),\dots,\varepsilon(m_k))$, where $f_i \in \oplus_J R$ and $m_j \in R^I$. Clearly we may assume there is an injective map, say $j : I \to J$ (for this it suffices to choose, from the beginning, an infinite set I such that $|I| > R$). For each $i \in I$ denote by $\pi_i(\pi_{j(i)})$ the natural projection $R^I \to R_i = R$ $(\oplus_J R \to R_{j(i)} = R)$ and let $e_i : R = R_{j(i)} \to \oplus_J R$ be the natural embedding. Set $t_i = e_i\pi_i$ then, by hypothesis, there exists $u_i \in \mathrm{End}_R(\oplus_J R)$ such that $u_i\varepsilon = t_i$. Thus we have $t_i(m_\alpha) =$ $= u_i(f_1 a_{1\alpha} + \dots + f_p a_{p\alpha})$, for $1 \leq \alpha \leq k$ and so $\pi_i(m_\alpha) = \pi_{j(i)} t_i(m_\alpha) =$

$(\pi_{j(i)}u_i(f_1))a_{1\alpha} + \dots + (\pi_{j(i)}u_i(f_p))a_{p\alpha}$. If we define $g_s \in R^I$, the element whose i th component is $\pi_{j(i)}u_i(f_s)$, $s = 1,\dots,p$, then

$$(\varepsilon(g_1),\dots, \varepsilon(g_p))A = (\varepsilon(m_1),\dots, \varepsilon(m_k)).$$

Hence $\varepsilon(R^I)$ is pure in $\oplus_J R$. It follows from Chase's Lemma that R is right perfect. Since a pure submodule of a flat module is flat, we see that R^I is flat, for any set I, and so R is left coherent. □

Example. Let $F = \oplus_I Z$ be a free Z-module (Z denotes the ring of rational integers). By Lenzing's Theorem, the ring $\mathrm{End}_Z(F)$ is right coherent.

If $|I| \leq \aleph_0$ then every torsionless Z-module, M, generated by $|I|$-elements is contained (as a submodule) in $\prod_{i=1}^{\infty} Z$ and so M is free by Specker's Theorem [5]. It follows from proposition 1 that $\mathrm{End}_Z(F)$ is left coherent.

If $|I| \geq \aleph_1$ it follows from the fact that $\prod_{i=1}^{\infty} Z$ is not Z-free and proposition 1 that $\mathrm{End}_Z(F)$ is not left coherent.

Now we shall characterize those rings R such that H is left self-FP-injective or right IF-ring. First we need a lemma.

Lemma 4. *If* $\eta \in H$ *then* ηH *is the right annihilator of a subset* S *of* H *if and only if* $\bigcap_{\varphi\in S} \mathrm{Ker}\,\varphi = \mathrm{Im}\,\eta$.

Proof. Suppose $\eta H = r(S)$, then $S\eta = 0$ and so $\mathrm{Im}\,\eta \leq \bigcap_{\varphi\in S} \mathrm{Ker}\,\varphi$. On the other hand, let $x \in \bigcap_{\varphi\in S} \mathrm{Ker}\,\varphi$ and take any element $f \in F$ belonging to an R-basis of F. If $fR \oplus G = F$, define $t \in H$ by $t(f) = x$ and $t(G) = 0$. Then $\mathrm{Im}\, t \leq \bigcap_{\varphi\in S} \mathrm{Ker}\,\varphi$ and hence $St = 0$. By hypothesis $t \in \eta H$ and thus $x = t(f) \in \mathrm{Im}\,\eta$.

Conversely, if $\bigcap_{\varphi\in S} \mathrm{Ker}\,\varphi = \mathrm{Im}\,\eta$, then $S\eta = 0$ and so $\eta H \leq r(S)$. Let $t \in H$ such that $St = 0$. Then we have $\mathrm{Im}\, t \leq \mathrm{Im}\,\eta$. Set $(f_i)_{i\in C}$ a basis of F over R, then there exist elements $(s_i)_{i\in C}$ of R such that $t(f_i) = \eta(s_i)$. If we define $u \in H$ by $u(f_i) = s_i$ we obtain $t = \eta u$ as required. □

Theorem 5. (i) H *is left self-FP-injective if and only if every right R-module defined by a set of relations of cardinality* $\leq |C|$ *is torsionless.*

