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A COMBINATORIAL APPROACH TO NONINVOLUTIVE SET-THEORETIC SOLUTIONS OF THE YANG–BAXTER EQUATION

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Abstract: We study noninvolutive set-theoretic solutions (X,r) of the Yang–Baxter equations in terms of the properties of the canonically associated braided monoid $S(X,r)$, the quadratic Yang–Baxter algebra $A = A(\mathbf{k},X,r)$ over a field $\mathbf{k}$, and its Koszul dual $A^!$. More generally, we continue our systematic study of *nondegenerate quadratic sets* (X,r) *and their associated algebraic objects*. Next we investigate the class of (noninvolutive) square-free solutions (X,r). This contains the self distributive solutions (quandles). We make a detailed characterization in terms of various algebraic and combinatorial properties each of which shows the contrast between involutive and noninvolutive square-free solutions. We introduce and study a class of finite square-free braided sets (X,r) of order $n \geq 3$ which satisfy *the minimality condition*, that is, $\dim_{\mathbf{k}} A_2 = 2n-1$. Examples are some simple racks of prime order p. Finally, we discuss general extensions of solutions and introduce the notion of *a generalized strong twisted union of braided sets*. We prove that if (Z,r) is a nondegenerate 2-cancellative braided set splitting as a generalized strong twisted union of r-invariant subsets $Z = X \natural^* Y$, then its braided monoid S_Z is a generalized strong twisted union $S_Z = S_X \natural^* S_Y$ of the braided monoids S_X and S_Y. We propose a construction of a generalized strong twisted union $Z = X \natural^* Y$ of braided sets (X, r_X) and (Y, r_Y), where the map r has a high, explicitly prescribed order.

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1. Introduction

It was established in the last three decades that solutions of the linear braid or Yang–Baxter equations (YBE) on a vector space of the form $V^{\otimes 3}$ lead to remarkable algebraic structures. We will use the notation $r\colon V\otimes V \to V\otimes V$, $r^{12} = r\otimes \mathrm{id}$, and $r^{23} = \mathrm{id}\otimes r$. These structures include coquasitriangular bialgebras $A(r)$, their quantum group (Hopf

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algebra) quotients, quantum planes and associated objects, at least in the case of specific standard solutions; see [**30, 36**]. On the other hand, the variety of all solutions on vector spaces of a given dimension has remained rather elusive in any degree of generality. It was proposed by V. G. Drinfeld ([**11**]) to consider the same equations in the category of sets and in this setting numerous results were found. It is clear that a set-theoretic solution extends to a linear one, but more important than this is that set-theoretic solutions lead to their own remarkable algebraic and combinatoric structures, only somewhat analogous to quantum group constructions. In the present paper we continue our systematic study of set-theoretic solutions based on the associated quadratic algebras and monoids that they generate.

More generally, we study quadratic sets and their algebraic objects. The notions of a quadratic set (X, r) and its related algebraic objects were introduced by the author and studied first in [**15**]; see also [**22, 17**] for more results on quadratic sets (X, r). We shall use the terminology, notation, and some results from [**15, 17, 21, 19, 22**].

Definition 1.1 ([**15**])**.** Let X be a nonempty set (possibly infinite) and let $r\colon X \times X \to X \times X$ be a bijective map. In this case we use notation (X, r) and refer to it as *a quadratic set*. The image of (x, y) under r is presented as

$$r(x, y) = ({}^x y, x^y).$$

This formula defines a *left action* $\mathcal{L}\colon X \times X \to X$ and a *right action* $\mathcal{R}\colon X \times X \to X$ on X as: $\mathcal{L}_x(y) = {}^x y$, $\mathcal{R}_y(x) = x^y$, for all $x, y \in X$.

(1) (X, r) is *nondegenerate* if the maps $\mathcal{L}_x$ and $\mathcal{R}_x$ are bijective for each $x \in X$.
(2) (X, r) is *involutive* if $r^2 = \mathrm{id}_{X \times X}$.
(3) (X, r) is *square-free* if $r(x, x) = (x, x)$ for all $x \in X$.
(4) (X, r) is *quantum binomial* if it is nondegenerate, square-free, and involutive.
(5) (X, r) is *a set-theoretic solution of the Yang–Baxter equation* (YBE) if the braid relation

$$r^{12} r^{23} r^{12} = r^{23} r^{12} r^{23}$$

holds in $X \times X \times X$, where $r^{12} = r \times \mathrm{id}_X$ and $r^{23} = \mathrm{id}_X \times r$. In this case we refer to (X, r) also as *a braided set*. A braided set (X, r) with r involutive is called *a symmetric set*.

In this paper we always assume that r is nondegenerate. As a notational tool, we shall often identify the sets $X^{\times m}$ of ordered m-tuples, $m \geq 2$, and X^m, the set of all monomials of length m in the free monoid $\langle X \rangle$.

As in our previous works ([**15, 17, 21, 19, 22**]), to each quadratic set (X, r) we associate canonically several algebraic objects (see Definition 2.4) generated by X and with quadratic defining relations naturally determined as

$xy = y'x' \in \Re(r)$ iff $r(x, y) = (y', x')$ and $(x, y) \neq (y', x')$ hold in $X \times X$.

Note that in the case when X is finite, the set $\Re(r)$ of defining relations is also finite, and therefore the associated algebraic objects are finitely presented.

We continue our systematic study of the close relations between the combinatorial properties of the defining relations, i.e. of the map r, and the structural properties of the associated algebraic objects.

In the first half of the paper we investigate nondegenerate quadratic sets (X, r) of finite order, their quadratic graded algebras A, and the monoid $S(X, r)$. Section 2 contains preliminary material on quadratic sets. In Section 3 we study nondegenerate quadratic sets (X, r), with 2-cancellation. Proposition 3.10 provides upper and lower bounds for the dimension $\dim A_2$ and shows that the upper bound is attained whenever r is involutive. The main result of the section is Theorem 3.16. It implies, in particular, that a square-free nondegenerate quadratic set (X, r) with $|X| = n$ is a symmetric set if and only if its quadratic algebra A has Hilbert series $H_A(z) = \frac{1}{(1-z)^n}$. The theorem improves an old result of the author; see [**18**, Theorem 2]. In Section 4 we pay special attention to square-free quadratic sets with cyclic conditions. We find some new combinatorial results, see Theorem 4.7, and use them to show that, surprisingly, a square-free quadratic set (X, r) of finite order $|X| = n$ which satisfies the cyclic conditions is a symmetric set if and only if $\dim_{\mathbf{k}} A^!_3 = \binom{n}{3}$; see Proposition 4.8. In Section 5 we study square-free braided sets and the contrast between the involutive and noninvolutive cases. We show that every square-free braided set (of arbitrary cardinality) satisfies the cyclic conditions. Theorem 5.5 characterizes the involutive braided sets (X, r) in terms of various equivalent properties of the algebra A, its Koszul dual $A^!$, and the monoid $S(X, r)$. Corollary 5.6 provides a characterization of *noninvolutive* square-free braided sets. In Section 6 we introduce quadratic sets (X, r) which satisfy *the minimality condition* $\mathbf{M}$, that is, $\dim_{\mathbf{k}} A_2 = 2n - 1$; see Definition 6.1. We first investigate (general) square-free 2-cancellative quadratic sets (X, r) with minimality condition and prove Proposition 6.5. We make some initial steps in the study of braided sets and, in particular, quandles with minimality condition $\mathbf{M}$. Corollary 6.18 implies that every square-free self distributive solution (X, r) (see Definition 6.6) corresponding to a dihedral quandle of prime order $|X| = p > 2$ satisfies the minimality condition **M**. In Section 7 we propose a construction which generates noninvolutive extensions (Z, r) of braided (or symmetric) sets, where the map r has high, explicitly prescribed order; see Theorem 7.2. In Section 8 braided monoids $S(X, r)$ and extensions of solutions are studied. We consider *general* extensions of braided sets. In Subsection 8.4 we introduce *generalized strong twisted unions* $Z = X \natural^* Y$ *of nondegenerate braided sets*; see Definition 8.8. The main result of the section is Theorem 8.13. Finally, in Section 9 we give a list of questions and problems. Some of these are still open questions, other were posed in earlier versions of our work and have stimulated recent results of other authors.

2. Preliminaries

During the last two decades the study of set-theoretic solutions of the Yang–Baxter equation and related structures has notably intensified; a relevant selection of works for the interested reader is [**11, 24, 12, 29, 15, 5, 6, 37, 40, 22, 18, 21, 19, 8, 41, 26, 3, 28, 38, 39, 4**], and the references therein. In this section we recall basic notions and results which will be used in the paper. We shall use the terminology, notation, and some results from [**15, 17, 21, 19, 22**].

Remark 2.1. Let (X, r) be a quadratic set, and let ${}^{x}\bullet$, and $\bullet^{x}$ be the associated left and right actions. Then

(1) The map r is involutive *iff* the actions satisfy:

$$ {}^{u^v}(u^v) = u \text{ and } ({}^{u}v)^{u^v} = v, \quad \forall u, v \in X. \tag{2.1}$$

(2) r is square-free if and only if ${}^{x}x = x$, and $x^x = x$, $\forall x \in X$.

(3) If r is nondegenerate and square-free, then

$$\begin{array}{lllll} {}^{z}t = {}^{z}u & \Longrightarrow & t = u & \Longleftarrow & t^z = u^z, \\ {}^{z}t = z & \Longleftrightarrow & t = z & \Longleftrightarrow & t^z = z. \end{array} \tag{2.2}$$

Remark 2.2 ([**12**]). Let (X, r) be quadratic set. Then r obeys the YBE, that is, (X, r) is a braided set *iff* the following conditions hold for all $x, y, z \in X$:

$$\text{l1}: \ {}^{x}({}^{y}z) = {}^{{}^{x}y}({}^{x^y}z), \quad \text{r1}: \ (x^y)^z = (x^{{}^{y}z})^{y^z},$$

$$\text{lr3}: \ ({}^{x}y)^{({}^{x^y}z)} = {}^{(x^{{}^{y}z})}(y^z).$$

Convention 2.3. In this paper by *a solution* we mean *a nondegenerate braided set* (X, r), where X is a set of arbitrary cardinality. We shall also refer to it as *a braided set*, keeping the convention that we consider only nondegenerate braided sets. *An involutive solution* means *a nondegenerate symmetric set*. In most cases we shall also assume that r is 2-cancellative but this will be indicated explicitly.

2.1. Quadratic sets and their algebraic objects. Let X be a nonempty set, and let $\mathbf{k}$ be a field. We denote by $\langle X\rangle$ and ${}_{\mathrm{gr}}\langle X\rangle$, respectively, the free monoid and the free group generated by X, and by $\mathbf{k}\langle X\rangle$ the free associative $\mathbf{k}$-algebra generated by X. For a set $F \subseteq \mathbf{k}\langle X\rangle$, we denote by (F) the two sided ideal of $\mathbf{k}\langle X\rangle$ generated by F.

For $m \geq 1$, the length of a monomial $u = x_1 \cdots x_m \in X^m$ will be denoted by $|u| = m$.

As in our works [**15, 16, 17, 22, 23, 19**], we use the following.

Definition 2.4. To each quadratic set (X, r) we canonically associate algebraic objects generated by X and with quadratic relations $\Re = \Re(r)$ naturally determined as

$$xy = y'x' \in \Re(r) \text{ iff } r(x,y) = (y',x') \text{ and } (x,y) \neq (y',x') \text{ hold in } X \times X.$$

The monoid $S = S(X,r) = \langle X; \Re(r)\rangle$ with a set of generators X and a set of defining relations $\Re(r)$ is called *the monoid associated with* (X,r). The *group* $G = G(X,r) = G_X$ *associated with* (X,r) is defined analogously.

For an arbitrary fixed field $\mathbf{k}$, *the* $\mathbf{k}$-*algebra associated with* (X,r) is defined as

$$A = A(\mathbf{k}, X, r) = \mathbf{k}\langle X\rangle/(\Re_0) \simeq \mathbf{k}\langle X; \Re(r)\rangle,$$
$$\text{where } \Re_0 = \Re_0(r) = \{xy - y'x' \mid xy = y'x' \in \Re(r)\}.$$

Clearly, the quadratic algebra A generated by X and with defining relations $\Re_0(r)$ is isomorphic to the monoid algebra $\mathbf{k}S(X,r)$.

Definition 2.5. We shall call a quadratic set (X,r) *injective* if the set X is embedded in $G(X,r)$.

Recall that when (X,r) is a braided set, its monoid $S = S(X,r)$ is a graded braided monoid ([**22**]) and the group $G(X,r)$ is a braided group ([**29**]); see details in Section 8. Moreover, the associated quadratic algebra $A = A(\mathbf{k}, X, r)$ is also called *a Yang–Baxter algebra*; see [**32**].

Remark 2.6 ([**17**, Proposition 2.3]). If (X,r) is a nondegenerate and involutive quadratic set of finite order $|X| = n$, then the set $\Re(r)$ consists of precisely $\binom{n}{2}$ quadratic relations. Clearly, in this case the associated algebra $A = A(\mathbf{k}, X, r)$ satisfies

$$\dim A_2 = \binom{n+1}{2}.$$

Various equivalent conditions are given in Proposition 3.10.

Remark 2.7. Suppose (X,r) is a finite quadratic set. Then A is a quadratic algebra generated by X and with quadratic defining relations $\Re(r)$. Clearly, A is *a connected graded* $\mathbf{k}$-algebra (naturally graded by length),

$A = \bigoplus_{i \geq 0} A_i$, where $A_0 = \mathbf{k}$, A is generated by $A_1 = \operatorname{Span}_{\mathbf{k}} X$, so each graded component A_i is finite dimensional. Moreover, the associated monoid $S = S(X, r)$ *is naturally graded by length*:

$$S = \bigsqcup_{m \geq 0} S_m;$$

$$\text{where } S_0 = 1,\ S_1 = X,\ S_m = \{u \in S \mid |u| = m\},\ S_m \cdot S_t \subseteq S_{m+t}.$$

In the sequel, by *a graded monoid* S, we shall mean that S is generated by $S_1 = X$ and graded by length. The grading of S induces a canonical grading of its monoid algebra $\mathbf{k}S(X, r)$. The isomorphism $A \cong \mathbf{k}S(X, r)$ agrees with the canonical gradings, so there is an isomorphism of vector spaces $A_m \cong \operatorname{Span}_{\mathbf{k}} S_m$.

Remark 2.8 ([**16**]). Let (X, r) be a quadratic set and let $S = S(X, r)$ be the associated monoid.

(1) By definition, two monomials $w, w' \in \langle X \rangle$ are equal in S *iff* w can be transformed to w' by a finite sequence of replacements, each of the form

$$axyb \longrightarrow ar(xy)b \text{ or } axyb \longrightarrow ar^{-1}(xy)b, \quad \text{where } x, y \in X,\ a, b \in \langle X \rangle.$$

Clearly, every such replacement preserves monomial length, which therefore descends to $S(X, r)$. Furthermore, replacements coming from the defining relations are possible only on monomials of length ≥ 2, hence $X \subset S(X, r)$ is an inclusion. For monomials of length 2, $xy = zt$ holds in $S(X, r)$ *iff* $zt = r^k(xy)$ is an equality of words in X^2 for some $k \in \mathbb{Z}$.

(2) It is convenient, for each $m \geq 2$, to refer to the subgroup D_m of the symmetric group $\operatorname{Sym}(X^m)$ generated concretely by the maps

(2.3) $r^{ii+1}\colon X^m \longrightarrow X^m$, $r^{ii+1} = \mathrm{id}_{X^{i-1}} \times r \times \mathrm{id}_{X^{m-i-1}}, \quad i = 1, \dots, m-1.$

One can also consider the free groups

$$\mathcal{D}_m(r) = {}_{\mathrm{gr}}\langle r^{ii+1} \mid i = 1, \dots, m-1 \rangle,$$

where the r^{ii+1} are treated as abstract symbols, as well as various quotients depending on the further type of r of interest. These free groups and their quotients act on X^m via the actual maps r^{ii+1} so that the image of $\mathcal{D}_m(r)$ in $\operatorname{Sym}(X^m)$ is $D_m(r)$. In particular, $D_2(r) = \langle r \rangle \subset \operatorname{Sym}(X^2)$ is the cyclic group generated by r. It follows straightforwardly from part (1) that $w, w' \in \langle X \rangle$ are equal as words in $S(X, r)$ *iff* they have the same length, say m, and belong to the same orbit of $\mathcal{D}_m(r)$ in X^m. Clearly, in this case the equality $w = w'$ holds in the group $G(X, r)$ and in the algebra $A(\mathbf{k}, X, r)$.

An effective part of our combinatorial approach is the exploration of the actions of the group $\mathcal{D}_2(r) = \langle r \rangle$ on X^2, the group $\mathcal{D}_3(r) = {}_{\rm gr}\langle r^{12}, r^{23}\rangle$ on X^3, and, in particular, the properties of the corresponding orbits. In the literature a $\mathcal{D}_2(r)$-orbit $\mathcal{O}$ in X^2 is often called *an r-orbit* and we shall use this terminology.

If r is involutive, the bijective maps r^{12} and r^{23} are involutive as well, so in this case $\mathcal{D}_3(r)$ is *the infinite dihedral group*

$$\mathcal{D}_3(r) = \mathcal{D}(r) = {}_{\rm gr}\langle r^{12}, r^{23} \mid (r^{12})^2 = e, (r^{23})^2 = e\rangle.$$

Remark 2.9. In notation and assumption as above, let (X, r) be a finite quadratic set $S = S(X, r)$ graded by length. Then the order of S_2 equals the number of $\mathcal{D}_2(r)$-orbits in X^2.

