Pub. Mat. UAB
Vol. 26 Nº 2 Juny 1982

THE ALGEBRA OF THIN OPERATORS IS DIRECTLY FINITE

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Rebut el 15 de Juliol del 1981

Let H be an infinite-dimensional Hilbert space, K the algebra of compact operators on H, $T = K + \mathbb{C}1$ the algebra of all operators $x = a + \lambda 1$ with $a \in K$ and λ complex. (Such operators x are said to be *thin* [4] (because their essential spectrum reduces to a single point).) The aim of this note is to prove that every element of T that is left-invertible in T is in fact invertible in T.

We begin with the observation that T is a C*-algebra with unity; thus for every $x \in T$ one has $(x^*x)^{1/2} \in T$, so that T satisfies the "square root" axiom (SR) of [6,p.90].

LEMMA 1. *If* $x \in T$ *and* $xx^* \leq x^*x$, *then* $xx^* = x^*x$.

Proof [1, p.1175, Corollary 7]. The essential point is that a compact operator satisfying the inequality is normal, a result due originally to C.R. Putnam [7, p.1029, Corollary 3].

LEMMA 2. *The idempotens of* T *are the operators* e, 1-e, *where* e *runs over the idempotent operators of finite rank.*

Proof. By "operator" we mean bounded linear operator. The essential point of the proof is that an idempotent compact operator has finite rank.

Idempotents e, f of a ring R are said to be *equivalent* (in R), written $e \sim f$, if there exist elements x, y in R such that $xy = e$ and $yx = f$ (replacing x, y by exf, fye, one can suppose $x \in eRf$, $y \in fRe$) [6, p.22]. Projections (= self-adjoint idempotents) e, f of a ring with involution are said to be **-equivalent* if there exists an element x such that $xx^* = e$ and $x^*x = f$.

PROPOSITION. *If* x, y *are thin operators such that* $xy = 1$, *then* $yx = 1$.

<u>Proof</u>. In the language of ring theory, we are asserting that the ring T is "directly finite" [5, p.49]. Let F be the algebra of operators on H of finite rank, $A = F + \mathbb{C}1$; thus A is a*-subalgebra of T and, by Lemma 2, A contains every idemptent of T. Since F and $A/F \cong \mathbb{C}$ are both regular rings, A is a regular ring [5, p.2, Lemma 1.3]; since, moreover, the involution of A is proper ($aa^* = 0$ implies $a = 0$), A is *-regular in the sense of von Neumann [2, p.229].

If x, y are elements of T such that $xy = 1$, then $e = yx$ is an idempotent of T such that $e \sim 1$ in T. As noted above, $e \in A$; since A is *-regular, there exists a projection $f \in A$ such that $fA = eA$ [2, p.229, Proposition 3]. Then $f \sim e$ in A [6, p.21, Theorem 14], a fortiori $f \sim e$ in T; already $e \sim 1$ in T, so $f \sim 1$ in T by transitivity. Since T contains square roots of its positive elements, it follows that the projections f, 1 are *-equivalent in T [6, p.35, Theorem 27], say $x \in T$ with $xx^* = f$, $x^*x = 1$. By Lemma 1, $f = 1$; then $eA = fA = A$ shows that $e = 1$, that is, $yx = 1$. We remark that the ring A is studied in detail in [3].

The proposition can obviously be reformulated as follows: if a and b are compact operators such that $a + b + ab = 0$, then $ab = ba$.

Addendum. 1. Israel Halperin has generalized the Proposition to operators in Banach space [C.R. Math. Rep. Acad. Sci. Canada 3 (1981), 33-35].

2. A referee has pointed out that a brief alternate proof can be based on the index theory of Fredholm operators.

3. G.A. Elliott observes (in a letter) that the proposition extends to any AF-algebra with unity (indeed, that every matrix algebra over such an algebra is directly finite).

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