(ii) H *is a right* IF-*ring if and only if every right* R-*module defined by a set of relations of cardinality* $\leq |C|$ *is contained in a free module.*

Proof. (i) Suppose H is left self-FP-injective and assume that M is a right R-module such that $0 \to U \to L \to M \to 0$, where L is free and U is generated by a set of cardinality $\leq |C|$. It is then clear that $M \cong (F/W) \oplus L'$, where W is a homomorphic image of F and L' is free. In order to prove that M is torsionless it suffices to prove that F/W so is. Let $\eta \in H$ such that $\operatorname{Im}\eta = W$. Since H is left self-FP-injective we know that ηH is the right annihilator of a subset S of H. It follows from lemma 4 that $\operatorname{Im}\eta = \bigcap_{\varphi \in S} \operatorname{Ker}\varphi$ and so $F/\operatorname{Im}\eta \hookrightarrow \prod_{\varphi \in S} F/\operatorname{Ker}\varphi \hookrightarrow \prod_{\varphi \in S} F$. Thus F/W is torsionless.

Conversely, suppose that every right R-module defined by a set of relations of cardinality $\leq |C|$ is torsionless. We need only to prove that ηH is a right annihilator for each $\eta \in H$. Since $F/\operatorname{Im}\eta$ is defined by a set of relations of cardinality $\leq |C|$ we see that it is torsionless. Hence there is a homomorphism $t : F \to \prod_{i \in S} F$ with $\operatorname{Im}\eta$ as kernel. If $\pi_i : \prod_{i \in S} F \to F$ denotes the natural projection we see that $\operatorname{Im}\eta = \operatorname{Ker} t = \bigcap_{i \in S} \operatorname{Ker} \pi_i t$. Now the result follows from lemma 4. □

The proof of (ii) is similar.□

Faith-Walker [3] and Sandomierski [10] have shown that H is right self-injective if and only if R is QF. In our next result we prove that R is QF by assuming only that H is right self-FP-injective, this allows to us to characterize the rings R such that H is right self-FP-injective and then we obtain examples of rings H that are left but not right self-FP-injective.

Theorem 6. *The following statements are equivalent*

(i) H *is a left* IF-*ring*

(ii) H *is right self-* FP-*injective*

(iii) R *is* QF

(iv) H *is right self-injective*

Proof. Trivially (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Suppose H is right self-FP-injective. First we prove that for each right ideal I of R there exists a finite subset $J \leq I$ such that $\mathcal{l}(I) = \mathcal{l}(J)$. Suppose this is not the case and choose $x_0 \in I$, then $\mathcal{l}(x_0) \neq \mathcal{l}(I)$ so there exists $y \in \mathcal{l}(x_0)$ with $yI \neq 0$. If $x_1 \in I$ and $yx_1 \neq 0$ we have $\mathcal{l}(x_1, x_0) < \mathcal{l}(x_0)$, by this procedure we can construct an infinite descending chain $\mathcal{l}(S_0) > \mathcal{l}(S_1) > \dots$, where $S_0 < S_1 < \dots$ and each S_n is a finitely generated right ideal of R. Set $T = \bigcup_{i \geq 0} S_i$, then T is a countably generated right ideal of R. Now we claim that every R-homomorphism $t : L \to F$, where L is a countably generated right ideal of R may be extended to R. Let us fix $i_0 \in C$ and consider $\alpha_0 : R \to F$ defined by $\alpha_0(r) = (r_i)$ with $r_{i_0} = r$ and $r_i = 0$ if $i \neq i_0$. Set $L' = \alpha_0(L)$ so that L' is countably generated and hence there is an R-homomorphism $\beta : F \to F$ such that Im $\beta = L'$. Define $\delta : F \to F$ by $\delta = t\pi_0\beta$, where $\pi_0 : F \to R$ is the natural projection on the i_0 th component. By hypothesis H is right self-FP-injective so $H\beta = \mathcal{l}(S)$ where S is contained in H. Since $\beta S = 0$ we have that $\delta S = 0$ thus $\delta \in H\beta$. Let $h_0 \in H$ such that $\delta = h_0\beta$. Now we prove that $h : R \to F$ defined by $h = h_0\alpha_0$ is an extension of t. If $x \in L$ then $x = \pi_0(\beta(y))$, $y \in F$. Thus $h(x) = h_0(\beta(y)) = \delta(y) = t\pi_0(\beta(y)) = t(x)$ as claimed. Now choose a sequence $x_n \in R$ such that $x_n \in \mathcal{l}(S_n) \setminus \mathcal{l}(S_{n+1})$. Define $\varphi : T \to \bigoplus_{i=1}^{\infty} R \leq F$ by $r \to (x_n r)$, clearly φ is well-defined and, by the above φ is left multiplication by some element of F, so there exists $m \geq 1$ with $x_n r = 0$ for all $r \in T$ and $n \geq m$. But this contradicts the choice of the x_n's.