For positive integers $i < n$, the maps $r^{ii+1}\colon X^n \to X^n$ are defined by (2.3). Recall that the braid group B_n is generated by elements b_i, $1 = i = n-1$, with defining relations

$$b_ib_j = b_jb_i,\ |i-j| > 1, \quad b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1},$$

and the symmetric group S_n is the quotient of B_n by the relations $b_i^2 = 1$. It is well known (and straightforward) that for every $n \geq 3$ the following hold:

(1) The assignment $b_i \mapsto r^{ii+1}$ extends to a (left) action of B_n on X^n if and only if (X, r) is a braided set.
(2) The assignment $b_i \mapsto r^{ii+1}$ extends to an action of S_n on X^n if and only if (X, r) is a symmetric set.

3. Nondegenerate quadratic sets with 2-cancellation and their quadratic algebras

3.1. Basics on quadratic algebras. Our main reference for this subsection is [**34**].

A quadratic algebra is an associative graded algebra $A = \bigoplus_{i\geq 0} A_i$ over a ground field $\mathbf{k}$ determined by a vector space of generators $V = A_1$ and a subspace of homogeneous quadratic relations $R = R(A) \subset V \otimes V$. We assume that A is finitely generated, so $\dim A_1 < \infty$. Thus $A = T(V)/(R)$ inherits its grading from the tensor algebra $T(V)$. The Koszul dual algebra of A, denoted by $A^!$, is the quadratic algebra $T(V^*)/(R^\perp)$; see [**31, 32**]. The algebra $A^!$ is also referred to as *the quadratic dual algebra to a quadratic algebra A*; see [**34**, p. 6].

Following the classical tradition (and a recent trend), we take a combinatorial approach to study A. The properties of A will be read off a presentation $A = \mathbf{k}\langle X\rangle/(\Re)$, where by convention X is a fixed finite set of generators of degree 1, $|X| = n$, $\mathbf{k}\langle X\rangle$ is the unital free associative algebra generated by X, and $(\Re)$ is the two-sided ideal of relations, generated by a *finite* set $\Re$ of homogeneous polynomials of degree two.

A quadratic algebra A is *a PBW algebra* if there exists an enumeration of X, $X = \{x_1, \dots, x_n\}$, such that the quadratic relations $\Re$ form a (noncommutative) Gröbner basis with respect to the degree-lexicographic ordering $<$ on $\langle X\rangle$ extending $x_1 < x_2 < \cdots < x_n$. In this case the set of normal monomials (mod $\Re$) forms a $\mathbf{k}$-basis of A called a *PBW basis* and $x_1, \dots, x_n$ (taken exactly with this enumeration) are called *PBW-generators of A*. The notion of a *PBW* algebra was introduced by Priddy [**35**]. His *PBW basis* is a generalization of the classical Poincaré–Birkhoff–Witt basis for the universal enveloping of a finite dimensional Lie algebra. PBW algebras form an important class of Koszul algebras. The interested reader can find information on quadratic algebras and, in particular, on Koszul algebras and PBW algebras in [**34**].

There are various equivalent definitions of a Koszul algebra; see for example [**34**, p. 19]. We recall one of them. A graded $\mathbf{k}$-algebra A is *Koszul* if A is quadratic and $\operatorname{Ext}^*_A(\mathbf{k},\mathbf{k}) \simeq A^!$. It is known that if (X, r) is a finite square-free involutive solution, then its quadratic algebra $A(\mathbf{k}, X, r)$ is Koszul; see [**24**]. We shall prove that, conversely, if (X, r) is a (general) square-free nondegenerate braided set and its algebra $A(\mathbf{k}, X, r)$ is Koszul, then r is involutive. This follows from our more general result, Proposition 3.12.

The following results can be used to test whether a quadratic algebra is Koszul.

Fact 3.1. (1) *(*[**35**, Theorem 5.3]*) Every quadratic PBW algebra is Koszul.*

(2) *(*[**34**, Corollary 2.2]*) If A is a quadratic Koszul algebra, with Koszul dual $A^!$, then their Hilbert series satisfy*

$$H_A(z) \cdot H_{A^!}(-z) = 1. \tag{3.1}$$

Note that (3.1) *is a necessary but not a sufficient condition for Koszulity of A* [**34**].

3.2. Quadratic set with 2-cancellation and their quadratic algebras.

To proceed further, we require some cancellation conditions.

Definition 3.2 ([**22**, Definition 2-10])**.** A quadratic set (X, r) is 2-*cancellative* if for every positive integer k, less than the order of r, the following two conditions hold:

$$r^k(x, y) = (x, z) \Longrightarrow z = y, \quad r^k(x, y) = (t, y) \Longrightarrow x = t.$$

The monoid $S = S(X, r)$ has cancellation on monomials of length 2 *if and only if* r is 2-cancellative; see [**22**, Proposition 2.11(1)]. Moreover, every injective quadratic set (X, r) (see Definition 2.5) is 2-cancellative. Note that if $x, y, z \in X$, $y \neq z$, each of the equalities $r^k(x, y) = (x, z)$, or $r^k(y, x) = (z, x)$ implies $y = z$ in $G(X, r)$.

Remark 3.3. (1) Every nondegenerate involutive quadratic set (X, r) is 2-cancellative; see [**22**, Corollary 2.13]. Recall that when X is a finite (nondegenerate) symmetric set, the monoid $S = S(X, r)$ is embedded in the group $G(X, r)$ and therefore S is a monoid with cancellation.
(2) nondegenerate braided set (X, r) may fail to be 2-cancellative; see Example 3.6.
(3) There exist various examples of (noninvolutive) nondegenerate braided sets (X, r), where r is 2-cancellative but the corresponding monoid $S(X, r)$ fails to be 3-cancellative.

We shall prove that if (X, r) is a finite square-free braided set, then the monoid $S(X, r)$ is cancellative *iff* r is involutive; see Proposition 5.4.

Notation 3.4. Denote by Δ_m the diagonal of $X^{\times m}$, $m \geq 2$:

$$\Delta_m := \mathrm{diag}(X^m) = \{x^m \mid x \in X\}.$$

One has $\Delta_3 = (\Delta_2 \times X) \cap (X \times \Delta_2)$.

Notation 3.5. Suppose (X, r) is a quadratic set. The element $(x, y) \in X^2$ is *an r-fixed point* if $r(x, y) = (x, y)$. *The set of r-fixed points* in X^2 will be denoted by $\mathcal{F}(X, r)$, that is:

$$\mathcal{F}(X, r) = \{xy \in X^2 \mid r(x, y) = (x, y)\}. \tag{3.2}$$

Examples 3.6. (1) [**22**, Example 2.14]. Let $X = \{x, y, z\}$ and let $\rho = (x\ y\ z)$ be a cycle of length three in $\mathrm{Sym}(X)$. Define $r(a, b) := (\rho(b), a)$, that is, $(x, x) \to^r (y, x) \to^r (y, y) \to^r (z, y) \to^r (z, z) \to^r (x, z) \to^r (x, x)$, $(x, y) \to^r (z, x) \to^r (y, z) \to^r (x, y)$. It is easy to check that (X, r) is a nondegenerate braided set (this is a permutation solution).

Note that $r(a, b) \neq (a, b)$, $\forall a, b \in X$, so $\mathcal{F}(X, r) = \emptyset$, and that r is not 2-cancellative, since $xx = yx$ in S. Moreover, (X, r) is not injective, since in the group $G(X, r)$ all generators are equal: $x = y = z$.

(2) [**22**, Example 2.17]. Let $X = \{x, y, z\}$ and let r be the map

$$(x, y) \longrightarrow^r (x, z) \longrightarrow^r (y, z) \longrightarrow^r (y, y) \longrightarrow^r (x, y),$$

$$(x, x) \longrightarrow^r (z, z) \longrightarrow^r (y, x) \longrightarrow^r (z, y) \longrightarrow^r (x, x), \quad r(z, x) = (z, x).$$

The bijective map r is nondegenerate, $\mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_z = (x\ z\ y)$; $\mathcal{R}_x = \mathcal{R}_y = \mathcal{R}_z = (x\ z)$, r is not 2-cancellative, conditions l1 and r1 are satisfied, but (X, r) is not a braided set. Here $\mathcal{F}(X, r) = \{(z, x)\}$.

Lemma 3.7. *Suppose (X, r) is a nondegenerate quadratic set (possibly infinite). Then*

(1) *If X has finite order, then*

$$0 \leq |\mathcal{F}(X, r)| \leq |X|. \tag{3.3}$$

(2) *If (X, r) is square-free, then $\mathcal{F}(X, r) = \Delta_2$, the diagonal of X^2. In particular, if X has finite order, then $|\mathcal{F}(X, r)| = |X|$.*

(3) *Suppose (X, r) is nondegenerate and 2-cancellative. Then the following conditions hold:*

(a) *For every $y \in X$ there exists a unique $x \in X$ such that $r(x, y) = (x, y)$. In other words there exist a bijective map $t\colon X \to X$ such that $r(t(y), y) = (t(y), y)$, for every $y \in X$.*

(b) *For every $x \in X$ there exists a unique $y \in X$ such that $r(x, y) = (x, y)$.*

(c) *If X is finite, $X = \{x_1, \dots, x_n\}$, then*

$$\mathcal{F} = \mathcal{F}(X, r) = \{xy \in X^2 \mid r(x, y) = (x, y)\} = \{x_1y_1, \dots, x_ny_n\}, \tag{3.4}$$

where $y_i \in X$, is the unique element with $r(x_i, y_i) = (x_i, y_i)$, $1 \leq i \leq n$. In particular, $|\mathcal{F}| = |X| = n$.

Proof: (1) The equality $r(x, y) = ({}^xy, x^y)$ implies

$$(x, y) \in \mathcal{F} \text{ if and only if } x = {}^xy \text{ and } x^y = y.$$

It follows from nondegeneracy that for each $y \in X$ there exists a unique $x \in X$, with $x^y = y$. In general, it is possible that ${}^xy \neq x$, and in this case $r(x, y) = ({}^xy, y) \neq (x, y)$. This implies (3.3).

(2) Suppose (X, r) is square-free. Then, by definition, $\Delta_2 \subseteq \mathcal{F}(X, r)$. Assume $xy \in \mathcal{F}(X, r)$. Then ${}^xy = x = {}^xx$, and hence $y = x$ by nondegeneracy. This gives $\mathcal{F}(X, r) \subseteq \Delta_2$, and therefore $\mathcal{F}(X, r) = \Delta_2$. Moreover, $|\mathcal{F}(X, r)| = |X|$ whenever X is a finite set.

(3) Assume (X, r) is 2-cancellative and nondegenerate. Suppose $y \in X$. Then, by nondegeneracy there exists a unique $x \in X$ such that $x^y = y$. Consider the equality $r(x, y) = ({}^xy, x^y) = ({}^xy, y)$. Then by the 2-cancellation law, one has ${}^xy = x$, and hence $r(x, y) = (x, y)$, as desired. Assume now that $r(z, y) = (z, y)$ for some $z \in X$. Then $r(z, y) = ({}^zy, z^y) = (z, y)$ implies $z^y = y = x^y$, which by nondegeneracy gives $z = x$. This

proves part (3)(a). Part (3)(b) is analogous. Part (3)(c) follows straightforwardly from (3)(a) and (3)(b). □

Corollary 3.8. *A nondegenerate quadratic set is square-free if and only if for every pair* $x, y \in X$ *one has*

$$r(x,y) = (x,y) \Longleftrightarrow x = y.$$

Remark 3.9. (1) Suppose (X, r) is a 2-cancellative nondegenerate quadratic set of finite order $|X| = n$. One can apply the theory of (noncommutative) Gröbner bases. We enumerate X, as $X = \{x_1 < x_2 < \cdots < x_n\}$ and consider the degree-lexicographic ordering $\leq$ on $\langle X \rangle$. Let $\mathcal{O}_j$, $1 \leq j \leq q$, be the set of all nontrivial r-orbits. Each r-orbit $\mathcal{O}_j$, $1 \leq j \leq q$, has length $l_j = |\mathcal{O}_j| \geq 2$ and contains a unique monomial $z_j t_j \in \mathcal{O}_j$, which is minimal (in $\mathcal{O}_j$) with respect to the ordering $<$ on $\langle X \rangle$. Then the subset of defining relations determined by $\mathcal{O}_j$, namely $\{xy - r(xy) \mid xy \in \mathcal{O}_j\}$, reduces to exactly $l_j - 1$ relations with explicit pairwise distinct highest terms

$$xy - z_j t_j = 0, \quad x, y \in X, \quad xy \in \mathcal{O}_j, \quad xy > z_j t_j.$$

The set of reduced relations $\mathrm{R}(r)$ is defined as

$$\mathrm{R}(r) = \{xy - z_j t_j \mid xy \in \mathcal{O}_j,\, xy > z_j t_j,\, 1 \leq j \leq q\}, \quad \text{and}$$
$$|\mathrm{R}(r)| = s = \sum_{1 \leq j \leq q} (l_j - 1) = \left(\sum_{1 \leq j \leq q} l_j \right) - q \geq q.$$

There is an equality of sets $\Re_0(r) = \mathrm{R}(r)$ if and only if r is involutive. The two sets $\Re_0(r)$ and $\mathrm{R}(r)$ generate the same two-sided ideal I of $\mathbf{k}\langle X \rangle$. Hence the algebra $A = A(\mathbf{k}, X, r) \cong \mathbf{k}\langle X \rangle / (\Re_0(r))$ has a finite presentation as $A = \mathbf{k}\langle X \rangle / (\mathrm{R}(r))$. The set of reduced relations $\mathrm{R}(r)$ is exactly the quadratic part of the (minimal) *reduced Gröbner basis of* I, denoted $\mathrm{GR}(I)$ (with respect to the degree-lexicographic ordering $<$ on $\langle X \rangle$). The set $\mathrm{R}(r)$ is linearly independent, so $\dim I_2 = |\mathrm{R}(r)| = s$. In general, $\mathrm{R} \subset \mathrm{GR}(I)$, and the reduced Gröbner basis $\mathrm{GR}(I)$ may be infinite. It follows from the theory of Gröbner bases that the set $\mathcal{N}$ of all monomials of length 2 which are normal modulo I (with respect to the degree-lexicographic ordering $<$ on $\langle X \rangle$) projects to a basis of A_2. For every integer $m \geq 2$ denote by $\mathcal{N}_m$ the set of all monomials in X^m which are normal modulo $I = (\mathrm{R}(r))$. Then

$$\begin{aligned} \dim_{\mathbf{k}} A_m &= |\mathcal{N}_m| = |S_m| \\ &= \text{the number of all disjoint } \mathcal{D}_m(r)\text{-orbits in } X^m. \end{aligned}$$

(2) The *Koszul dual algebra* $A^!$ has a presentation $A^! = \mathbf{k}\langle \xi_i, \dots, \xi_n\rangle/(\mathrm{R}^\perp)$, where $\mathrm{R}^\perp$ consists of $s+n$ relations and splits into two disjoint sets

$$\begin{aligned} \mathrm{R}^\perp &= \mathrm{R}_0^\perp \cup \mathrm{R}_1^\perp, \quad \text{where} \\ \mathrm{R}_0^\perp &= \{\xi_j\xi_i + \xi_{i'}\xi_{j'} \mid x_jx_i - x_{i'}x_{j'} \in \mathrm{R}\}, \quad |\mathrm{R}_0^\perp| = s, \\ \mathrm{R}_1^\perp &= \{\xi_j\xi_i \mid (x_jx_i) \in \mathcal{F}(X,r)\}, \quad |\mathrm{R}_1^\perp| = n. \end{aligned}$$

There are equalities

$$\dim A_2 = n^2 - s, \quad \dim A_2^! = n^2 - s - n. \tag{3.5}$$

Suppose now that (X,r) is a nondegenerate square-free quadratic set (we do not assume 2-cancellativity). Then $\mathcal{F}(X,r) = \Delta_2$ and $|\mathcal{F}(X,r)| = n$. Moreover,

$$\mathrm{R}_1^\perp = \{\xi_i\xi_i \mid 1 \le i \le n\}.$$

Proposition 3.10. *Let (X,r) be a nondegenerate quadratic set of finite order $|X| = n \ge 3$, and let $A = A(\mathbf{k},X,r)$ be its associated quadratic $\mathbf{k}$-algebra, naturally graded by length. Suppose that X^2 contains exactly q nontrivial r-orbits, $\mathcal{O}_1,\dots,\mathcal{O}_q$, $|\mathcal{O}_j| = l_j \ge 2$, $1 \le j \le q$.*

(1) *If (X,r) is 2-cancellative, then the following conditions hold:*
 (a) *X^2 has exactly n one-element r-orbits, that is, $|\mathcal{F}(X,r)| = n$.*
 (b) *The following inequalities hold:*

$$2n-1 \le \dim_{\mathbf{k}} A_2 = n+q \le \binom{n+1}{2} \ \text{and}\ n-1 \le q \le \binom{n}{2}, \tag{3.6}$$

 where the upper bounds for $\dim_{\mathbf{k}} A_2$ and for q are exact for all $n \ge 3$, and the lower bounds are exact whenever $n = p > 2$ is a prime number.

(2) *Suppose $|\mathcal{F}(X,r)| = n$. Then the following conditions are equivalent:*
 (a) *The map r is involutive.*
 (b) $\dim_{\mathbf{k}} A_2 = \binom{n+1}{2}$.
 (c) $q = \binom{n}{2}$.
 (d) $\dim I_2 = |\Re(r)| = \binom{n}{2}$.
 (e) $\dim_{\mathbf{k}} A_2^! = \binom{n}{2}$.
 Each of these conditions implies that (X,r) is 2-cancellative.

(3) *For every integer $m \ge 2$, $\dim_{\mathbf{k}} A_m = |S_m| =$ the number of all disjoint D_m-orbits in X^m.*

Proof: (1) Suppose (X,r) is 2-cancellative. Part (1)(a) follows from Lemma 3.7.

