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SUMS OF INDEPENDENT RANDOM VARIABLES AND SUMS OF THEIR SQUARES

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Let $\{X_{nj}: j=1,\dots,k_n,\ n\in\mathbb{N}\}$ be a triangular array of row-wise independent random variables, $S_n=\Sigma_j X_{nj}$ the row sums and $T_n=\Sigma_j X_{nj}^2$ the row sums of squares. Raikov (1938) proved that S converges weakly to a Gaussian law if and only if T_n converges in probability to a constant. Hall (1978) shows that if S_n converges to a Poisson law with parameter λ, then so does T_n. In this note we give the exact relation between tightness and convergence of $\{\mathcal{L}(S_n)\},\{\mathcal{L}(T_n)\}$ and $\{\min(1,x^2)\Sigma_j d\mathcal{L}(X_{nj})\}$ for infinitesimal arrays; these results contain those of Raikov and Hall as particular cases. The tightness relations proved to be useful in some work with M.B.Marcus on the central limit theorem in C(S). I acknowledge Prof. M.Marcus for the correspondence that led to this note (as a byproduct).

The notation will be as follows: $\{X_{nj}: j=1,\dots,k_n,\ n\in\mathbb{N}\}$ will be a triangular array of row-wise independent random variables ($\{X_{nj}\}$ for short), $S_n=\Sigma_j X_{nj}$, $T_n=\Sigma_j X_{nj}^2$, $X_{nj\tau}=X_{nj}I_{\{|X_{nj}|\le\tau\}}$, $S_{n,\tau}=\Sigma_j X_{nj\tau}$, $(\tau>0)$, and $\{\tilde{X}_{nj}\}$ will denote independent symmetrizations of $\{X_{nj}\}$. $\{X_{nj}\}$ is <u>infinitesimal</u> if $\lim_n \max_j P\{|X_{nj}|>\varepsilon\}=0$ for all $\varepsilon>0$.

We refer to Gnedenko and Kolmogorov (1968, Theorem 25.1) or to Araujo and Giné (1980, Theorem 2.4.7) for the general central limit theorem on the line (CLT).

The following is our main observation. It elaborates on exercise 2.5.3 of Araujo and Giné (1980) and its proof is inspired on their proof of the converse CLT.

<u>Theorem 1</u>. Let $\{X_{nj}\}$ be a triangular array of row-wise independent rv's. Then $\{\mathcal{L}(\Sigma_j X_{nj}^2)\}$ is tight if and only if the family measures

$$d\nu_n(x)=\min(1,x^2)\Sigma_j d\mathcal{L}(X_{nj})(x)$$

is uniformly bounded and tight.

Proof. Assume $\{\mathcal{L}(\Sigma_j X_{nj}^2)\}$ tight. Then by positivity, so are $\{\mathcal{L}(\Sigma_j X_{nj\tau}^2)\}_{n=1}^{\infty}$ and $\{\mathcal{L}(X_{nj}^2)\}_{n,j}$. The converse Kolmogorov and Lévy inequalities give

$$E(\Sigma_j(X_{nj\tau}^2-EX_{nj\tau}^2))^2 \leq c_{\tau,d}/[1-2P\{|\Sigma_j(X_{nj\tau}^2)^{\sim}|>d\}]$$

where $c_{\tau,d}$ is a finite constant for each τ, $d>0$. The tightness of $\{\mathcal{L}(\Sigma_j(X_{nj\tau}^2)^{\sim})\}$ therefore implies $\sup_n E(\Sigma_j(X_{nj\tau}^2-EX_{nj\tau}^2))^2<\infty$, hence the tightness of $\{\mathcal{L}(\Sigma_j X_{nj\tau}^2-\Sigma_j EX_{nj\tau}^2)\}$ by Chebyshev. $\{\mathcal{L}(\Sigma_j X_{nj\tau}^2)\}$ being tight, we conclude

$$\sup_n \Sigma_j EX_{nj\tau}^2<\infty \tag{1}$$

for all $\tau>0$. As is well known, Lévy's ineqiality gives that if $\{\eta_i\}$ are independent symmetric, then