In order to prove that R is QF it suffices to prove that F is self-injective as right R-module, cf [2, 24.18, 24.20]. With the above notation suppose $\eta : I \to F$ is an R-homomorphism. Since J is finitely generated η/J is left multiplication by some $t \in F$. Let $x \in I$, then $J + xR$ is finitely generated so that $\eta/J + xR$ is left multiplication by some $t' \in F$. Clearly $t - t' \in$

[illegible]$_F$(J, [illegible]$_F$(i). Hence [illegible] (x). This shows that [illegible] is left multiplication by t.

(iii) ⇒ (iv) Is due to Sandomierski [10] and (iv) ⇒ (iii) to Faith and Walker [3]. Since trivially (iv) ⇒ (ii), the result will follow if we prove that H is right coherent whenever R is QF. Obviously R is right coherent so H is right coherent by theorem 2. □

Corollary 7. *Let* R *be a ring such that every right* R*-module is torsionless but* R *is not* QF. *Then the endomorphism ring of any free right* R*-module of infinite rank is left*-FP*-injective but not right self*-FP*-injective.*

Proof. It follows from theorem 5(i) and theorem 6. □

Notice that the rings of corollary 7 occur in nature. For example if R is an injective cogenerator in mod-R (that is R is PF) it is clear that every right R-module is torsionless but there are examples due to Osofsky, cf [2, pp. 213-216], of PF rings not QF.

I suspect there are rings R with the property that for some infinite cardinal c the endomorphism ring of a free right R-module over a set of c-elements is left but not right IF-ring. In view of theorem 5(ii) and theorem 6 this is true if the following question has negative answer.

Question 1. *Let* R *be a ring and let c be an infinite cardinal. If every right* R*-module defined by a set of relations of cardinality* $\leq$ c *is contained in a free module. Is* R QF?

Theorem 5(ii) says that if the endomorphism ring of every free right R-module is a right IF-ring then every right R-module is contained in a free module. By a well-known theorem of Faith and Walker [2, 24.12] this happens if and only if R is QF. It seems to be unknown if R is QF by assuming only that R is a right FGF-ring (any finitely generated right R-module embeds in a free R-module), (the reader is referred to [4] for a discussion on this problem). We conjecture that R is not QF even in the case that every countably generated right R-module embeds in a free module. If this is true then Question 1 would have a negative answer.

We shall see as the proof of Osofsky's theorem [9, Theorem 1] may be slightly modified in order to prove that if R is a right FGF-ring (in fact, we need only that every cyclic right R-module embeds in a free module) such that $E(R)$, the injective hull of R, (as right R-module) embeds in a free module, then R is QF. In particular, this says that for a given ring R there is a cardinal c such that if every right R-module defined by a set of relations of cardinality $\leq c$ is contained in a free module, then R is QF.

For any ring R denote by $\Omega(R)$ the set of isomorphism classes of simple right R-modules, and if M is a right R-module we denote by $C(M)$ the set of isomorphism classes of simple submodules of M.

Theorem 8. *Let R be a ring which posseses a finitely generated projective and injective right R-module P with $|\Omega(R)|\leq|C(P)|$ then $|\Omega(R)|<\infty$.*

Proof: By the theorem of Morita we need only consider the case where P is cyclic, say $P=eR$ for some idempotent e in R. Since $|\Omega(R)|\leq|C(P)|$ there exists an injective map $F:\Omega(R)\to C(P)$. Assume $\Omega=\Omega(R)$ is infinite. Using Tarski's Theorem Ω can be decomposed into a class Γ of subsets of Ω with $|\Gamma|>|\Omega|$ and for all $X, Y\in\Gamma$ $|X|=|Y|>|X\cap Y|$ if $X\neq Y$.