(1)(b) The action of the cyclic group $\langle r\rangle$ on X^2 splits X^2 into disjoint r-orbits $\mathcal{O}$. We shall analyze the possible number of orbits and their lengths. It is clear that the map r is involutive *iff* every nontrivial orbit $\mathcal{O}(xy)$ has precisely two elements.

Let $X = \{x_1, \dots, x_n\}$ be an arbitrary enumeration on X. By Lemma 3.7, X^2 contains exactly n elements fixed under r (see (3.4)), so there are exactly n one-element r-orbits $\mathcal{O}(x_iy_i) = \{x_iy_i\}$, $1 \le i \le n$.

By assumption, the complement $X^2 \setminus \mathcal{F}(X,r)$ splits into q disjoint orbits:

$$X^2 \setminus \mathcal{F}(X,r) = \bigcup_{1\le j\le q} \mathcal{O}_j, \quad \text{where } |\mathcal{O}_j| = l_j \ge 2.$$

Then

$$|X^2 \setminus \mathcal{F}(X,r)| = n^2 - n = \sum_{1\le j\le q} l_j.$$

By the 2-cancellativity of r, a nontrivial orbit $\mathcal{O}$ does not contain distinct monomials of the shape $xu, xv, u \ne v$, or $xu, yu, x \ne y$, hence

$$2 \le |\mathcal{O}_j| = l_j \le n, \quad \forall 1 \le j \le q.$$

Therefore

$$2q \le \sum_{1\le j\le q} l_j = n^2 - n \le nq.$$

But $2q \le n^2 - n$ is equivalent with $q \le n(n-1)/2 = \binom{n}{2}$, moreover $n^2 - n \le nq$ implies $n - 1 \le q$. This proves the right-hand side inequalities in (3.6).

Recall that $a, b \in \mathcal{O}_j$ if and only if $a = b$ in the algebra A or, equivalently, in the monoid S. We argue with the number of distinct words of length 2 in S which is the same as the number of all orbits

$$|S_2| = n + q.$$

One has $A_2 = \operatorname{Span}_{\mathbf{k}} S_2$, and since every set of pairwise distinct words in S_2 is linearly independent, we obtain

$$2n - 1 \le \dim_{\mathbf{k}} A_2 = |S_2| = n + q \le n + \binom{n}{2} = \binom{n+1}{2},$$

which proves the left-hand side inequalities in (3.6). (One may also use the theory of Gröbner basis for a detailed proof; see Remark 3.9.) We shall discuss the exactness of the bounds after the proof of part (2).

(2) The equality $\dim_{\mathbf{k}} A_2 = n + q$ implies the equivalence of (b) and (c). The equivalence of (b) and (e) follows from (3.5).

Each $w \in X^2 \setminus \mathcal{F}$ belongs to a nontrivial orbit $\mathcal{O}(w)$, and $|X^2 \setminus \mathcal{F}| = n(n-1)$. It is clear that $q = \binom{n}{2}$ if and only if each nontrivial orbit has exactly two elements, which is equivalent to $r^2 = 1$. Thus (a) and (c) are equivalent. One has

$$\mathbf{k}\langle X\rangle_2 = I_2 \oplus A_2, \quad \dim(\mathbf{k}\langle X\rangle)_2 = \dim I_2 + \dim A_2,$$

so

$$n^2 - \dim A_2 = \dim I_2 = \sum_{1\le j\le q} (l_j - 1) = \Big(\sum_{1\le j\le q} l_j\Big) - q$$

and $\dim I_2 = \binom{n}{2}$ if and only if $\dim A_2 = \binom{n+1}{2}$, which gives the equivalence of (b) and (d).

It follows from [**22**, Corollary 2.13] that every nondegenerate involutive quadratic set (X, r) is 2-cancellative. Therefore each of the equivalent conditions (a) through (d) implies that (X, r) is 2-cancellative.

Part (2) implies that the upper bounds in (3.6) are exact. If $n = p > 2$ is a prime number, then by Corollary 6.18 every square-free self distributive solution (X, r) corresponding to a dihedral quandle of prime order $|X| = p > 2$ satisfies what we call *the minimality condition*: $\dim_{\mathbf{k}} A_2 = 2n - 1$, which is equivalent to $q = n - 1$; see Definition 6.1. This proves the exactness of the lower bound, whenever $n = p > 2$ is a prime number.

(3) The distinct elements of the monoid $S = S(X, r)$ form a $\mathbf{k}$-basis of the monoid algebra $\mathbf{k}S \simeq A(\mathbf{k}, X, r)$. In particular, $\dim A_m$ equals the number of distinct monomials of length m in S which is exactly the number of $\mathcal{D}_m(r)$-orbits in X^m; see Remark 3.9. □

Corollary 3.11. *Let (X, r) be a nondegenerate quadratic set of finite order $|X| = n \geq 3$. Suppose one of the following two conditions holds:*

(1) $|\mathcal{F}(X, r)| = n$.
(2) *(X, r) is a square-free quadratic set.*

Then the map r is involutive if and only if $\dim_{\mathbf{k}} A_2 = \binom{n+1}{2}$. In this case (X, r) is 2-cancellative.

Proposition 3.12. *Retaining the above notation, suppose that (X, r) is a nondegenerate quadratic set of finite order $|X| = n$ and $|\mathcal{F}(X, r)| = n$. Let $A = A(\mathbf{k}, X, r)$ be its associated quadratic algebra $A = \mathbf{k}\langle X\rangle/(\mathrm{R}(r))$.*

(1) *If A is Koszul, then $r^2 = \mathrm{id}$ and $|\mathrm{R}(r)| = \binom{n}{2}$.*
(2) *In particular, if there exists an enumeration of X such that the set of quadratic relations $\mathrm{R}(r)$ is a Gröbner basis or, equivalently, A is a PBW algebra, then (X, r) is involutive.*
(3) *In each of the cases:*
 (a) *(X, r) is 2-cancellative, or*
 (b) *(X, r) is square-free,*
 there is an equality $|\mathcal{F}(X, r)| = n$.

Proof: (1) Denote $s := |\mathrm{R}(r)|$. Suppose A is Koszul, so its dual algebra $A^!$ is also Koszul and their Hilbert series satisfy (3.1). By (3.5) one has

$$H_A(z) = 1 + nz + (n^2 - s)z^2 + (\dim A_3)z^3 + \cdots ,$$
$$H_{A^!}(-z) = 1 - nz + (n^2 - s - n)z^2 - (\dim A^!_3)z^3 + \cdots$$

We replace these in (3.1) and compute the coefficient for z^2 to yield:

$$(n^2 - s - n)z^2 - n^2z^2 + (n^2 - s)z^2 = 0,$$

which implies

$$|\mathrm{R}(r)| = s = \binom{n}{2}. \tag{3.7}$$

Each word $xy \in (X^2 \setminus \mathcal{F}(X, r))$ belongs to a nontrivial r-orbit, so xy occurs once in a relation in $\mathrm{R}(r)$. One has $|X^2 \setminus \mathcal{F}(X, r)| = n^2 - n = 2\binom{n}{2}$. This, together with (3.7), implies that each nontrivial r-orbit $\mathcal{O}$ in X^2 has length $|\mathcal{O}| = 2$, and therefore (X, r) is involutive. Clearly, in this case $\Re(r) = \mathrm{R}(r)$.

(2) Assume now that there exists an enumeration of X, such that the set of quadratic relations $\mathrm{R}(r)$ is a Gröbner basis. Then A is a PBW algebra (in the sense of Priddy), so A is Koszul and, by part (1), (X, r) is involutive. □

Corollary 3.13. *Let (X, r) be a nondegenerate quadratic set of finite order $|X| = n$, let $A = A(\mathbf{k}, X, r) = \mathbf{k}\langle X\rangle/(\Re(r))$, and let $I = (\Re(r))$ be the corresponding ideal of $\mathbf{k}\langle X\rangle$. Consider the following conditions:*

(1) *(X, r) is involutive.*
(2) *(X, r) is 2-cancellative.*
(3) *The set of fixed points $\mathcal{F}(X, r)$ has cardinality n.*
(4) *The number q of nontrivial r-orbits in X^2 is $q = \binom{n}{2}$.*
(5) $\dim A_2 = \binom{n+1}{2}$.
(6) $\dim I_2 = \binom{n}{2}$.
(7) *The algebra A has exactly $\binom{n}{2}$ defining relations $|\Re(r)| = \binom{n}{2}$.*
(8) $\dim A^!_2 = \binom{n}{2}$.
(9) *The algebra A is Koszul.*

The following implications hold.

(1) $\Longrightarrow$ (2), (3), (4), (5), (6), (8).

(2) $\Longrightarrow$ (3) *and* $n - 1 \leq q \leq \binom{n}{2}$, $2n - 1 \leq \dim A_2 \leq \binom{n+1}{2}$.

Assume (2) holds. Then

$$(1) \iff (4) \iff (5) \iff (6) \iff (7) \iff (8) \quad (9) \implies (1).$$

Lemma 3.14. *Suppose $X = \{x_1, x_2, x_2\}$, and (X, r) is a nondegenerate, 2-cancellative, and square-free quadratic set.*

Then (up to isomorphism) there are exactly two nonisomorphic quadratic algebras $A^{(i)}$ corresponding to quadratic sets (X, r_i), $i = 1, 2$, which satisfy the hypothesis. These are

$$A^{(1)} = \mathbf{k}\langle X : x_3x_2 - x_2x_3,\ x_3x_1 - x_1x_3,\ x_2x_1 - x_1x_2\rangle,$$

$$A^{(2)} = \mathbf{k}\langle X : x_3x_2 - x_1x_3,\ x_3x_1 - x_2x_3,\ x_2x_1 - x_1x_2\rangle.$$

The algebras $A^{(1)}$ and $A^{(2)}$ are PBW *algebras with* $\operatorname{GK}\dim A = 3$ *(in fact these are binomial skew-polynomial rings). The quadratic set (X, r_1) is the trivial solution of the YBE. Up to isomorphism, (X, r_2) is the unique nontrivial square-free solution of the YBE of order $|X| = 3$.*

Question 3.15. Let (X, r) be a 2-cancellative nondegenerate quadratic set of finite order $|X| = n$. Suppose $A = A(\mathbf{k}, X, r)$ is *PBW*. (We know that this implies $r^2 = 1$.)

(1) Is it true that if (X, r) is square-free, then A has polynomial growth?

An equivalent question is:

(2) Is it true that if (X, r) is square-free, then A has finite global dimension?

This is so for $|X| = 3$; see Lemma 3.14. An affirmative answer would imply that (X, r) is a solution of the YBE, and A satisfies all conditions (1) through (8) in Theorem 3.16. We do not know of a counterexample.

In [**18**] we study the close relation between square-free nondegenerate symmetric sets (X, r) and a class of Artin–Schelter regular algebras. Our result ([**18**, Theorem 2]) investigates *quantum binomial quadratic sets* (X, r), that is, square-free nondegenerate involutive quadratic set, in terms of various algebraic, homological, and numerical properties of the algebra $A(\mathbf{k}, X, r)$. We have proven that each of these properties of A is equivalent to the fact that (X, r) is a solution of the YBE. Our new Theorem 3.16 is a similar but stronger result. We *weaken* the hypothesis, assuming only that (X, r) is a square-free nondegenerate quadratic set (we do not assume involutiveness of r) and give a list of similar algebraic and homological properties of A each of which is equivalent to saying that (X, r) is an involutive solution of the YBE (that is, a symmetric set).

Theorem 3.16. *Let (X,r) be a square-free nondegenerate quadratic set of finite order $|X| = n$. Let A be its associated quadratic algebra $A = \mathbf{k}\langle X\rangle/(\mathrm{R}(r))$. The following conditions are equivalent:*

(1) *The Hilbert series of A is*

$$H_A(z) = \frac{1}{(1-z)^n}. \tag{3.8}$$

(2) *A is a PBW algebra with a set $X = \{x_1, x_2, \dots, x_n\}$ of PBW generators and with polynomial growth.*

(3) *A is a PBW algebra with a set $X = \{x_1, x_2, \dots, x_n\}$ of PBW generators and with finite global dimension $\operatorname{gl\,dim} A < \infty$.*

(4) *A is a PBW Artin–Schelter regular algebra.*

(5) *There exists an enumeration $X = \{x_1, x_2, \dots, x_n\}$ such that the set*

$$\mathcal{N} = \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\}$$

is a $\mathbf{k}$-basis of A. We also have:

(6)

$$\dim_{\mathbf{k}} A_2 = \binom{n+1}{2} \quad \text{and} \quad \dim_{\mathbf{k}} A_3 = \binom{n+2}{3}.$$

(7)

$$\dim_{\mathbf{k}} A^!_2 = \binom{n}{2} \quad \text{and} \quad \dim_{\mathbf{k}} A^!_3 = \binom{n}{3}.$$

(8) *A is a binomial skew polynomial ring in the sense of* [**14**].

(9) *(X,r) is* a square-free symmetric set, *that is, an involutive solution of the YBE.*

In this case A is a Noetherian domain. Moreover,

$$\operatorname{GK\,dim} A = n = \operatorname{gl\,dim} A, \quad \dim_{\mathbf{k}} A_m = \binom{n+m-1}{m}, \quad m \geq 2. \tag{3.9}$$

Proof: We shall prove that each one of the conditions (1) through (9) implies that (X,r) is involutive.

Note first that the hypothesis of the theorem implies $|\mathcal{F}(X,r)| = n$. Then by Proposition 3.10, part (2), the map r is involutive *iff* $\dim_{\mathbf{k}} A_2 = \binom{n+1}{2}$.

Each of the conditions (6), (7), and (9) implies straightforwardly that (X,r) is involutive. Each of the conditions (2), (3), (4), (5), (8) gives that A is a PBW algebra. Hence, by Proposition 3.12, (X,r) is involutive.

Assume (1) holds. Then A and the algebra of polynomials $P^n = \mathbf{k}[x_1, \dots, x_n]$ have the same Hilbert series $H_A(z) = H_{P^n}(z)$; see (3.8). Therefore A and P^n have the same Hilbert functions

$$\dim A_m = h_A(m) = h_{P^n}(m) = \binom{n+m-1}{m}, \quad m \geq 1.$$

In particular, $\dim A_2 = \binom{n+1}{2}$, so r is involutive. Moreover, $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$.

We have shown that each one of the conditions (1) through (9) implies that (X, r) is a square-free nondegenerate involutive quadratic set, that is, (X, r) is *a quantum binomial quadratic set*; see Definition 1.1(4). Now our result [**18**, Theorem 1.2] implies straightforwardly the equivalence of conditions (1) through (9), the equalities (3.9), and the fact that A is a Noetherian domain. □

The 3-generated PBW algebras from Lemma 3.14 are particular cases of the class of PBW algebras described by Theorem 3.16.

4. Square-free quadratic sets with cyclic conditions

In this section we continue the study of square-free nondegenerate quadratic sets (X, r), the associated algebra $A(\mathbf{k}, X, r)$, and the monoid $S(X, r)$. In a series of works (see [**15, 22, 21, 7**]), we have shown that the combinatorial properties of a solution of the YBE (X, r) are closely related to the algebraic and combinatorial properties of its associated structures. Solutions satisfying some of the conditions defined below are of particular interest.

Definition 4.1 ([**22, 15**]). Let (X, r) be a quadratic set.

(1) The following are called *cyclic conditions on* (X, r).

$$\text{cl1}: {}^{(y^x)}x = {}^y x, \quad \text{for all } x, y \in X; \quad \text{cr1}: x^{({}^x y)} = x^y, \quad \text{for all } x, y \in X;$$

$$\text{cl2}: {}^{({}^x y)}x = {}^y x, \quad \text{for all } x, y \in X; \quad \text{cr2}: x^{(y^x)} = x^y, \quad \text{for all } x, y \in X.$$

(2) Condition lri on (X, r) is defined as

$$\text{lri}: \ ({}^x y)^x = y = {}^x(y^x), \quad \text{for all } x, y \in X.$$

In other words, lri holds if and only if (X, r) is nondegenerate and

$$\mathcal{R}_x = \mathcal{L}_x^{-1} \quad \text{and} \quad \mathcal{L}_x = \mathcal{R}_x^{-1}.$$

The cyclic conditions were introduced by the author in [**13, 14**] in the context of binomial skew polynomial algebras and were crucial for the proof that every binomial skew polynomial algebra defines canonically (via its relations) a set-theoretic solution of the YBE; see [**24**]. It is known that *every square-free nondegenerate symmetric set* (X, r) *satisfies the cyclic conditions cc and condition lri, so the map* r *is uniquely determined by the left action:* $r(x, y) = (\mathcal{L}_x(y), \mathcal{L}_y^{-1}(x))$; see [**15, 22**]. We shall prove that every square-free nondegenerate braided set (X, r) (not necessarily finite or involutive) satisfies the cyclic conditions cl1 and cr1; see Proposition 4.4. The main result of this section is Theorem 4.7.