$$\Sigma_{i=1}^n P\{|\eta_i|>\delta\} \leq -\log(1-2P\{|\Sigma_i\eta_i|>\delta\})$$

for all $\delta>0$(as observed by Feller (1971), page 149). Hence, using Fubini we can conclude that there exist $\beta>0$ and $x_{nj}\in\mathbb{R}$ such that

$$\sup_n\Sigma_j P\{|X_{nj}^2-x_{nj}|>\beta\} < 1/2.$$

Since $\{\mathcal{L}(X_{nj}^2)\}_{n,j}$ is tight, there exists $M>0$ such that $\sup_{n,j}P\{X_{nj}^2>M\}<1/2$, which implies that $|x_{nj}|\leq M+\beta$. So, there exists $\tau>0$ such that

$$\sup_n\Sigma_j P\{X_{nj}^2>\tau\} < 1/2.$$

If we apply this to r independent copies of S_n, $r\in\mathbb{N}$, we conclude that there exists $\tau_r>0$ such that

$$\sup_n r\Sigma_j P\{X_{nj}^2>\tau_r\} < 1/2.$$

This proves that $\{\Sigma_j\mathcal{L}(X_{nj})|\,\|x\|>\tau^{1/2}\}$ is uniformly bounded and tight, hence by (1), so is $\{\min(1,x^2/\tau)\Sigma_j\mathcal{L}(X_{nj})\}$. It is trivial to see that if for some $\tau>0$ these measures are uniformly bounded and tight, the same is true for each $\tau>0$, in particular for $\tau=1$.

Conversely, assume now that $\{\nu_n\}$ is uniformly bounded and tight. Then for all τ, $t>0$,

$$P\{T_n>t\}\leq P\{\Sigma_j X_{nj\tau}^2>t,\ |X_{nj}|\leq\tau,\ j=1,\dots,k_n\}$$
$$+\Sigma_j P\{|X_{nj}|>\tau\}\leq \Sigma_j EX_{nj\tau}^2/t+\Sigma_j P\{|X_{nj}>\tau\} .$$

Given $\varepsilon > 0$ choose $\tau > 0$ such that the last sum is not greater than $\varepsilon/2$ and then t such that $\Sigma_j EX^2_{nj\tau}/t < \varepsilon/2$. Hence, $\{\mathcal{L}(T_n)\}$ is tight. ◻

Since

(2) $$E(X_{nj}-EX_{nj1})^2 I_{\{|X_{nj}| \le 1\}} - EX^2_{nj1} =$$

$$= -(1+P\{|X_{nj}| > 1\})(EX_{nj1})^2,$$

the previous theorem together with theorem 2.45 in Araujo and Giné(1980),give:

Corollary 2. Let $\{X_{nj}\}$ be an infinitesimal array such that

(3) $$\sup_n \Sigma_j (EX_{nj1})^2 < \infty.$$

Then, $\{\mathcal{L}(S_n - ES_{n,\tau})\}$ is tight if and only if $\{\mathcal{L}(\Sigma_j X^2_{nj})\}$ is.

Remark. Condition (3) is satisfied if:

(a) $\{X_{nj}\}$ is symmetric, but in this case it is not necessary to assume infinitesimality (use Araujo and Giné (1980). Cor. 2.5.7), and

(b) for $\{Z_{nj} = X_{nj} - EX_{nj\tau}\}$, any $\tau > 0$, if either $\{\mathcal{L}(S_n - ES_{n,\tau})\}$ or $\{\mathcal{L}(\Sigma_j (X_{nj} - EX_{nj\tau})^2)\}$ are tight for some $\tau \ge 0$. Let us see it for $\tau = 1$:

$$|EZ_{nj1}| \le |\int_{|X_{nj}| \le 1} Z_{nj} dP| + (1 + |EX_{nj1}|) P\{|X_{nj}| > 1 - |EX_{nj1}|\}$$

$$= |EX_{nj1}| P\{|X_{nj}| > 1\} + (1 + |EX_{nj1}|) P\{|X_{nj}| > 1 - |EX_{nj1}|\},$$

and since $\max_j |EX_{nj1}| \to 0$ as $n \to \infty$ by infinitesimality, we obtain that $\sup_n \Sigma_j (EZ_{nj1})^2 < c \sup_n \Sigma_j P\{|X_{nj}| > 1/2\}$ and this quantity is finite if either one of the two families of sums are tight, by the previous theorems. So we have:

Corollary 3. Let $\{X_{nj}\}$ be infinitesimal. Then $\{\mathcal{L}(S_n - ES_{n,\delta})\}$ is tight if and only if $\{\mathcal{L}(\Sigma_j (X_{nj} - EX_{nj\delta})^2)\}$ is tight for some (all) $\delta > 0$.

Next we examine convergence relations. We will let $T(x) = x^2$. Note that if μ and ν are σ-finite Borel measures the equation $\nu \circ T = \mu$, ν unknown, has a unique symmetric solution and a unique solution supported by $\mathbb{R}_+ (\mathbb{R}_-)$. This solution will be denoted $\nu = \mu \circ T$ and it will be symmetric or supported by $\mathbb{R}_+$ depending on the context.

Theorem 4. Let $\{X_{nj}\}$ be an infinitesimal array.

(a) Assume $\{X_{nj}\}$ satisfies condition (3) and

(4) $$\mathcal{L}(S_n - ES_{n,\delta}) \to_w N(0,\sigma^2) * c_\delta \mathrm{Pois}\,\mu$$

for some $\sigma^2 \geq 0$, Lévy measure μ and $\delta > 0$ such that $\mu\{-\delta,\delta\} = 0$.
Then,

(5) $$\mathcal{L}(\Sigma_j X_{nj}^2 - \Sigma_j EX_{nj\delta}^2) \to_w c_\delta 2\ \mathrm{Pois}(\mu o T^{-1})$$

In particular, if condition (3) is replaced by the stronger condition

(6) $$\lim_n \Sigma_j EX_{nj\delta}^2 = a < \infty,$$

then

(7) $$\mathcal{L}(\Sigma_j X_{nj}^2) \to_w \delta_a * c_\delta 2\ \mathrm{Pois}(\mu o T^{-1})$$

(b) Conversely if the X_{nj} are non-negative (symmetric) and

(8) $$\lim_{\delta \downarrow 0} \overline{\lim}_n \Sigma_j (EX_{nj\delta})^2 = 0,$$

then, the fact that

(9) $$\mathcal{L}(\Sigma_j X_{nj}^2) \to_w \delta_a * c_\delta 2\ \mathrm{Pois}\mu$$

for some Lévy measure μ and δ such that $\mu\{-\delta^2,\delta^2\} = 0$, implies

(10) $$\mathcal{L}(S_n - ES_{n,\delta}) \to_w N(0,\sigma^2) * c_\delta \mathrm{Pois}(\mu o T),$$

where $\sigma^2 = a - \int_0^{\delta^2} x d\mu(x)$. ($\mu o T$ is symmetric if the X_{nj} are symmetric and with support in $\mathbb{R}_+$ if the X_{nj} are non-negative). Also, $\Sigma_j EX_{nj\delta}^2 \to a$.

(b') If in (9) $\mu = 0$, then (b) is true without the variables X_{nj} being non-negative or symmetric.