For each $A\leq\Omega$ set $S(A)=\Sigma U$ where the summation is taken over all simple submodules U of P such that $U\in F(M)$ for some $M\in A$. Notice that $PS(A)=S(A)$. Let $E(A)$ be an injective hull of $S(A)$ contained in P.

CLAIM I. $E(A)=fR$ *where* $f\in eRe$ *is an idempotent.*

Since $E(A)$ is a direct summand of R it is generated by an idempotent $g\in R$, that is $E(A)=gR\leq eR$ so $g=eg$. On the other hand $geS(A)=S(A)$ so $geR\leq_e gR$. But ge is idempotent, hence $geR=gR$ and $f=ege$ is the desired idempotent.

CLAIM II. If $E(A)=fR$ *with* $f\in eRe$ *an idempotent then* $\overline{f}$ *is central in*

$\overline{eRe}$. For each $\bar{a}$ R, a denotes $a+J$ where J is the Jacobson radical of R.

If $x \in eRe$ then $(e-f)xfS(A) \leq (e-f)s(A)=0$. Furthermore $(e-f)xf(e-f)=0$. Inasmuch $S(A) \leq_e fR$ we have $S(A) \oplus (e-f)R \leq_e fR \oplus (e-f)R=eR$. Hence $\underline{r}_{eR}((e-f)xf) \leq_e eR$, that is $(e-f)xf \leq J(eRe) \leq J(R)$ and so $(\bar{e}-\bar{f})\bar{R}\bar{f} = \bar{0}$. Since $\bar{R}$ is semiprime also $\bar{f}\bar{R}(\bar{e}-\bar{f}) = \bar{0}$. From these it follows that $\bar{f}$ is central in $\overline{eRe}$.

CLAIM III. If $E(A)=fR$ *and* gR *is an injective hull of* $S(A)$ *contained in* eR *with* f,g *idempotents in* eRe *then* $\bar{f}=\bar{g}$.

Clearly $(g-gfg)S(A)=0$ and, since $S(A) \leq_e gR$ and gR is injective, it follows that $\bar{g}=\overline{gfg}$. According to CLAIM II $\bar{f}$ and $\bar{g}$ commute so that $\bar{g}=\overline{gf}$. By symmetry $\bar{f}=\overline{fg}$ and thus $\bar{f}=\bar{g}$.

If $E(A)=fR$ where f is an idempotent of eRe we set $e_A=\bar{f}$. According to CLAIM I,II,III, e_A depends on A only.

We shall prove the following

(i) $e_A\bar{R} \leq e_B\bar{R}$ if and only if $A \subseteq B$

(ii) $e_Ae_B=e_{A\cap B}$ for all $A,B \subseteq \Omega$

(iii) $e_A\bar{R} + e_B\bar{R} \subseteq e_{A\cup B}\bar{R}$

(i) If $A \subseteq B$ then $S(A) \subseteq S(B)$. Choose an injective hull $E(A)$ such that $E(A) \subseteq E(B) \leq eR$. Then $e_A\bar{R} \leq e_B\bar{R}$.

Conversely, if $e_A\bar{R} \leq e_B\bar{R}$ then $e_A\bar{R}$ is a direct summand of $e_B\bar{R}$ and thus there exist $\pi: e_B\bar{R} \to e_A\bar{R}$ and $\varepsilon: e_A\bar{R} \to e_B\bar{R}$ with $\pi\varepsilon=1$. Since $E(B)$ is projective we obtain a commutative diagram

$$\begin{tikzcd}
 & E(A) \arrow[d, "\pi_A"'] & E(B) \arrow[l, "f"'] \arrow[d, "\pi_B"] \\
0 & e_A R \arrow[l] \arrow[r, shift left, "\varepsilon"] & e_B R \arrow[l, shift left, "\pi"]
\end{tikzcd}$$

where π_A, π_B denote the natural projections. Then $f(E(B)) + E(A)J=E(A)$

and by Nakayama's Lemma $f(E(B))=E(A)$. If $M\in A$ choose $U\in F(M)$ Since $E(A)\lesssim E(B)$ there exists $V\in F(N)$ with $N\in B$ and $V\approx U$. Therefore $F(M)=F(N)$ and since F is 1-1 $M=N\in B$. Thus $A\subseteq B$ and (i) follows.