4.1. Combinatorics in square-free quadratic sets with cyclic conditions. We recall the following useful result.

Fact 4.2 ([**22**, Proposition 2.25])**.** *Suppose* (X, r) *is a quadratic set.*

(1) *Any two of the following conditions imply the remaining third condition:*
   (a) (X, r) *is involutive.*
   (b) (X, r) *is nondegenerate and cyclic.*
   (c) (X, r) *satisfies lri.*
(2) *In particular, if* (X, r) *satisfies cl1 and cr1, then* (X, r) *is involutive* iff *condition lri holds.*

Sketch of the proof: For convenience of the reader we shall sketch the proof of (2). Assume lri. We shall prove that r is involutive or, equivalently, (2.1) holds. We apply first cr1 and lri, and next cl1 and lri to yield:

$${}^{{}^u v}(u^v) = {}^{{}^u v}(u^{{}^u v}) = u, \quad ({}^u v)^{u^v} = ({}^{u^v} v)^{u^v} = v, \quad \forall u, v \in X.$$

Conversely, assume that (X, r) is involutive. We shall prove that lri holds. Let $u, t \in X$. We have to show ${}^t(u^t) = u$ and $({}^t u)^t = u$. By nondegeneracy, there exist $v, w \in X$ such that $t = {}^u v = w^u$. Then we use cr1, cl1, and (2.1) to yield:

$${}^t(u^t) = {}^{{}^u v}(u^{{}^u v}) = {}^{{}^u v}(u^v) = u, \quad ({}^t u)^t = ({}^{w^u} u)^{w^u} = ({}^w u)^{w^u} = u. \qquad \square$$

Suppose (X, r) is a finite nondegenerate quadratic set $S = S(X, r)$. As we discussed in the preliminaries, for every integer $m \geq 2$, the group $\mathcal{D}_m(r) = {}_{\mathrm{gr}}\langle r^{ii+1}, 1 \leq i \leq m-1 \rangle$ acts on the left on X^m. Each element $a \in S$ can be presented as a monomial $a = \zeta_1 \zeta_2 \cdots \zeta_n$, $\zeta_i \in X$. Two words $a, b \in \langle X \rangle$ are equal in S if they have the same length, say $a, b \in X^m$, and belong to the same orbit of $\mathcal{D}_m(r) = {}_{\mathrm{gr}}\langle r^{ii+1}, 1 \leq i \leq m-1 \rangle$. Clearly, (X, r) is square-free if and only if $\mathcal{D}_m(r)$ acts trivially on Δ_m, $m \geq 2$.

Corollary 4.3. *Suppose* (X, r) *is a square-free quadratic set. Let* $x, y \in X$*, let* m *be an integer,* $m \geq 2$*. If* $x^m = y^m$ *is an equality in* S*, then* $x = y$*.*

Proposition 4.4. *Let (X, r) be a square-free nondegenerate braided set of arbitrary cardinality. Then (X, r) satisfies the cyclic conditions cl1 and cr1. Moreover, (X, r) is involutive* iff *condition lri holds.*

Proof: Let $a, x \in X$. Consider the *Yang–Baxter* diagram on monomials of length 3 in X^3.

$$\begin{array}{ccc} axx & \xrightarrow{r^{23}} & axx \\ {\scriptstyle r^{12}}\downarrow & & \downarrow{\scriptstyle r^{12}} \\ ({}^a x)(a^x)x & & ({}^a x)(a^x)x \\ {\scriptstyle r^{23}}\downarrow & & \downarrow{\scriptstyle r^{23}} \\ ({}^a x)({}^{(a^x)}x)(a^{xx}) & \xrightarrow{r^{12}} & ({}^a x)({}^{(a^x)}x)(a^{xx}) \end{array}$$

It follows that $r({}^a x, {}^{(a^x)}x) = ({}^a x, {}^{(a^x)}x)$ and therefore, by Corollary 3.8, ${}^{(a^x)}x = {}^a x$, which proves cl1.

Similarly, a YB diagram starting with the monomial xxa implies $r(x^{({}^x a)}, x^a) = (x^{({}^x a)}, x^a)$, hence $x^{({}^x a)} = x^a$, which proves cr1. Now Fact 4.2 implies straightforwardly that (X, r) is involutive *iff* lri holds. □

The action of the infinite dihedral group $\mathcal{D}$ on X^3 is of particular importance in this section. Assuming that (X, r) is a nondegenerate square-free quadratic set we shall find some counting formulae and inequalities involving the orders of the $\mathcal{D}$-orbits in X^3 and their number. As usual, the orbit of a monomial $\omega \in X^3$ under the action of $\mathcal{D}$ will be denoted by $\mathcal{O} = \mathcal{O}(\omega)$.

Definition 4.5. We call a $\mathcal{D}$-orbit $\mathcal{O}$ *square-free* if

$$\mathcal{O} \cap (\Delta_2 \times X \cup X \times \Delta_2) = \emptyset.$$

A monomial $\omega \in X^3$ is *square-free* in S if its orbit $\mathcal{O}(\omega)$ is square-free.

Notation 4.6. Denote $\mathrm{E}(\mathcal{O}) = \mathcal{O} \cap ((\Delta_2 \times X \cup X \times \Delta_2) \backslash \Delta_3)$.

Theorem 4.7. *Suppose (X, r) is a nondegenerate square-free quadratic set of finite order.*

(1) *Let $\mathcal{O}$ be a $\mathcal{D}$-orbit in X^3. The following implications hold:*
   (a) $\mathcal{O} \cap \Delta_3 \neq \emptyset \Leftrightarrow |\mathcal{O}| = 1$.
   (b) $\mathrm{E}(\mathcal{O}) \neq \emptyset \Rightarrow |\mathcal{O}| \geq 3$.
   In this case we say that $\mathcal{O}$ is a $\mathcal{D}$-orbit of type (b)*.*
   (c) $\mathcal{O} \cap (\Delta_2 \times X \cup X \times \Delta_2) = \emptyset \Rightarrow |\mathcal{O}| \geq 6$.
   Recall that in this case $\mathcal{O}$ is called a square-free orbit*; see Definition 4.5.*

(2) *The following two conditions are equivalent:*
   (a) *(X, r) is involutive and satisfies the cyclic conditions cl1 and cr1.*
   (b) *Every orbit $\mathcal{O}$ of type (b) contains exactly three distinct elements.*

Proof: Condition (1)(a) is clear.

(1)(b) Assume that $\mathrm{E}(\mathcal{O}) \neq \emptyset$. Then $\mathcal{O}$ contains an element of the shape $\omega = xxy$, or $\omega = xyy$, where $x, y \in X$, $x \neq y$. Without loss of generality we can assume $\omega = xxy \in \mathcal{O}$. We look at a fragment of the Yang–Baxter diagram starting with ω:

$$\omega = \omega_1 = xxy \longrightarrow^{r^{23}} \omega_2 = x({}^x y)(x^y) \longrightarrow^{r^{12}} \omega_3 = ({}^{x^2} y)(x^{{}^x y})(x^y) \longrightarrow \cdots$$

Note that the first three elements ω_1, ω_2, ω_3 are distinct monomials in X^3. Indeed, $x \neq y$ implies $r(xy) \neq xy$ in X^2 (see Corollary 3.8), so $\omega_2 \neq \omega_1$. By assumption, (X, r) is square-free, so ${}^x x = x$, and $y \neq x$, implies ${}^x y \neq x$, by nondegeneracy. Therefore $r(x({}^x y)) \neq x({}^x y)$ and $\omega_3 \neq \omega_2$. Furthermore, $\omega_3 \neq \omega_1$. Indeed, if we assume $x = {}^{x^2} y = {}^x({}^x y)$, then by (2.2) one has ${}^x y = x$, and therefore $y = x$, a contradiction. It follows that $|\mathcal{O}| \geq 3$.

(1)(c) Suppose $\mathcal{O} = \mathcal{O}(xyz)$ is a square-free $\mathcal{D}$-orbit in X^3. Consider the set

$$O_1 = \{v_i \mid 1 \leq i \leq 6\} \subseteq \mathcal{O}$$

consisting of the first six elements of the Yang–Baxter diagram

$$\begin{array}{ccc}
v_1 = xyz & \xrightarrow{\;\;r^{12}\;\;} & ({}^x y x^y)z = v_2 \\
\Big\downarrow{\scriptstyle r^{23}} & & \Big\downarrow{\scriptstyle r^{23}} \\
v_3 = x({}^y z y^z) & & ({}^x y)({}^{x^y} z)(x^y)^z = v_5 \\
\Big\downarrow{\scriptstyle r^{12}} & & \Big\downarrow{\scriptstyle r^{12}} \\
v_4 = {}^x({}^y z)(x^{{}^y z})(y^z) & & [{}^{{}^x y}({}^{x^y} z)][({}^x y)^{({}^{x^y} z)}][(x^y)^z] = v_6
\end{array}$$

Clearly,

$$O_1 = U_1 \cup U_3 \cup U_5, \quad \text{where } U_j = \{v_j,\, v_{j+1} = r^{12}(v_j)\}, \quad j = 1, 3, 5.$$

We claim that U_1, U_3, U_5 are pairwise disjoint sets, and each of them has order 2. Note first that since v_j is a square-free monomial, for each $j = 1, 3, 5$, one has $v_j \neq r_{12}(v_j) = v_{j+1}$, therefore $|U_j| = 2$, $j = 1, 3, 5$. The monomials in each U_j have the same tail. More precisely, $v_1 = (xy)z$ and $v_2 = r(xy)z$ have a tail z; the tail of v_3 and v_4 is y^z; and the tail of v_5 and v_6 is $(x^y)^z$. It will be enough to show that the three elements $z, y^z, (x^y)^z \in X$ are pairwise distinct. But $\mathcal{O}(xyz)$ is square-free, so $y \neq z$ and by (2.2), $y^z \neq z$. Furthermore $v_2 = ({}^x y)(x^y)z \in \mathcal{O}(xyz)$, so $x^y \neq y$ and $x^y \neq z$. Now by nondegeneracy one has

$$x^y \neq z \Longrightarrow (x^y)^z \neq z, \quad x^y \neq y \Longrightarrow (x^y)^z \neq y^z.$$

Therefore, the three elements $z, y^z, (x^y)^z \in X$ occurring as tails in U_1, U_3, U_5, respectively, are pairwise distinct, so the three sets are pairwise disjoint. This implies $6 = |O_1| \leq |\mathcal{O}|$.

(2)(a) $\Rightarrow$ (b) Suppose (X, r) is involutive and satisfies cl1 and cr1. Let $\mathcal{O}$ be an orbit of type (b). Without loss of generality we may assume $\mathcal{O} = \mathcal{O}(xxy)$. Then (since r is involutive) each arrow in diagram (4.1) is pointed in both directions, i.e. the arrows have the shape $\longleftrightarrow$ or $\updownarrow$.

$$\begin{array}{ccc}
v_1 = xxy & \xrightarrow{\;r^{12}\;} & xxy = v_1 \\
\downarrow{\scriptstyle r^{23}} & & \downarrow{\scriptstyle r^{23}} \\
v_2 = x({}^x y x^y) & & x({}^x y x^y) = v_2 \\
\downarrow{\scriptstyle r^{12}} & & \downarrow{\scriptstyle r^{12}} \\
v_3 = ({}^x({}^x y))(x^{{}^x y})(x^y) = ({}^{xx}y)x^y x^y & \xrightarrow{\;r^{23}\;} & ({}^x({}^x y))(x^{{}^x y})(x^y) = ({}^{xx}y)(x^y)(x^y) = v_3
\end{array} \tag{4.1}$$

It is clear that the diagram contains all elements of $\mathcal{O}$, hence $|\mathcal{O}| = 3$.

(b) $\Rightarrow$ (a) Suppose every orbit $\mathcal{O}$ of type (b) contains exactly three elements. The diagram

$$\begin{array}{ccc}
v_1 = xxy & \xrightarrow{\;r^{12}\;} & xxy = v_1 \\
\downarrow{\scriptstyle r^{23}} & & \downarrow{\scriptstyle r^{23}} \\
v_2 = x({}^x y x^y) & & x({}^x y x^y) = v_2 \\
\downarrow{\scriptstyle r^{12}} & & \downarrow{\scriptstyle r^{12}} \\
v_3 = {}^x({}^x y)(x^{{}^x y})(x^y) & & {}^x({}^x y)(x^{{}^x y})(x^y) = v_3
\end{array}$$

contains three distinct elements of $\mathcal{O}$, and therefore it contains the whole orbit $\mathcal{O}$.

The element $r^{23}(v_3) = {}^x({}^x y) r((x^{{}^x y})(x^y))$ belongs to $\mathcal{O} = \{v_1, v_2, v_3\}$. It is clear that $r^{23}(v_3) \neq v_1$ and $r^{23}(v_3) \neq v_2$, so $r^{23}(v_3) = v_3$. This implies that $r((x^{{}^x y})(x^y)) = (x^{{}^x y})(x^y)$, hence $(x^{{}^x y})(x^y) \in \mathcal{F}(X, r) = \Delta_2$. Therefore

$$x^{{}^x y} = x^y, \quad \forall x, y \in X,$$

that is, cr1 holds. An analogous argument proves cl1 (in this case we work with a YB diagram with a left top element $v_1 = xyy$).

Notice that if there exists a pair (x, y) with $r^2(xy) \neq xy$, then the orbit $\mathcal{O}(xxy)$ contains (but is not limited to) the following four distinct elements

$$\begin{aligned} &v_1 = xxy, && v_2 = r^{23}(v_1) = xr(xy) = x({}^x y x^y), \\ &v_3 = r^{12}(v_2) = ({}^{xx}y)(x^{{}^x y})(x^y), && v_4 = r^{23}(v_2) = xr^2(xy), \end{aligned}$$

which contradicts (b). It follows that r is involutive. We have proven (b) $\Rightarrow$ (a). □

4.2. More on square-free quadratic sets with cyclic conditions. We end up the section with new results on square-free quadratic sets which will be used to describe the contrast between involutive and non-involutive solutions of the YBE in the next section.

Proposition 4.8. *Suppose (X, r) is a finite nondegenerate square-free quadratic set with $|X| = n$ and that satisfies cl1 and cr1. Then (X, r) is a symmetric set if and only if $\dim_{\mathbf{k}} A^!_3 = \binom{n}{3}$.*

Proof: By hypothesis, cl1 and cr1 hold. Assume $\dim_{\mathbf{k}} A^!_3 = \binom{n}{3}$. We have to show that (X, r) is a symmetric set. We shall prove first that (X, r) is involutive, and therefore it is a quantum binomial set; see Definition 1.1(4).

As usual, we study the $\mathcal{D}_3$-orbits $\mathcal{O}$. Our assumption implies that X^3 contains exactly $\binom{n}{3}$ square-free orbits $\mathcal{O}^{(s)}$, $1 \leq s \leq \binom{n}{3}$. By Theorem 4.7, part (1)(c), the length of each square-free orbit satisfies

$$|\mathcal{O}^{(s)}| = l_s \geq 6, \quad \forall 1 \leq s \leq \binom{n}{3}. \tag{4.2}$$

Denote by W the set of all words $w \in X^3 \setminus \Delta_3$ which vanish in $A^!_3$. Note first that if $y, b \in X$, $y \neq b$, the orbit $\mathcal{O}(yyb) \subset W$ contains the three distinct monomials occurring in the following diagram

$$u = yyb \longrightarrow_{r^{23}} y({}^y b y^b) \longrightarrow_{r^{12}} ({}^{yy}b)(y^{{}^y b} \cdot y^b) = ({}^{yy}b)(y^b \cdot y^b).$$

We shall call the word $r^{23}(yyb) = y({}^y b y^b)$ *the transition element for the pair of words* $u = yyb, czz = r^{12} \circ r^{23}(u) \in W$. It is clear that each pair $y, b \in X$, $y \neq b$, determines uniquely the three elements yyb, $r^{23}(yyb), czz = r^{12} \circ r^{23}(yyb) \in W$.

Note that if $(y, b) \neq (t, c)$, then the transition elements $y({}^y b y^b) \neq t({}^t c t^c)$. Indeed, the inequality is straightforward if $t \neq y$. If $t = y, b \neq c$, then by nondegeneracy one has ${}^y b \neq {}^y c = {}^t c$. So W contains $n(n-1)$ disjoint triples $yyb, r^{23}(yyb), r^{12} \circ r^{23}(yyb) = czz$. Therefore $|W| \geq 3n(n-1)$.

Assume that (X, r) is not involutive, we shall prove that $|W| > 3n(n-1)$.

Clearly, there exist a pair $x, a \in X$, $x \neq a$, such that $r^2(x,a) \neq (x,a)$ so the words xa, $r(xa)$, $r^2(xa)$ are distinct elements of X^2. Then the orbit $\mathcal{O} = \mathcal{O}(xxa)$ contains at least the set O of four distinct monomials given below:

$$\begin{aligned} \mathrm{O} = \{v_1 = xxa,\ & v_2 = r^{23}(xxa) = x({}^x a x^a), \\ & v_3 = r^{12} \circ r^{23}(v_1) = ({}^{xx}a)(x^a x^a), \\ & v_4 = (r^{23})^2(v_1) = xr^2(xa)\}. \end{aligned}$$

Moreover, the set O contains the word $v_4 = xr^2(xa)$ which is square-free, but is not a transition element for any triple $yyb, r^{23}(yyb), r^{12} \circ r^{23}(yyb) = czz$. This implies that

$$|W| > 3n(n-1). \tag{4.3}$$

The set X^3 splits into the following disjoint subsets

$$X^3 = \Delta_3 \cup W \cup \Bigg(\bigcup_{1 \le s \le \binom{n}{3}} \mathcal{O}^{(s)} \Bigg).$$

This, together with (4.2) and (4.3), implies

$$n^3 = |X^3| = |\Delta_3| + |W| + \sum_{1 \le s \le \binom{n}{3}} |\mathcal{O}^{(s)}| > n + 3n(n-1) + 6\binom{n}{3} = n^3,$$

which gives a contradiction. It follows that r is involutive, hence (X, r) is a quantum binomial set. Now our result [**18**, Theorem 1.2] implies that (X, r) is a solution of the YBE, and therefore, it is a symmetric set. The converse implication follows again from [**18**, Theorem 1.2]. □

Lemma 4.9. *Let (X, r) be a finite square-free nondegenerate quadratic set and let $S = S(X, r)$. Suppose (X, r) satisfies the cyclic conditions cl1 and cr1. The following conditions are equivalent:*

(1) *(X, r) is involutive.*
(2) *S satisfies the following conditions:*

$$\begin{aligned} & axx = byy \text{ holds in } S, \quad a,b,x,y \in X \Longrightarrow a = b,\ x = y; \\ & xxc = yyd \text{ holds in } S, \quad c,d,x,y \in X \Longrightarrow c = d,\ x = y. \end{aligned} \tag{4.4}$$

(3)

$$\begin{aligned} & byy \in \mathcal{O}(axx), \quad a,b,x,y \in X \Longrightarrow a = b,\ x = y; \\ & yyd \in \mathcal{O}(xxc), \quad c,d,x,y \in X \Longrightarrow c = d,\ x = y. \end{aligned}$$

Proof: The equivalence (2) $\Leftrightarrow$ (3) is clear.