Proof. (a) By the CLT , $\Sigma_j \mathcal{L}(X_{nj})|_{\{|x|>\delta\}} \to_w \mu|_{\{|x|>\delta\}}$ if $\mu\{-\delta,\delta\} = 0$. Then if $\mu\{-\delta^{1/2},\delta^{1/2}\} = 0$, $\Sigma_j \mathcal{L}(X_{nj}^2)|_{\{|x|>\delta\}} \to_w \mu o T^{-1}|_{\{|x|>\delta\}}$. On the other hand,

$$\lim_{\delta \downarrow 0} \overline{\lim}_n \Sigma_j E[X_{nj}^4 I_{\{X_{nj}^2 \leq \delta\}} - (EX_{nj}^2 I_{\{X_{nj}^2 \leq \delta\}})^2]$$

$$\leq \lim_{\delta \downarrow 0} \overline{\lim}_n \Sigma_j (\delta - E(X_{nj}^2)_\delta) E(X_{nj}^2)_\delta = 0$$

because by condition (3) and the CLT , $\sup_n \Sigma_j EX_{nj1}^2 < \infty$ (see (2)). Hence (5) follows from the CLT.

(b) Assume now that (8) and (9) hold. Then (3) holds and therefore Corollary 3 gives that $\{\mathcal{L}(S_n - ES_{n,\delta})\}$ is tight. But obviously all the subsequential limits have the same Lévy measure $\mu o T$, hence $\Sigma_j \mathcal{L}(X_{nj})|_{\{|x|>\delta\}} \to_w \mu o T|_{\{|x|>\delta\}}$

if $\mu o T\{-\delta,\delta\} = 0$. Now, the CLT and (9) give $\lim_n \Sigma_j EX^2_{nj\delta} = a$, and therefore, condition (8) implies

$$\lim_{\tau\downarrow 0} \left(\begin{matrix}\overline{\lim}\\ \underline{\lim}\end{matrix}\right)_n \Sigma_j (EX_{nj\tau})^2) = \lim_{\tau\downarrow 0} \left(\begin{matrix}\overline{\lim}\\ \underline{\lim}\end{matrix}\right)_n \Sigma_j EX^2_{nj\tau}$$

$$= \lim_{\tau\downarrow 0,\mu\{\tau^2\}=0} \lim_n \Sigma_j EX^2_{nj\tau} = \lim_{\tau\downarrow 0,\mu\{\tau^2\}=0} [a - \int_{\tau^2}^{\delta^2} x\,d\mu(x)]$$

$$= a - \int_0^{\delta^2} x\,d\mu(x).$$

So, (10) follows by the CLT . (b') also follows from the CLT, Gaussian convergence, and from (ii) with $\mu = 0$.

Remarks.(1) Condition (8) is satisfied in the symmetric case and also for $Z_{nj} = X_{nj} - EX_{nj1}$ in general (from remark (b) after Corollary 3 we obtain that if $\{\mathcal{L}(\Sigma_j X^2_{nj})\}$ or $\{\mathcal{L}(\Sigma_j Z^2_{nj})\}$ are shift tight, then $\overline{\lim}_n \Sigma_j E|Z_{nj\delta}|^2 \leq$ $\leq \overline{\lim}_n \Sigma_j (P\{|X_{nj}| > \delta\})^2 \leq \overline{\lim}_n \max_j P\{|X_{nj}| > \delta\} \; \Sigma_j (P\{|X_{nj}| > \delta\} = 0)$.

(2) Let us finally remark that if both $\{\mathcal{L}(S_n)\}$ and $\{\mathcal{L}(\Sigma_j X^2_{nj})\}$ converge, then the p-th moment of $|S_n|$ converges if and only if the (p/2)-th moment of $\Sigma_j X^2_{nj}$ does:both conditions are equivalent to

$$\lim_{t\to\infty} \sup_n \Sigma_j E|X_{nj}|^p I_{\{|X_{nj}| > t\}} = 0$$

(de Acosta and Giné (1978)). With this remark, Theorem 5 contains the result in Hall (1978) as a particular case (the cases $\mu = 0$ and $\mu = \lambda\delta_1$).

(3) It is also clear from the foregoing that the power 2 is basic only if $\lim_n \Sigma_j EX^2_{nj\delta} \neq 0$ and the Lévy measure μ (or $\mu o T^{-1}$) gives positive mass to intervals arbitrarily near to zero. Otherwise the previous results hold for $\Sigma_j |X_{nj}|$, for any $p > 0$ (as observed by Hall (1978) in the particular case $\mu = \lambda\delta_1$).

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