(ii) If $A,B\subseteq\Omega$ then it is clear that $S(A\cap B)=S(A)\cap S(B)\leq_e E(A)\cap E(B)$. Then $E(A)\cap E(B)\leq E(A\cap B)$ and so $fR\cap gR\leq hR$ with f,g,h idempotents of eRe such that $\bar{f}=e_A$, $\bar{g}=e_B$ and $\bar{h}=e_{A\cap B}$. Let $0\neq x\in e_Ae_B\bar{R}$. Since $\bar{R}$ is semiprime $x\bar{R}e_A\neq 0$. Hence $x\bar{R}e_A$ is a nonzero right ideal of the regular ring $e_A\bar{R}e_A$ so that it contains a nonzero idempotent say $\bar{u}\in x\bar{R}e_A$. Inasmuch $e_A\bar{R}e_A$ is right self-injective we can choose $u\in fRf$ to be an idempotent. But then uR is a direct summand of fR and hence injective. On the other hand $S(A)\leq_e fR$ implies $uR\geq U$ where $U\in F(M)$ for some $M\in A$. Thus uR contains an injective hull, $\hat{U}$, of U and so $\bar{u}\bar{R}$ contains the simple module $\hat{U}+J/J\cong\hat{U}/\hat{U}J$ where $U\in F(M)$ for some $M\in A$. According to CLAIM II, $e_Ae_B=e_Be_A$ so, by symmetry, $\hat{U}/\hat{U}J\approx\hat{V}/\hat{V}J$ where $V\in F(N)$ for some $N\in B$. But then $\hat{U}\approx\hat{V}$ and so $U\approx V$, which implies $F(M)=F(N)$ and, since F is injective, $M=N$. Hence $U\leq hR$. Choose an injective hull, H, of $S(A\cap B)$ such that $\hat{U}\leq H\leq eR$. Then, according to CLAIM III, $H+J/J=e_{A\cap B}\bar{R}$ and thus $\hat{U}+J/J\leq e_{A\cap B}\bar{R}$. Therefore we have shown that $\mathrm{Soc}(e_Ae_B\bar{R})$ is essential and contained in $e_{A\cap B}\bar{R}$. We conclude that $e_Ae_B\bar{R}=e_{A\cap B}\bar{R}$, but this implies $e_Ae_B(\bar{e}\bar{R}\bar{e})=e_{A\cap B}(\bar{e}\bar{R}\bar{e})$. Inasmuch e_Ae_B and $e_{A\cap B}$ are central in $\bar{e}\bar{R}\bar{e}$ we see that $e_Ae_B=e_{A\cap B}$.

(iii) It follows from (i)

Let $I=\Sigma e_A\bar{R}$, where the summation is taken over all subsets A of Ω such that $|A|<|X|$ for $X\in\Gamma$. Now for each $X\in\Gamma$ set $I_X=I+(e-e_X)\bar{R}$. Since X is infinite it is not contained in a finite union sets of cardinality $<|X|$. By (iii) and (i), $e_X\notin I$ and so it is clear that $e_X\notin I_X$. On the other hand, if $Y\in\Gamma$ and $Y\neq X$ then $|X\cap Y|<|X|$ and so $e_Xe_Y=e_{X\cap Y}\in I$. Thus $e_Y=(1-e_X)e_Y+e_Xe_Y\in I_X$. For each $X\in\Gamma$ fix a maximal submodule of $\bar{e}\bar{R}$, J_X, containing I_X so that we produce a family $M_X=\bar{e}\bar{R}/J_X$, $X\in\Gamma$ of simple right

$\overline{R}$-modules. By the above $M_X e_Y=0$ and $M_Y e_Y \neq 0$ for all $X, Y \in \Gamma$, $X \neq Y$. So $|\Omega(\overline{R})| > |\Omega(R)|$ noting that $\Omega(\overline{R}) = \Omega(R)$ we get a contradiction. The theorem is proved.

Corollary 9. *Let* R *be a ring such that every cyclic right module is contained in a free right* R*-module and the injective hull of* R *is projective. Then* R *is* QF.

Proof: By hypothesis R contains a copy of each simple right R-module. Hence E, the injective hull of R, is an injective cogenerator; moreover E is projective so it is finitely generated. By theorem 8 $|\Omega(R)| < \infty$. It follows from the proof of [2, Proposition 24-9] that R is right self-injective. By the proof of Theorem 3.5 A of [4] we conclude that R is QF. □

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