(1) $\Rightarrow$ (2) Suppose (X, r) is involutive. Theorem 4.7 implies that, for each $a \neq x$, the orbit $\mathcal{O} = \mathcal{O}(axx)$ is of type (b) and $\mathcal{O} \cap (X \times \Delta_2) = \{axx\}$. In other words, there is no element of the shape $byy \neq axx$ such that $byy \in \mathcal{O}$ which gives the first implication in (4.4). Analogous argument gives the second implication in (4.4).

(2) $\Rightarrow$ (1) Conversely, assume that conditions (4.4) hold. We have to show that r is involutive. By Fact 4.2 it will be enough to prove that (X, r) satisfies condition lri, that is,

$$({}^t x)^t = x \text{ and } {}^t(x^t) = x, \quad \forall x, t \in X. \tag{4.5}$$

Let $a, x \in X$. We consider the elements of the $\mathcal{D}$-orbit $\mathcal{O}(axx)$ in X^3 and deduce the following equalities of elements in $S = S(X, r)$:

$$\begin{aligned}
a.xx &= ({}^a x)(a^x)x \\
&= ({}^a x)({}^{a^x} x)a^{xx} = ({}^a x)({}^a x)(a^{xx}) \\
&= ({}^a x)({}^{{}^a x}(a^{xx}))(({}^a x)^{a^{xx}}) \\
&= ({}^{({}^a x}({}^{{}^a x}(a^{xx})))(({}^a x)^{({}^{{}^a x}(a^{xx}))}(({}^a x)^{(a^{xx})}) \\
&= b(({}^a x)^{a^{xx}})(({}^a x)^{a^{xx}}) \quad \text{where } b = {}^{({}^a x)}({}^{({}^a x)}(a^{xx})) \\
&= byy, \quad \text{where } y = [({}^a x)]^{(a^{xx})}.
\end{aligned}$$

We have obtained that, for $a \neq x$ the following equalities hold in S

$$axx = byy, \quad \text{where } y = [({}^a x)]^{(a^{xx})}.$$

Now the first condition in (4.4) implies

$$y = [({}^a x)]^{(a^{xx})} = x$$

and

$$\begin{aligned}
{}^a x &= {}^{a^x} x = {}^{(a^x)^x} x = {}^{(a^{(xx)})} x, \\
x &= [({}^a x)]^{(a^{xx})} = [{}^{(a^{(xx)})} x]^{(a^{xx})} = ({}^t x)^t, \quad \text{where } t = a^{(xx)}.
\end{aligned} \tag{4.6}$$

Suppose $t, x \in X$. By nondegeneracy there exists $a_1 \in X$ such that $t = a_1^x$ and, similarly, there exists $a \in X$ with $a^x = a_1$, hence $t = a^{xx}$ for some $a \in X$. Then (4.6) implies $({}^t x)^t = x$. The second equality ${}^t(x^t) = x$ in (4.5) is proven by an analogous argument. Therefore (X, r) satisfies condition lri. $\square$

5. Square-free braided sets and the contrast between the involutive and noninvolutive cases

Braided monoids were introduced and studied in [**22**]. For convenience of the reader we recall basic definitions and results in Section 8. Recall that if (X, r) is a braided set, then its monoid $S(X, r)$ is a graded braided M3-monoid. We denote it by (S, r_S); see Definitions 8.1 and 8.2, in particular, S satisfies condition ML2. More details can be found in Section 8.

Notation 5.1. Let (X, r) be a nondegenerate quadratic set. Let $a, x \in X$ and let m be a positive integer. We shall use the following notation

$$^{(x^m)}a := (\mathcal{L}_x^m)(a), \quad a^{(x^m)} := (\mathcal{R}_x^m)(a).$$

This formulae agree with the natural left and right actions of S upon itself.

Remark 5.2. Suppose (X, r) is a quadratic set with cl1 and cr1. Then the following equalities hold in X:

$$^{a^{(x^m)}}x = {}^a x, \quad x^{^{(x^m)}a} = x^a, \tag{5.1}$$

for all $a, x \in X$ and all positive integers m.

The formulae in (5.1) are easy to prove using induction on m.

Proposition 5.3. *Let (X, r) be a square-free nondegenerate quadratic set satisfying the cyclic conditions cl1 and cr1, and let $S = S(X, r)$. Then the following conditions hold:*

(1) *For every pair $a, x \in X$ and every positive integer m the following equalities hold in S:*

$$a \cdot (x^m) = ((^a x)^m) \cdot (a^{(x^m)}), \quad (x^m) \cdot a = (^{(x^m)}a)((x^a)^m). \tag{5.2}$$

(2) *Assume that (X, r) is a braided set. Then the following are equalities in the braided monoid S:*

$$^a(x^m) = (^a x)^m, \quad (x^m)^a = (x^a)^m, \tag{5.3}$$

for all $a, x \in X$ and all positive integers m.

(3) *Suppose that (X, r) is a finite braided set, and let p be the least common multiple of the orders of all actions $\mathcal{L}_x$ and $\mathcal{R}_x$, $x \in X$, so $^{(x^p)}a = a = a^{(x^p)}$, $\forall a, x \in X$. Then the following equalities hold in S:*

$$a \cdot (x^p) = a \cdot ((^a x)^a)^p, \quad (x^p) \cdot a = (^a(x^a))^p \cdot a, \quad \forall a, x \in X. \tag{5.4}$$

$$(x^p)(y^p) = (y^p)(x^p), \quad \forall x, y \in X. \tag{5.5}$$

Proof: (1) We shall use induction on m to prove the first equality in (5.2). Clearly, for $m = 1$, one has $ax = {}^a x{\cdot}a^x$. Assume $a{\cdot}(x^k) = (({}^a x)^k){\cdot}(a^{(x^k)})$, $\forall 1 \leq k \leq m$, and all $a, x \in X$. Let $a, x \in X$. Then

$$\begin{aligned} a\cdot(x^{m+1}) &= (a\cdot x^m)x = (({}^a x)^m)\cdot[(a^{(x^m)})x] = ({}^a x)^m\cdot({}^{a^{(x^m)}}x)\cdot(a^{(x^m)})^x \\ &= ({}^a x)^{m+1})(a^{(x^{m+1})}), \end{aligned}$$

as claimed. For the last equality we have used (5.1).

This verifies the first equality in (5.2). An analogous argument verifies the second equality in (5.2).

(2) Assume that (X, r) is a braided set. Then (X, r) satisfies cl1 and cr1; see Proposition 4.4. We shall prove (5.3) using induction on m. The base for induction is clear. Assume the formula is true for all $k \leq m-1$, where $m \geq 2$. We use the inductive assumption, ML2, and (5.1) to get

$${}^a(x^m) = {}^a((x^{m-1})\cdot x) = ({}^a(x^{m-1}))\cdot({}^{(a^{(x^{m-1})})}x) = ({}^a x)^{m-1})({}^a x) = ({}^a x)^m.$$

This proves the first equality in (5.3). Analogous argument verifies the second equality in (5.3).

(3) Assume now that (X, r) is a finite braided set and p is the least common multiple of the orders of all actions. We use successively M3, (5.3), and M3 again to obtain

$$a\cdot(x^p) = {}^a(x^p)\cdot(a^{(x^p)}) = ({}^a x)^p\cdot a = ({}^{(({}^a x)^p)}a)\cdot(({}^a x)^p)^a = a\cdot(({}^a x)^a)^p.$$

This gives the first equality in (5.4). An analogous argument proves the second equality in (5.4). The equality (5.5) is straightforward. □

Proposition 5.4. *Let (X, r) be a square-free nondegenerate braided set of finite order $|X| = n$, let $S = S(X, r) = (S, r_S)$ be the associated braided monoid, and let $A = A(\mathbf{k}, X, r)$. Let p be the least common multiple of the orders of all actions $\mathcal{L}_x$ and $\mathcal{R}_x$, $x \in X$. The following conditions are equivalent:*

(1) *The equality $ax^p = ay^p$ in S, for $a, x, y \in X$, implies $x = y$.*
(2) *The equality $(x^p)a = (y^p)a$ in S, for $a, x, y \in X$, implies $x = y$.*
(3) *The monoid S is cancellative.*
(4) *The quadratic algebra A has* Gelfand–Kirillov dimension $\operatorname{GK}\dim A = n$.
(5) *The solution (X, r) is involutive, that is, (X, r) is a symmetric set.*

In this case $S(X, r)$ is embedded in the braided group $G(X, r) = (G, r_G)$. Moreover, both (S, r_S) and (G, r_G) are also (nondegenerate) involutive solutions.

Proof: The implications (3) $\Rightarrow$ (1), (2), are clear.

(1) $\Rightarrow$ (2) Assume $(x^p)a = (y^p)a$, where $a, x, y \in X$. Then we use (5.2) to obtain

$$(x^p)a = a(x^a)^p, \quad (y^p)a = a(y^a)^p.$$

It follows that $a(x^a)^p = a(y^a)^p$, so by our assumption (1), $x^a = y^a$. Hence, by nondegeneracy, $x = y$. The implication (2) $\Rightarrow$ (1) is proven analogously.

(1) $\Rightarrow$ (5) Suppose condition (1) holds (hence (2) is also true). Proposition 5.3 implies the following equalities in S:

$$a \cdot (x^p) = a \cdot (({}^a x)^a)^p, \quad (x^p) \cdot a = ({}^a(x^a))^p \cdot a, \quad \forall a, x \in X.$$

It follows from (1) that

$$x = (({}^a x)^a), \quad x = ({}^a(x^a)), \quad \forall a, x \in X.$$

Therefore the braided set (X, r) satisfies condition lri. By Fact 4.2(2), (X, r) is involutive, so (X, r) is a nondegenerate symmetric set.

(5) $\Rightarrow$ (3) It is known (see [**12**]) that if (X, r) is a nondegenerate symmetric set, then its monoid $S(X, r)$ is embedded in the group $G(X, r)$, and therefore S is left and right cancellative.

The implication (5) $\Rightarrow$ (4) is known; see [**18**, Theorem 1.2] or [**24**].

(4) $\Rightarrow$ (1) Suppose A has *Gelfand–Kirillov dimension* $\operatorname{GK}\dim A = n$. Assume, on the contrary, that condition (1) is not satisfied. Then there exist three elements $a, x, y \in X$, $a \neq x, y$ such that

$$ax^p = ay^p, \quad x \neq y.$$

This implies that $ax^{Mp} = ay^{Mp}$, for all positive integers M, hence $\operatorname{GK}\dim A < n$, a contradiction. $\square$

The following result shows the close relations between various algebraic and combinatorial properties of a finite square-free solution (X, r), the YB-algebra $A = A(\mathbf{k}, X, r)$, and its braided monoid $S = S(X, r)$. Each of these conditions describes the contrast between a square-free symmetric set and a square-free noninvolutive braided set.

Theorem 5.5. *Let (X, r) be a square-free nondegenerate braided set of order $|X| = n$. Let $S = S(X, r)$ be its braided monoid, let $A = A(\mathbf{k}, X, r)$ be its graded $\mathbf{k}$-algebra over a field $\mathbf{k}$, and let $A^!$ be the Koszul dual algebra of A. The following conditions are equivalent:*

1. *The solution (X, r) is involutive, that is, (X, r) is a symmetric set.*
2. *(X, r) satisfies lri.*
3. *The Hilbert series of A is $H_A(z) = \frac{1}{(1-z)^n}$.*
4. *The quadratic algebra A is Koszul.*

(5) *There exists an enumeration of X such that the set of quadratic relations* $\mathrm{R}(r)$ *is a Gröbner basis, that is, A is a PBW algebra.*
(6) *A is a binomial skew polynomial ring (in the sense of* [**14**]*) with respect to an enumeration of X.*
(7) $\dim_{\mathbf{k}} A_2 = \binom{n+1}{2}$.
(8) $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$.
(9) $\dim_{\mathbf{k}} A^!_3 = \binom{n}{3}$.
(10) *The algebra A has* Gelfand–Kirillov dimension $\operatorname{GK}\dim A = n$. *(This means that the integer-valued function $i \mapsto \dim_{\mathbf{k}} A_i$ is bounded by a polynomial in i of degree n.)*
(11) *If $ax^p = ay^p$ holds in S, $a, x, y \in X$, where p is the least common multiple of the orders of all actions $\mathcal{L}_x$ and $\mathcal{R}_x$, $x \in X$, then $x = y$.*
(12) *The monoid S satisfies conditions* (4.4)*.*
(13) *The monoid S is cancellative.*
(14) *A is a domain.*

Each of these conditions implies that A is an Artin–Schelter regular algebra of global dimension n.

Proof: Note first that (X, r) satisfies cl1 and cr1, by Proposition 4.4. Moreover, $|\mathcal{F}(X, r)| = n$. It is known that a finite square-free nondegenerate symmetric set (X, r) satisfies all conditions (2) through (14) in the theorem, so (1) implies all conditions (2) through (14). These implications have been published in various works of the author, but one can find them all in [**18**, Theorem 1.2]. The remaining implications with references to the corresponding results are listed below:

(1) $\Longleftrightarrow$ (2)	: Fact 4.2
(1) $\Longleftrightarrow$ (3)	: Theorem 3.16
(6) $\Longrightarrow$ (5) $\Longrightarrow$ (4)	: Clear, see Section 2
(4) $\Longrightarrow$ (1)	: Proposition 3.12
(7) $\Longrightarrow$ (1)	: Proposition 3.10, part (2)
(8) $\Longleftrightarrow$ (9)	: Easy to prove, we leave it to the reader
(9) $\Longrightarrow$ (1)	: Proposition 4.8
(10) $\Longleftrightarrow$ (11) $\Longleftrightarrow$ (13) $\Longleftrightarrow$ (1)	: Proposition 5.4
(12) $\Longrightarrow$ (1)	: by Lemma 4.9
(14) $\Longrightarrow$ (13)	: Clear
(1) $\Longrightarrow$ (14)	: see [**24**].

□

Artin–Schelter regular algebras were introduced and first studied in [**2**]. The study of Artin–Schelter regular algebras, their classification, and finding new classes of such algebras is one of the basic problems in noncommutative geometry.

Corollary 5.6 (Characterization of *noninvolutive* square-free braided sets)**.** *Let (X,r) be a square-free nondegenerate braided set of order $|X| = n$ and denote $S(X,r)$, $A = A(\mathbf{k},X,r)$, and $A^!$ as in Theorem 5.5. Suppose $r^2 \neq \mathrm{id}_{X\times X}$. Then the following conditions hold:*

(1) *(X,r) does not satisfy condition lri.*
(2) *The algebra A is not Koszul.*
(3) *The set of quadratic relations $\mathrm{R}(r)$ is not a Gröbner basis with respect to any enumeration of X.*
(4) *A is not a binomial skew polynomial ring with respect to any enumeration of X.*
(5) $2n-1 \leq \dim_{\mathbf{k}} A_2 \leq \binom{n+1}{2} - 1$.
(6) $\dim_{\mathbf{k}} A_3 < \binom{n+2}{3}$.
(7) $0 \leq \dim_{\mathbf{k}} A^!_3 < \binom{n}{3}$ *and* $A^!_3 = 0$*, whenever* $\dim_{\mathbf{k}} A_2 = 2n-1$.
(8) $\mathrm{GK}\dim A < n$.
(9) *Suppose p is the least common multiple of the orders of all actions $\mathcal{L}_x$ and $\mathcal{R}_x$, $x \in X$. Then there exist pairwise distinct elements $a,x,y \in X$ such that $ax^p = ay^p$ holds in S.*
(10) *There exist $x,y \in X$ such that $x \neq y$, and $x^p = y^p$ holds in the group $G(X,r)$.*
(11) *There exist $a,b,x,y \in X$ such that $x \neq y$, $x \neq a$, $y \neq b$, and the equality $axx = byy$ holds in S.*
(12) *The monoid $S = S(X,r)$ is not cancellative. In particular, $S(X,r)$ is not embedded in the group $G(X,r)$.*
(13) *The algebra A is not a domain.*

Remark 5.7. The lower bound $2|X|-1 \leq \dim_{\mathbf{k}} A_2$ is exact, whenever $|X| = p$, p is a prime number. More precisely, a Dihedral quandle $(X,\triangleright)$ of prime order $|X| = p$ satisfies condition $\mathbf{M}$; see Lemma 6.16.

6. Square-free braided sets (X,r) satisfying the minimality condition

6.1. Square-free 2-cancellative quadratic sets (X,r) with minimality condition.

Definition 6.1. We say that a finite quadratic set (X,r) *satisfies the minimality condition* $\mathbf{M}$ if

$$\mathbf{M}: \ \dim_{\mathbf{k}} A_2 = 2|X| - 1. \tag{6.1}$$

Example 6.2. Every square-free self distributive solution (X,r), corresponding to a dihedral quandle of prime order $|X| = p > 2$, satisfies the minimality condition $\mathbf{M}$; see Corollary 6.18.

Let (X, r) be a square-free nondegenerate quadratic set of order $|X| = n$. Assume that (X, r) is 2-cancellative. Let $S = S(X, r)$ be its graded monoid, let $A = A(\mathbf{k}, X, r)$ be its graded $\mathbf{k}$-algebra over a field $\mathbf{k}$, and let $A^!$ be the Koszul dual algebra. Consider the action of $\mathcal{D}_3(r) = {}_{\mathrm{gr}}\langle r^{12}, r^{23}\rangle$ on X^3. The following useful remarks are straightforward.

Remark 6.3. The following are equivalent:

(1) Each $\mathcal{D}_3(r)$-orbit in X^3 contains a word of the type xxy, $x, y \in X$ (and ztt, $z, t \in X$).
(2) $A^!_3 = 0$.

Remark 6.4. Let (X, r) be a square-free nondegenerate quadratic set of order $|X| = n$. Assume that (X, r) is 2-cancellative. Suppose $\mathcal{O}$ is a nontrivial r-orbit in X^2 of order $|\mathcal{O}| = n$. Then

(1) For every $x \in X$ there exists $y \in X$ such that $xy \in \mathcal{O}$.
(2) For every $y \in X$ there exists $x \in X$ such that $xy \in \mathcal{O}$.

Proposition 6.5. *Let (X, r) be a square-free nondegenerate quadratic set of order $|X| = n$, say $X = \{x_1, \dots, x_n\}$. Write $S = S(X, r)$, $A = A(\mathbf{k}, X, r)$, and $A^!$ as before (e.g. Theorem 5.5). Assume that (X, r) is 2-cancellative. Let $\mathcal{O}_i$, $1 \leq i \leq q$, be the set of all nontrivial r-orbits in X^2 (these are exactly the square-free r-orbits in X^2).*

(1) *The following three conditions are equivalent:*
 (a) *(X, r) satisfies the minimality condition $\boldsymbol{M}$; see* (6.1)*.*
 (b) *Each nontrivial orbit $\mathcal{O}_i$ has order $|\mathcal{O}_i| = n$.*
 (c) *The algebra A has a finite presentation $A \cong \mathbf{k}\langle X\rangle/(\mathrm{R}_0)$, where R_0 is a set of exactly $(n-1)^2$ quadratic square-free binomial relations:*

$$\mathrm{R}_0 = \{x_{in}y_{in} - x_1x_i,\ x_{in-1}y_{in-1} - x_1x_i, \dots, x_{i2}y_{i2} - x_1x_i \mid 2 \leq i \leq n\}, \tag{6.2}$$

 where $x_{ij} \neq y_{ij}$, $1 \leq i, j \leq n$, and the following two conditions hold for every $2 \leq i \leq n$:
 (c1) *$x_{in}y_{in} > x_{in-1}y_{in-1} > \cdots > x_{i2}y_{i2} > x_{i1}y_{i1} = x_1x_i$;*
 (c2) *there are equalities of sets*

$$\{x_{ij} \mid 2 \leq j \leq n\} = X \setminus \{x_1\}, \quad \{y_{ij} \mid 2 \leq j \leq n\} = X \setminus \{x_i\}.$$

 In this case, after a possible re-enumeration of the orbits, one has

$$\mathcal{O}_i = \mathcal{O}(x_1x_i) = \{x_1x_i := x_{i1}y_{i1} < x_{i2}y_{i2} < \cdots < x_{in-1}y_{in-1} < x_{in}y_{in}\}, \quad 2 \leq i \leq n.$$

(2) *Moreover, each of conditions (1)(a), (1)(b), (1)(c) implies that*
 (a) $A^!_3 = 0$, *and in particular* X^3 *does not contain square-free* $\mathcal{D}_3(r)$*-orbits.*
 (b) $\mathrm{GK}\dim A \leq 2$.

Proof: Recall first that for an arbitrary quadratic set (X, r) the number of distinct words of length 2 in S is exactly the number of $\mathcal{D}_2(r)$-orbits in X^2, so one has

$$\dim A_2 = |S_2| = \sharp(\mathcal{D}_2\text{-orbits in } X^2).$$

(1)(a) $\Leftrightarrow$ (b) Suppose the minimality condition (6.1) holds. Then X^2 splits into exactly $2n-1$ orbits. More precisely, there are n one element orbits, which are the elements of the diagonal Δ_2 and $n-1$ square-free orbit $\mathcal{O}_i$, $1 \leq i \leq n-1$. Due to 2-cancellativity one has $|\mathcal{O}_i| \leq n$. At the same time one has:

$$n^2 - n = \left| \bigcup_{1 \leq i \leq n-1} \mathcal{O}_i \right| = \sum_{1 \leq i \leq n-1} |\mathcal{O}_i| \leq n(n-1).$$

Therefore $|\mathcal{O}_i| = n$, for all $1 \leq i \leq n-1$.

Conversely, suppose each nontrivial orbit $\mathcal{O}$ has length n, and let q be the number of square-free orbits $\mathcal{O}$. There are exactly $(n-1)n$ square-free words in X^2, each one contained in some $\mathcal{O}$, so $n(n-1) = nq$, and therefore $q = n-1$. Thus, the total number of disjoint orbits in X^2 is $n + (n-1) = 2n-1$. It follows that $|S_2| = 2n-1$ and $\dim_{\mathbf{k}} A_2 = |S_2| = 2n-1$, so the minimality condition $\mathbf{M}$ holds.

(b) $\Rightarrow$ (c) Suppose each nontrivial $\mathcal{D}_2(r)$-orbit $\mathcal{O}_i$ in X^2 has order $|\mathcal{O}_i| = n$, $1 \leq i \leq n-1$. It follows from Remark 6.4 that for each $1 \leq i \leq n-1$ there exists a unique $x \in X$ such that $x_1 x \in \mathcal{O}_i$. We re-enumerate the orbits (if necessary), so that $x_1 x_i \in \mathcal{O}_i$. Let $1 \leq i \leq n-1$. We order lexicographically the n (distinct) words in $\mathcal{O}_i$:

$$\mathcal{O}_i = \{x_{in}y_{in} > x_{in-1}y_{in-1} > \cdots > x_{i2}y_{i2} > x_{i1}y_{i1} =: x_1 x_i\}.$$

Each two monomials in $\mathcal{O}_i$, considered as elements of S, are equal. This information is encoded by the set R_i of exactly $n-1$ *reduced* relations determined by $\mathcal{O}_i$:

$$\mathrm{R}_i: \; x_{in}y_{in} = x_1 x_i;\; x_{in-1}y_{in-1} = x_1 x_i;\; \ldots, x_{i2}y_{i2} = x_1 x_i.$$

As discussed in Section 3 the set of defining relations $\Re(r)$ is equivalent to the set of reduced relations

$$\mathrm{R} = \bigcup_i \mathrm{R}_i$$

and the corresponding "algebra-type" relations are exactly the $n(n-1)$ reduced relations R_0 given in (6.2). It follows from the properties of the orbits $\mathcal{O}_i$ that the relations in R_0 satisfy all additional conditions in part (c).

(c) $\Rightarrow$ (b) The set of relations R_0 splits into $n-1$ disjoint subsets R_i, $1 \leq i \leq n-1$. Note that the properties of the relations given in part (c) imply that (X, r) is 2-cancellative. It is clear that a relation $a = b \in \mathrm{R}_i$ implies that a, b belong to the same orbit $\mathcal{O}$ in X^2. We denote this orbit by $\mathcal{O}_i$ (one has $x_1x_i \in \mathcal{O}_i$). One can also read off from the properties of R_i that $\mathcal{O}_i$ has exactly n-elements. Note that the sets $\Re_0(r)$ and R_0 generate the same two-sided ideal I of $\mathbf{k}\langle I\rangle$, so we get $A \cong \mathbf{k}\langle X\rangle/(\mathrm{R}_0)$. It follows from the theory of Gröbner bases that the ideal I has unique reduced Gröbner basis $\mathrm{GR}(I)$ (w.r.t $\leq$). Moreover, R_0 is a proper subset of $\mathrm{GR}(I)$ (we assume $n \geq 3$) and all additional elements of $\mathrm{GR}(I) \setminus \mathrm{R}_0$ are homogeneous polynomials of degree ≥ 3. Therefore, the set of normal monomials of length 2:

$$\mathcal{N}_2 = \{x_1x_2, x_1x_3, \ldots, x_1x_n\} \cup \{x_ix_i \mid 1 \leq i \leq n\}$$

projects to a $\mathbf{k}$-basis of $A_2 \cong \mathbf{k}S_2$, so this again implies $\dim_{\mathbf{k}} A_2 = 2n-1$.

(2) Suppose (X, r) satisfies the minimality condition $\mathbf{M}$. It follows from the argument in part (1) that the normal basis $\mathcal{N}$ of A satisfies

$$\mathcal{N} \subseteq \{x_1^{\alpha}x_i^{\beta} \mid 2 \leq i \leq n,\ \alpha \geq 0,\ \beta \geq 0\}.$$

This implies that $\operatorname{GK}\dim A \leq 2$.

We shall prove that $A^!_3 = 0$. By Remark 6.3 it will be enough to show that each $\mathcal{D}_3(r)$-orbit in X^3 contains a word of the type xxy, $x, y \in X$ (and ztt, $z, t \in X$). Let $a, b, c \in X$. Without loss of generality, we may assume that $a \neq b$ and $b \neq c$. Clearly, $bc \in \mathcal{O}(bc) \subset X^2$, and by Remark 6.4 the (square-free) orbit $\mathcal{O}(bc)$ contains an element of the shape at, $t \in X$, thus $bc = at$ is an equality in S_2, so $abc = aat$ holds in S_3. This implies that the $\mathcal{D}_3$-orbit $\mathcal{O}(abc)$ in X^3 contains the monomial aat. It follows then that there are no square-free orbits in X^3, hence $A^!_3 = 0$. □

Let (X, r) be a quadratic set. A subset $Y \subset X$ is *r-invariant* if $r(x, y) \in Y \times Y$, $\forall x, y \in Y$. In this case the restriction (Y, r_Y), where $r_Y := r_{|Y \times Y}$ is a quadratic set. A quadratic set (X, r) is *decomposable* if $X = Y \cup Z$ is a decomposition into nonempty disjoint r-invariant subsets. Clearly, if $|Y| \geq 2$, then the restriction (Y, r_Y) inherits from (X, r) properties like nondegeneracy, 2-cancellativity, or being square-free.

Definition 6.6. We call a quadratic set (X, r) *(left) self distributive*, or SD for short, if it satisfies

$$\text{SD}: \ r(x, y) = ({}^{x}y, x), \quad \forall x, y \in X.$$

Lemma 6.7. *Suppose (X, r) is a finite square-free nondegenerate quadratic set which is 2-cancellative and satisfies the minimality condition* ***M****. Let $|X| = n \geq 3$. Then (X, r) is indecomposable. Moreover, if (X, r) is a self distributive quadratic set, then for every $x \in X$ the permutation $\mathcal{L}_x$ has exactly one fixed point, so*

$$\mathcal{L}_x(x) = x, \quad \mathcal{L}_x(y) \neq y, \quad \forall x, y \in X, \ y \neq x.$$

Proof: By Proposition 6.5 (X, r) satisfies the minimality conditions *iff* the set of square-free words of length 2, $X^2 \setminus \Delta_2$, splits into $n-1$ disjoint $\mathcal{D}_2$-orbits $\mathcal{O}_i$, $1 \leq i \leq n-1$, each of which contains n distinct monomials.

We shall prove first that (X, r) is indecomposable. Suppose $X = Y \cup Z$ is a decomposition into nonempty disjoint r-invariant subsets, say $|Y| = k \geq 2$, $|Z| = s \geq 1$, $k + s = n$. The restriction (Y, r_Y) is a nondegenerate, square-free, and 2-cancellative quadratic set of order $k < n$. Let $x, y \in Y$, $x \neq y$, then the $\mathcal{D}_2$-orbit $\mathcal{O}(xy)$ in X^2 is contained entirely in Y^2, and by the 2-cancellativity of (Y, r_Y), $|\mathcal{O}(xy)| \leq k < n$. At the same time $\mathcal{O}(xy)$ is a $\mathcal{D}_2$-orbit in X^2, so $\mathcal{O}(xy) = \mathcal{O}_i$ for some $1 \leq i \leq n-1$, and by Proposition 6.5 $|\mathcal{O}(xy)| = |\mathcal{O}_i| = n$, a contradiction.

Suppose now that (X, r) is a self distributive quadratic set with minimality condition. Let $x, y \in X$, $x \neq y$. If we assume that $\mathcal{L}_x(y) = y$, then $r(xy) = yx$ and $r^2(xy) = r(yx) = {}^{y}xy$, hence (due to 2-cancellativity) ${}^{y}x = x$. It follows that $\mathcal{O}(xy) = \{xy, yx\}$, so $|\mathcal{O}(xy)| = 2 < n$, in contradiction with Proposition 6.5. $\square$

It follows from Corollary 6.18 that every square-free self distributive solution (X, r), corresponding to a dihedral quandle of prime order $|X| = p > 2$, satisfies the minimality condition **M**. We do not know examples where (X, r) is a nondegenerate, square-free, and 2-cancellative quadratic set of order $n \geq 3$ which satisfies the minimality condition **M**, but (X, r) is not a solution of the YBE.

Example 6.24 gives a square-free braided set (X, r) which is indecomposable (and injective), but does not satisfy the minimality condition **M**.

Lemma 6.8. *Suppose (X, r) is a square-free self distributive quadratic set of finite order $|X| \leq 5$. If (X, r) is 2-cancellative and satisfies the minimality condition* ***M****, then (X, r) is a braided set.*

More precisely, (up to isomorphism) either

(1) (X, r) *is the quadratic set corresponding to the dihedral quandle of order* 3, *or*
(2) (X, r) *is the quadratic set corresponding to the dihedral quandle of order* 5.

Sketch of the proof: Our assumptions imply that (X, r) is nondegenerate and $\mathcal{L}_x(y) \neq y$, $\forall x, y \in X$, $x \neq y$; see Lemma 6.7. The statement is straightforward for $|X| = 3$. If $|X| = 4$, then, $\forall x \in X$, the map $\mathcal{L}_x = (y_1\ y_2\ y_3)$ is a cycle of length 3. Then a single r-orbit of length 4 determines each map $\mathcal{L}_x$ uniquely, which in turn determines r and all r-orbits in X^2 uniquely. A combinatorial argument shows that a 2-cancellative square-free quadratic set (X, r) with $|X| = 4$ and the minimality condition does not exist. Assume $|X| = 5$. Then each map $\mathcal{L}_x$ is either of the shape (a) $\mathcal{L}_x = (y_1\ y_2\ y_3\ y_4)$, a cycle of length 4, where $y_i \neq x$, $1 \leq i \leq 4$, or (b) $\mathcal{L}_x = (y_1\ y_2)(y_3\ y_4)$, a product of disjoint transpositions where $y_i \neq x$, $1 \leq i \leq 4$. Using a combinatorial argument one shows that if some $\mathcal{L}_x = (y_1\ y_2\ y_3\ y_4)$, then (X, r) is not 2-cancellative. It follows that only case (b) is possible. Then using an argument similar to the proof of Proposition 6.21 one shows that (X, r) is a braided set isomorphic to the dihedral quandle of order 5. □

6.2. Some basics on indecomposable injective racks.

Lemma 6.9. *Suppose (X, r) is an SD quadratic set.*

(1) *If (X, r) is 2-cancellative, then*
  (a) (X, r) *is nondegenerate.*
  (b) ${}^x x = x$, $\forall x \in X$, *so* (X, r) *is square-free.*
  (c) ${}^y x = x \Leftrightarrow {}^x y = y$, $x, y \in X$.
(2) (X, r) *is involutive* iff (X, r) *is the trivial solution.*
(3) (X, r) *is a braided set* iff *the condition laut holds:*

$$\mathit{laut}(x, y, z): \ {}^x({}^y z) = {}^{{}^x y}({}^x z), \quad \forall x, y, z \in X.$$

Self distributive braided sets are closely related to racks. We recall some basics on racks; see [**1**].

Definition 6.10 ([**1**])**.** *A rack* is a pair $(X, \triangleright)$, where X is a nonempty set and $\triangleright\colon X \times X \to X$ is a map (a binary operation on X) such that

(R1) The map $\varphi_i\colon X \to X$, $x \mapsto i \triangleright x$ is bijective for all $i \in X$, and
(R2) $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$.

A quandle is a rack which also satisfies $i \triangleright i = i$, for all $i \in X$.

A crossed set is a quandle such that $j \triangleright i = i$, whenever $i \triangleright j = j$.

Remark 6.11. Suppose (X, r) is an SD quadratic set. Define $\triangleright\colon X \times X \to X$ as $x \triangleright y := {}^x y$. It is clear that $(X, \triangleright)$ is a rack *iff* (X, r) is a nondegenerate braided set. Moreover, (X, r) is a square-free braided set *iff* $(X, \triangleright)$ is a quandle.

Conversely, every rack $(X, \triangleright)$ defines canonically a nondegenerate SD braided set (X, r), where $r(x, y) = (x \triangleright y, y)$. Moreover (X, r) is square-free *iff* $(X, \triangleright)$ is a quandle. It follows from Lemma 6.9 that every rack (X, r) which is 2-cancellative is a quandle. Moreover (X, r) is a crossed set.

To simplify notation and terminology, *a self distributive nondegenerate braided set* (X, r) will be referred to as *a rack* and if, in addition, (X, r) is square-free, it will be also referred to as a *quandle*. Under this convention we shall keep our usual notation and shall write "${}^x y$", instead of "$x \triangleright y$". In this case there is an equality of maps:

$$\varphi_x = \mathcal{L}_x, \quad \forall x \in X.$$

The inner group $\mathrm{Inn}(X)$ of a rack X is the subgroup of $\mathrm{Sym}(X)$ generated by all permutations $\mathcal{L}_x$, $x \in X$. A rack (X, r) is *faithful* if the map $X \to \mathrm{Inn(X)}$, $x \mapsto \mathcal{L}_x$ is injective. In fact, X is indecomposable if and only if $\mathrm{Inn}(X)$ acts transitively on X.

Example 6.12 (Dihedral quandles)**.** Let n be a positive integer. Over the ring $\mathbb{Z}/n\mathbb{Z}$ of integers mod n define $x \triangleright y = 2x - y$. This is a quandle known as *the dihedral quandle* of order n. This is an Alexander quandle; see for example [**33**]. If we assume that n is odd, we can identify the elements of this quandle with the conjugacy class of involutions of D_n, the dihedral group of order $2n$. Classification of Alexander quandles of prime order p can be found for example in [**33**]. These are particular cases of affine racks. Let X be an abelian group and let g be an automorphism of X. Then $x \triangleright y = (1 - g)(x) + g(y)$ is a rack, *an affine rack over* X.

The following results are extracted from [**25**].

Fact 6.13. *Suppose (X, r) is a finite SD braided set, and assume that the corresponding rack $(X, \triangleright)$ is indecomposable. Then*

1. $(X, \triangleright)$ *is faithful* iff *it is injective; see* [**25**, Lemma 2.10].
   Clearly, in this case the solution (X, r) is also injective.
2. *Suppose $X = \{x_1, \dots, x_n\}$. Then all permutations $\mathcal{L}_i$, $i \in X$, have the same order m. Moreover, the equalities $x_i^m = x_j^m$ hold in G_X for all $1 \le i, j \le n$; see* [**25**, Lemma 2.18].
3. *For all $x \in X$ the permutation $\mathcal{L}_x$ has exactly $1 + k_2$ fixed points, where k_2 is the number of elements $j \in X$ such that $\mathcal{O}(1, j)$ has 2 elements; see* [**25**, Lemma 2.25.3].

Remark 6.14. Suppose that (X, r) is an indecomposable quandle of order $|X| \geq 3$, and every nontrivial r-orbit $\mathcal{O} \subset X^2$ has order $3 \leq |\mathcal{O}| \leq |X|$. Then

(1) For every $x \in X$ the permutation $\mathcal{L}_x$ has a unique fixed point, namely x.
(2) Moreover, if $\mathcal{L}_x^2 = \mathrm{id}$, $\forall x \in X$, then X has odd order $|X| = 2k+1$. In this case $\mathcal{L}_x$ is a product of k disjoint transpositions.

6.3. Quandles with minimality condition M.

Corollary 6.15. *Suppose (X, r) is a 2-cancellative SD braided set of finite order $n = |X| \geq 3$, and assume that $\mathcal{L}_x^2 = \mathrm{id}$ $\forall x \in X$.*

*If (X, r) satisfies the minimality condition **M** (hence X is indecomposable), then*

(1) *X has odd order $n = 2k + 1$, and*
(2) *each $\mathcal{L}_x$, $x \in X$, is a product of k disjoint transpositions.*

The following result is well known to experts.

Lemma 6.16. *If (X,r) is a dihedral quandle of prime order $|X| = p$, then each nontrivial r-orbit $\mathcal{O}$ in X^2 has length exactly p.*

Proof: Let $x, y \in X$, $x \neq y$. Then, by definition, $r(x, y) = (2x - y, x)$. One proves by induction that

$$r^k(x, y) = ((k + 1)x - ky, kx - (k - 1)y).$$

But all maps r^k are bijections and $(k+1)x - ky = x$ if and only if $k = 0$, mod p, which implies that the r-orbit $\mathcal{O}(x, y)$ in X^2 has size p. □

Recall that the dihedral quandles and the Alexander quandles are well known for decades. A classification of Alexander quandles of prime order p can be found for example in [**33**]. The particular proof of Lemma 6.16 was kindly provided to us by Leandro Vendramin.

Definition 6.17. If (X, r) is a dihedral quandle of prime order $|X| = p$, it is called *an Alexander quandle*. It can be identified with the set of reflections of a regular p-gon (elements of the dihedral group D_{2p}).

Corollary 6.18. *Suppose that (X, r) is a square-free SD solution, corresponding to a dihedral quandle of order $|X| = p > 2$, where p is a prime number. Then (X, r) satisfies the minimality condition **M**.*

6.4. Concrete examples of quandles. We have applied our results on square-free solutions (X, r) to find various examples as solutions on the following natural problem.

Problem 6.19. Consider the following data: (a) A set X of odd cardinality $n = 2k + 1$; (b) a cyclic permutation $r_0 \in \mathrm{Sym}(X^2 \setminus \Delta_2)$ of order n

$$\mathcal{O}\colon a_1b_1 \longrightarrow_{r_0} a_2b_2 \longrightarrow_{r_0} \cdots \longrightarrow_{r_0} a_nb_n \longrightarrow_{r_0} a_1b_1,$$

where $a_i \neq b_i$, $1 \leq i \leq n$, $a_i \neq a_j$, $b_i \neq b_j$, whenever $i \neq j$, $1 \leq i, j \leq n$.

Find an extension $r\colon X \times X \to X \times X$ of r_0 (equivalently, *find all maps* $\mathcal{L}_x$, $x \in X$, *explicitly*), so that

(1) (X, r) is a 2-cancellative square-free SD *quadratic set* (we do not assume (X, r) is a solution);
(2) $\mathcal{L}_x^2 = \mathrm{id}$, $\forall x \in X$.

Analyze the quadratic set obtained. In particular, decide (a) whether this data determines an SD solution of the YBE? (b) If moreover, n is a prime number and the quadratic set (X, r) satisfies the minimality condition **M**, does this imply that (X, r) is a braided set?

Remark 6.20. In earlier versions of this paper Problem 6.19 was posed for the case when (X, r) is a 2-cancellative square-free self distributive *braided set*, with $\mathcal{L}_x^2 = \mathrm{id}$; see [**20**, Problem 6.4.1]. Under the strong assumption that (X, r) is an SD braided set this problem has been solved in [**9**, Proposition 6.2]. It is interesting and more difficult to consider Problem 6.19 in its present version; see also Problem 9.3 in Subsection 9.1.

We give some concrete examples. The first illustrates a solution of Problem 6.19 on a quadratic set (X, r) of order 5.

Proposition 6.21. *Let X be a set of order $|X| = 5$. To simplify notation we set $X = \{1, 2, 3, 4, 5\}$ as it is often used for racks. Suppose (X, r) is a quadratic SD set, so $r(x, y) = ({}^x y, x)$, $x, y \in X$, and $\mathcal{L}_x^2 = \mathrm{id}$, $\forall x \in X$. Suppose $\langle r \rangle$ has an orbit of length 5, say:*

$$\mathcal{O}(12)\colon 54 \longrightarrow_r 35 \longrightarrow_r 23 \longrightarrow_r 12 \longrightarrow_r 41 \longrightarrow_r 54. \tag{6.3}$$

Then the following conditions hold:

(1) *The orbit (6.3) determines the maps $\mathcal{L}_i$, $i \in X$ (and r) uniquely, so that (X, r) is a nondegenerate 2-cancellative quadratic set with minimality condition. In this case the left actions are:*

$$\begin{array}{lll} \mathcal{L}_1 = (2\ 4)(3\ 5), & \mathcal{L}_2 = (1\ 3)(4\ 5), & \mathcal{L}_3 = (2\ 5)(1\ 4), \\ \mathcal{L}_4 = (1\ 5)(2\ 3), & \mathcal{L}_5 = (3\ 4)(1\ 2). & \end{array} \tag{6.4}$$

X^2 splits into four r-orbits of length 5: $\mathcal{O}(1\ i)$, $2 \leq i \leq 5$, and five one-element orbits for the elements of the diagonal Δ_2.

(2) *Consider the degree-lexicographic ordering $\leq$ on $\langle X\rangle$, induced by the ordering $1 < 2 < 3 < 4 < 5$ on X. The set of defining relations $\Re(r)$ reduces (and is equivalent) to the following set of sixteen quadratic relations:*

$$
\begin{array}{llll}
\mathrm{R} = \{54 = 12 & 41 = 12 & 35 = 12 & 23 = 12 \\
53 = 14 & 45 = 14 & 32 = 14 & 21 = 14 \\
52 = 15 & 43 = 15 & 31 = 15 & 24 = 15 \\
51 = 13 & 42 = 13 & 34 = 13 & 25 = 13\}.
\end{array} \tag{6.5}
$$

(3) *Moreover, (X, r) is a braided set isomorphic to the dihedral quandle of order 5. The solution (X, r) is also injective.*

(4) *Let $A = A(\mathbf{k}, X, r) = \mathbf{k}\langle X; \mathrm{R}\rangle \cong \mathbf{k}\langle X\rangle/(I)$ be the associated quadratic algebra graded by length. The ideal I is generated by the set*

$$\mathrm{R}_0 = \{u - v \mid u = v \in \mathrm{R}\} \subset \mathbf{k}\langle X\rangle.$$

It is not difficult to find that the reduced Gröbner basis $\mathrm{GB}(\mathrm{R}_0)$ contains four additional relations:

$$\mathrm{GB}(\mathrm{R}_0) = \mathrm{R}_0 \cup \{155 - 122,\ 144 - 122,\ 133 - 122,\ 1222 - 1112\}.$$

It follows that A is standard finitely presented.

(5) *A is left and right Noetherian.*

(6) $\operatorname{GK}\dim A = 1$; $A^!_3 = 0$.

(7) *The monoid S is not cancellative, S satisfies the relations (6.5) and the following relations derived from the Gröbner basis $\mathrm{GB}(\mathrm{R}_0)$*

$$155 = 122, \quad 144 = 122, \quad 133 = 122, \quad 1222 = 1112.$$

(8) *The group $G(X, r)$ satisfies the relations (6.5) which (only in the group case) give rise to the following new relations in G:*

$$55 = 44 = 33 = 22 = 11.$$

We have deduced these relations straightforwardly from the Gröbner basis (without using the theory of racks). Of course, they agree with Fact 6.13.

Sketch of the proof: Our assumption $\mathcal{L}_x^2 = \mathrm{id}$, $\forall x \in X$, and (6.3) imply that the permutations $\mathcal{L}_i$, $1 \leq i \leq 5$, are products of disjoint cycles of the shape

$$
\begin{array}{lll}
\mathcal{L}_1 = (2\ 4)\sigma_1, & \mathcal{L}_2 = (1\ 3)\sigma_2, & \mathcal{L}_3 = (2\ 5)\sigma_3, \\
\mathcal{L}_4 = (1\ 5)\sigma_4, & \mathcal{L}_5 = (3\ 4)\sigma_5, &
\end{array}
$$

where for each $1 \leq i \leq 5$, σ_i is either a transposition, or $\sigma_i = \mathrm{id}_X$. However, we assume that (X, r) satisfies the minimality condition and therefore by Lemma 6.7, $\mathcal{L}_x(x) = x$, $\mathcal{L}_x(y) \neq y$, $\forall x, y \in X$, $y \neq x$.

Therefore the maps $\mathcal{L}_x$, $x \in X$, are uniquely determined and satisfy (6.4). It is not difficult to check that (X, r) is a braided set, isomorphic to the dihedral quandle of order 5. One uses routine techniques of computation with Gröbner bases to verify the remaining properties of (X, r) listed above. □

Corollary 6.22. *Suppose (X, r) is a 2-cancellative SD quadratic set of order $|X| = 5$ and $\mathcal{L}_x^2 = \mathrm{id}$, $\forall x \in X$. The following conditions are equivalent:*

1. $\mathcal{L}_x(y) \neq y$, $\forall x, y \in X$, $x \neq y$.
2. *(X, r) satisfies the minimality condition.*
3. *(X, r) is a braided set.*
4. *(X, r) is isomorphic to the dihedral quandle of order 5.*

Example 6.23. Suppose (X, r) is an SD quadratic set of order 9. Assume that $(\mathcal{L}_x)^2 = 1$, $\forall x \in X$, and that the map r has a concrete orbit of order 9, say:

$$\begin{aligned}\mathcal{O}(98)\colon 98 &\longrightarrow_r 79 \longrightarrow_r 67 \longrightarrow_r 56 \longrightarrow_r 45 \\ &\longrightarrow_r 34 \longrightarrow_r 23 \longrightarrow_r 12 \longrightarrow_r 81 \longrightarrow_r 98.\end{aligned} \tag{6.6}$$

Then the orbit (6.6) determines the permutations $\mathcal{L}_x$, $x \in X$, uniquely, so that (X, r) is a nondegenerate 2-cancellative braided set with $\mathcal{L}_x^2 = 1$, $x \in X$. More precisely,

(1) (X, r) is a braided set *iff* the left actions satisfy

$$\begin{array}{ll} \mathcal{L}_9 = (1\,6)(2\,5)(3\,4)(7\,8), & \mathcal{L}_8 = (1\,9)(2\,7)(3\,6)(4\,5), \\ \mathcal{L}_7 = (1\,4)(2\,3)(5\,8)(6\,9), & \mathcal{L}_6 = (1\,2)(3\,8)(4\,9)(5\,7), \\ \mathcal{L}_5 = (1\,8)(2\,9)(3\,7)(4\,6), & \mathcal{L}_4 = (1\,7)(2\,6)(3\,5)(8\,9), \\ \mathcal{L}_3 = (1\,5)(2\,4)(6\,8)(9\,7), & \mathcal{L}_2 = (1\,3)(4\,8)(6\,7)(5\,9), \\ \mathcal{L}_1 = (2\,8)(3\,9)(5\,6)(4\,7). & \end{array}$$

(2) In this case the nontrivial r-orbits of X^2 are:

$$\begin{array}{ll} \mathcal{O}(12), \mathcal{O}(13), \mathcal{O}(15), \mathcal{O}(16), \mathcal{O}(18), \mathcal{O}(19) & \text{six orbits of order } 9, \\ \mathcal{O}(14), \mathcal{O}(41), \mathcal{O}(36), \mathcal{O}(63), \mathcal{O}(29), \mathcal{O}(92) & \text{six orbits of order } 3. \end{array}$$

Moreover, (X, r) decomposes as a union of three r-invariant subsets

$$X = X_1 \cup X_2 \cup X_3, \quad X_1 = \{1, 4, 7\}, \quad X_2 = \{3, 6, 8\}, \quad X_3 = \{2, 5, 9\},$$

and r induces maps

$$X_1 \times X_2 \longrightarrow X_3 \times X_1 \longrightarrow X_2 \times X_3 \longrightarrow X_1 \times X_2.$$

Each (X_i, r_i), $1 \leq i \leq 3$, is an SD solution whose quandle is isomorphic to the dihedral quandle of order 3.

Next we give an example of an indecomposable square-free solution (X, r) of order $|X| = 4$ which fails to satisfy the minimality condition $\mathbf{M}$.

Example 6.24. Suppose (X, r) is a square-free quadratic SD set of order $|X| = 4$, so $r(x, y) = ({}^{x}y, x)$. We again simplify notation setting $X = \{1, 2, 3, 4\}$. Suppose $\mathcal{L}_4$ is not involutive and $\langle r\rangle$ has an orbit of length 3, say:

$$\mathcal{O}(24)\colon 43 \longrightarrow_r 24 \longrightarrow_r 32 \longrightarrow_r 43. \tag{6.7}$$

The orbit (6.7) determines the maps $\mathcal{L}_i$, $i \in X$ (and r) uniquely, so that (X, r) is a 2-cancellative solution. More precisely,

(1) (X, r) is a braided set *iff* the left actions are:

$$\mathcal{L}_1 = (2\ 3\ 4), \quad \mathcal{L}_2 = (1\ 4\ 3),$$
$$\mathcal{L}_3 = (1\ 2\ 4), \quad \mathcal{L}_4 = (1\ 3\ 2).$$

(2) In this case X^2 has four r-orbits of length 3 (it is easy to write them explicitly), and four one-element orbits for the elements of $\operatorname{diag}(X^2)$.

(3) We consider the degree-lexicographic ordering $\leq$ on $\langle X\rangle$ induced by the ordering $1 < 2 < 3 < 4$ on X. The set of defining relations $\Re(r)$ reduces (and is equivalent) to the following set of eight quadratic relations:

$$\begin{aligned} \mathrm{R} = \{43 = 24 \quad 32 = 24 \quad 42 = 14 \quad 21 = 14 \\ 41 = 13 \quad 34 = 13 \quad 31 = 12 \quad 23 = 12\}. \end{aligned} \tag{6.8}$$

(4) Let $A = A(\mathbf{k}, X, r) = \mathbf{k}\langle X; \mathrm{R}\rangle \cong \mathbf{k}\langle X\rangle/(I)$ be the associated quadratic algebra graded by length. The ideal I is generated by the set

$$\mathrm{R}_0 = \{u - v \mid u = v \in \mathrm{R}\} \subset \mathbf{k}\langle X\rangle.$$

It is not difficult to show that the reduced Gröbner basis $\mathrm{GB}(\mathrm{R}_0)$ contains four additional relations:

$$\mathrm{GB}(\mathrm{R}_0) = \mathrm{R}_0 \cup \{244 - 133,\ 224 - 122,\ 1444 - 1222,\ 1333 - 1222\}.$$

It follows that A is standard finitely presented.

(5) The set $\mathcal{N}$ of normal (mod I) monomials, which projects to a $\mathbf{k}$-basis of A satisfies:

$$\mathcal{N} \supset X \cup \{12, 13, 14, 24\} \cup \{112, 113, 114, 122, 124, 133, 144\}$$
$$\cup \{1^k 2^m,\ k \geq 1,\ m \geq 3\} \cup \{x^k \mid x \in X,\ k \geq 2\}.$$

(6) $\dim_{\mathbf{k}} A_2 = 8 > 2|X| - 1$.

(7) $\operatorname{GK}\dim_A = 2$.

(8) (X, r) is indecomposable and injective, but (X, r) does not satisfy the minimality condition **M**.

(9) The monoid S satisfies the relations (6.8) and also the following relations derived from the Gröbner basis

$$244 = 133, \quad 224 = 122, \quad 1444 = 1222, \quad 1333 = 1222.$$

In particular S is 3-cancellative, but S is not cancellative.

(10) The group $G(X, r)$ satisfies the relations (6.8) which (only in the group case) give rise to the following new relations in G: $444 = 333 = 222 = 111$.

7. A class of special extensions

Remark 7.1. Let (X, r) be a quadratic set. A permutation $\tau \in \mathrm{Sym}(X)$ is *an automorphism of* (X, r) (or an *r-automorphism* for short) if $(\tau \times \tau) \circ r = r \circ (\tau \times \tau)$. The group of r-automorphisms of (X, r) is denoted by $\mathrm{Aut}(X, r)$.

In the hypothesis of the following theorem (X, r_X), (Y, r_Y) are most general disjoint braided sets. No restrictions like nondegeneracy or 2-cancellativeness are imposed.

Theorem 7.2. *Let (X, r_X) and (Y, r_Y) be disjoint braided sets and let $Z = X \cup Y$. Suppose $\sigma \in \mathrm{Sym}(X)$, $\sigma \neq 1$, $\tau \in \mathrm{Sym}(Y)$, $\tau \neq 1$. Define a bijective map $r\colon Z \times Z \to Z \times Z$ as follows*

$$r(y, x) := (\sigma(x), \tau(y)); \quad r(x, y) := (\tau(y), \sigma(x)), \quad \forall x \in X,\ y \in Y.$$
$$r(x_1, x_2) := r_X(x_1, x_2), \quad \forall x_1, x_2 \in X,$$
$$r(y_1, y_2) := r_Y(y_1, y_2), \quad \forall y_1, y_2 \in Y.$$

Then (Z, r) is a quadratic set which satisfies the following conditions:

(1) *(Z, r) is nondegenerate* iff *both (X, r_X) and (Y, r_Y) are nondegenerate.*

(2) *(Z, r) is 2-cancellative* iff
   (a) *both (X, r_X) and (Y, r_Y) are 2-cancellative, and*
   (b) *the maps σ and τ (considered as permutations) are products of disjoint cycles of the same length q. Clearly, in this case $|\sigma| = |\tau| = q$.*

(3) *Suppose conditions (2) are satisfied. For each pair $x \in X$, $y \in Y$, consider the r-orbit $\mathcal{O}(xy) = \{r^k(xy) \mid k \geq 0\}$ in Z^2.*
   (a) *If q is even, $q = 2m$, then $|\mathcal{O}(xy)| = q$. In this case the order $|r|$ of the map r is the least common multiple of the three orders, $\mathrm{lcm}(|r_X|, |r_Y|, q)$.*

(b) *If q is odd, $q = 2m+1$, then $|\mathcal{O}(xy)| = 2q$. In this case the order $|r|$ of r is the least common multiple* $\operatorname{lcm}(|r_X|, |r_Y|, 2q)$.

(4) *The quadratic set (Z, r) is* a regular extension *of (X, r_X) and (Y, r_Y), in the sense of* [**22**] *if and only if $\sigma^2 = \tau^2 = 1$. Moreover, (Z, r) is involutive* iff

(a) $\sigma^2 = \tau^2 = 1$, *and*

(b) *(X, r_X) and (Y, r_Y) are involutive.*

(5) *(Z, r) obeys the YBE if and only if the following conditions hold:*

(a) *$\sigma \in \operatorname{Aut}(X, r_X)$ and $\tau \in \operatorname{Aut}(Y, r_Y)$.*

(b) *The left and the right actions satisfy the following conditions.*

$$\begin{array}{ll} \mathcal{L}_{\sigma^2(x)} = \mathcal{L}_x, \quad \mathcal{R}_{\sigma^2(x)} = \mathcal{R}_x \text{ hold in } (X, r_X), & \forall x \in X, \\ \mathcal{L}_{\tau^2(y)} = \mathcal{L}_y, \quad \mathcal{R}_{\tau^2(y)} = \mathcal{R}_y \text{ hold in } (Y, r_Y), & \forall y \in Y. \end{array} \tag{7.1}$$

In this case $(Z, r) = (X, r_X)\natural^*(Y, r_Y)$ is a generalized strong twisted union of X and Y; see Definition 8.8.

Proof: Parts (1), (2), (3), and (4) are easy, and we leave their proof to the reader. We shall prove part (5). Assume (Z, r) obeys the YBE. We shall prove conditions (a) and (b). Consider diagram (7.2), where $\alpha \in Y$, $y, z \in X$. This diagram contains elements of the orbit of the monomial $\alpha yz \in Z^3$ under the action of the group $\mathcal{D}_3(r)$. All monomials occurring in this orbit are equal elements of S

$$\begin{array}{ccc} \alpha yz & \xrightarrow{\;r_{12}\;} & r(\alpha y)z = ({}^{\alpha}y\alpha^{y})z = \sigma(y)\tau(\alpha)z \\ \downarrow r_{23} & & \downarrow r_{23} \\ \alpha r(yz) = \alpha({}^{y}zy^{z}) & & (\sigma(y))r(\tau(\alpha)z) = (\sigma(y))(\sigma(z))\tau^2(\alpha) \\ \downarrow r_{12} & & \downarrow r_{12} \\ r(\alpha({}^{y}z))y^{z} = \sigma({}^{y}z)\tau(\alpha)y^{z} & & ({}^{\sigma(y)}\sigma(z))(\sigma(y)^{\sigma(z)})\tau^2(\alpha) \\ \downarrow r_{23} & & \\ \sigma({}^{y}z)\sigma(y^{z})\tau^2(\alpha) & & \end{array} \tag{7.2}$$

Therefore,

$$\begin{aligned} & r_{12}r_{23}r_{12}(\alpha yz) = r_{23}r_{12}r_{23}(\alpha yz), && \forall \alpha \in Y,\ y, z \in X, \\ \Longleftrightarrow\ & (\sigma \times \sigma) \circ r_X(yz) = r_X \circ (\sigma \times \sigma)(yz), && \forall y, z \in X, \\ \Longleftrightarrow\ & \sigma \in \operatorname{Aut}(X, r_X). && \end{aligned}$$

Similarly, a diagram starting with an arbitrary monomial of the shape $x\alpha\beta$, where $x \in X$, $\alpha, \beta \in Y$, shows that

$$
\begin{aligned}
& r_{12}r_{23}r_{12}(x\alpha\beta) = r_{23}r_{12}r_{23}(x\alpha\beta), && \forall x \in X,\ \alpha, \beta \in Y,\\
& \Longleftrightarrow (\tau \times \tau) \circ r_Y(\alpha\beta) = r_Y \circ (\tau \times \tau)(\alpha\beta), && \forall \alpha, \beta \in Y,\\
& \Longleftrightarrow \tau \in \operatorname{Aut}(Y, r_Y).
\end{aligned}
$$

We have proven (a). Next we shall prove (7.1). Consider the following diagram:

$$
\begin{array}{ccc}
x\alpha y & \xrightarrow{\ r_{12}\ } & \tau(\alpha)\sigma(x)y \\
\Big\downarrow{\scriptstyle r_{23}} & & \Big\downarrow{\scriptstyle r_{23}} \\
x\sigma(y)\tau(\alpha) & & \tau(\alpha)({}^{\sigma(x)}y)(\sigma(x)^y) \\
\Big\downarrow{\scriptstyle r_{12}} & & \Big\downarrow{\scriptstyle r_{12}} \\
{}^{x}\sigma(y)x^{\sigma(y)}\tau(\alpha) & & \sigma(({}^{\sigma(x)}y))\tau^2(\alpha)(\sigma(x)^y) \\
\Big\downarrow{\scriptstyle r_{23}} & & \\
{}^{x}\sigma(y)\tau^2(\alpha)\sigma(x^{\sigma(y)}) & &
\end{array}
$$

The following implication holds:

$$
\text{(7.3)}\quad
\begin{aligned}
& r_{12}r_{23}r_{12}(x\alpha y) = r_{23}r_{12}r_{23}(x\alpha y), && \forall x, y \in X,\ \alpha \in Y,\\
& \Longleftrightarrow \sigma(({}^{\sigma(x)}y)) = {}^{x}(\sigma(y)) \text{ and } \sigma(x^{(\sigma(y))}) = \sigma(x)^y, && \forall x, y \in X.
\end{aligned}
$$

But $\sigma \in \operatorname{Aut}(X, r_X)$, so it follows from (7.3) that

$$
{}^{x}(\sigma(y)) = \sigma(({}^{\sigma(x)}y)) = {}^{\sigma(\sigma(x))}\sigma(y) = {}^{(\sigma^2(x))}(\sigma(y)), \quad \forall x, y \in X.
$$

The map $\sigma\colon X \to X$ is bijective, hence

$$
{}^{(\sigma^2(x))}z = {}^{x}z, \quad \forall x, z \in X,
$$

which is equivalent to

$$
\mathcal{L}_{\sigma^2(x)} = \mathcal{L}_x \text{ holds in } (X, r_X), \quad \forall x \in X.
$$

Similarly, the equalities

$$
(\sigma(x))^{(\sigma^2(y))} = \sigma(x^{\sigma(y)}) = (\sigma(x))^y, \quad \forall x, y \in X,
$$

are equivalent to

$$
\mathcal{R}_{\sigma^2(y)} = \mathcal{R}_y \text{ holds in } (X, r_X), \quad \forall y \in X.
$$

This proves the first two equalities in (7.1). An analogous argument proves the remaining equalities in (7.1). We have shown that if (Z, r)

obeys the YBE, then conditions (a) and (b) hold. Conversely, assume that conditions (a) and (b) are satisfied. The above discussion implies easily that (Z, r) is a solution of the YBE. This proves part (5).

It is clear that our construction gives a particular case of a generalized strong twisted union $(Z, r) = (X, r_X) \natural^* (Y, r_Y)$; see Definition 8.8. $\square$

One may apply the results of Theorem 8.13 and get more information on the braided monoid $S(Z, r)$ and the braided group $G(Z, r)$.

To construct concrete extensions via automorphisms, and also for some kind of classification of this type of extensions, it may be practical to use results from [**10**].

Clearly, if (X, r) is a trivial solution, then $\mathrm{Aut}(X, r) = \mathrm{Sym}(X)$ and for every $\sigma \in \mathrm{Sym}(X)$ there is an equality $\mathcal{L}_{\sigma^2(x)} = \mathcal{L}_x = \mathrm{id}_X$. Hence we have at our disposal an easy method to construct nondegenerate 2-cancellative square-free braided sets (Z, r), $Z = X \cup Y$, where the order of the map r may vary as $2 \le |r| \le |Z|$.

Corollary 7.3. *Let (X, r_X) and (Y, r_Y) be disjoint trivial symmetric sets. Suppose that $|X| = m$, $|Y| = n$, and that $m \le n$. Let $Z = X \cup Y$. Suppose $\sigma \in \mathrm{Sym}(X)$, $\tau \in \mathrm{Sym}(Y)$. Define $r\colon Z \times Z \to Z \times Z$ as follows*

$$r(x_1, x_2) := r_X(x_1, x_2) = (x_2, x_1), \quad \forall x_1, x_2 \in X,$$
$$r(y_1, y_2) := r_Y(y_1, y_2) = (y_2, y_1), \quad \forall y_1, y_2 \in Y,$$
$$r(x, y) := (\tau(y), \sigma(x)), \quad r(y, x) := (\sigma(x), \tau(y)), \quad \forall x \in X,\ y \in Y.$$

(1) *(Z, r) is a nondegenerate square-free braided set.*
(2) *Moreover, (Z, r) is 2-cancellative* iff *the permutations σ and τ are products of disjoint cycles of the same length $q \le m$. In particular, $|\sigma| = |\tau| = q$. In this case, either*
   (a) *q is even and $|r| = q$, or*
   (b) *q is odd and $|r| = 2q$.*

Example 7.4. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ be disjoint sets and let (X, r_X), (Y, r_Y) be trivial solutions. Set $\sigma = (x_1\ x_2\ x_3) \in \mathrm{Sym}(X)$, $\tau = (y_1\ y_2\ y_3) \in \mathrm{Sym}(Y)$. Define $r\colon Z \times Z \to Z \times Z$ as follows

$$r(y, x) = (\sigma(x), \tau(y)), \qquad r(x, y) = (\tau(y), \sigma(x)), \quad \forall x \in X,\ y \in Y;$$
$$r(x_i, x_j) = (x_j, x_i), \qquad r(y_i, y_j) = (y_j, y_i), \qquad 1 \le i, j \le 3.$$

Then (Z, r) is a nondegenerate square-free braided set of order $|Z| = 6$ and $Z = X \natural^* Y$. We also have that (Z, r) is 2-cancellative and the order of r is $|r| = 6 = |Z|$. The algebra $A = A(\mathbf{k}, Z, r)$ satisfies $\dim A_2 = 2\binom{3+1}{2} + 3 = 15$. A more detailed computation shows that the associated graded algebra A does not have a finite Gröbner basis with respect to any ordering of Z.

8. The braided monoid $S(X,r)$ and extensions of solutions

8.1. The braided monoid $S(X,r)$ of a braided set (X,r). In [**22**] we introduced the notion of *a braided monoid* analogously to the term *braided group* in the sense of [**40, 29**]. We recall some definitions and results from [**22**].

To each braided set (X,r) with $S=S(X,r)$ we associate a matched pair (S,S) with left and right actions uniquely determined by r, which defines a unique *braided monoid* (S,r_S) associated with (X,r). This is not a surprise given the analogous results for the group $G(X,r)$ (see [**29**]), but our approach is necessarily different. In fact we first construct the matched pair of monoids which is a self-contained result and then consider the map $r_S\colon S\times S\to S\times S$; see [**22**, Theorem 3.6]. We prove (see [**22**, Theorem 3.14]) that r_S is bijective and obeys the YBE (as would be true in the group case). Moreover, we show that (S,r_S) is a graded braided monoid.

The reader should be aware that due to the possible lack of cancellation in S the proofs of our results for monoids are difficult and necessarily involve different computations and combinatorial arguments. In general, the results can not be extracted from the already known results for the group case. Nevertheless, the monoid case is the one naturally arising in this context. Both the monoid $S(X,r)$ and the quadratic algebra $A=A(\mathbf{k},X,r)$ over a field $\mathbf{k}$ are of particular interest. The theory of general braided monoids (S,r_S) gives interesting classes of braided objects. However it seems that the approach to these is different and more difficult from the approach to braided groups (equivalently, skew braces). We recall some basic definitions.

Definition 8.1 ([**22**])**.** The pair (S,T) is a matched pair of monoids if T acts from the left on S by ${}^{(\;)}\bullet$ and S acts on T from the right by $\bullet^{(\;)}$ and these two actions obey

$$\begin{aligned}
&\mathrm{ML0}: {}^{a}1=1,\quad {}^{1}u=u, && \mathrm{MR0}: 1^{u}=1,\quad a^{1}=a,\\
&\mathrm{ML1}: {}^{(ab)}u={}^{a}({}^{b}u), && \mathrm{MR1}: a^{(uv)}=(a^{u})^{v},\\
&\mathrm{ML2}: {}^{a}(u.v)=({}^{a}u)({}^{a^{u}}v), && \mathrm{MR2}: (a.b)^{u}=(a^{{}^{b}u})(b^{u}),
\end{aligned}$$

for all $a,b\in T$, $u,v\in S$.

Definition 8.2 ([**22**])**.** An *M3-monoid* is a monoid S forming part of a matched pair (S,S) for which the actions are such that

$$\mathrm{M3}:\ {}^{u}v u^{v}=uv$$

holds in S for all $u, v \in S$. We define the *associated map* $r_S\colon S \times S \to S \times S$ by $r_S(u,v) = ({}^u v, u^v)$. A *braided monoid* is an M3-monoid S where r_S is bijective and obeys the YBE.

Fact 8.3 ([**22**, Theorems 3.6 and 3.14])**.** *Let (X,r) be a braided set and $S = S(X,r)$ the associated monoid. Then*

(1) *The left and the right actions ${}^{(\ )}\bullet\colon X \times X \to X$ and $\bullet^{(\ )}\colon X \times X \to X$ defined via r can be extended in a unique way to a left and a right action*

$$ {}^{(\ )}\bullet\colon S \times S \longrightarrow S \quad \text{and} \quad \bullet^{(\ )}\colon S \times S \longrightarrow S, $$

which make S a strong graded M3-monoid. In particular, (S, r_S) is a set-theoretic solution of the YBE. The associated bijective map r_S restricts to r.

(2) *Moreover, the following conditions hold:*
   (a) *(S, r_S) is a graded braided monoid, that is, the actions agree with the grading of S: $|{}^a u| = |u| = |u^a|$, for all $a, u \in S$.*
   (b) *(S, r_S) is a nondegenerate solution of the YBE* iff *(X,r) is nondegenerate.*
   (c) *(S, r_S) is involutive* iff *(X,r) is involutive.*

Suppose (X,r) is a noninvolutive solution. The set X is always embedded in the braided monoid (S, r_S). Moreover, in contrast with the group $G(X,r)$, the monoid S preserves more detailed information about the solution (X,r). In particular, there is an equality $u = v$ in S if and only if $|u| = |v| = m$ and u and v are in the same $\mathcal{D}_m(r)$-orbit in X^m. In general, this is not true in $G(X,r)$, where a great portion of information about (X,r) is lost.

Corollary 8.4. *Suppose (X,r) is a self distributive braided set, $S = S(X,r)$, and $G = G(X,r)$. Then*

(1) *The braided monoid (S, r_S) is a self distributive solution.*
(2) *The braided group (G, r_G) is self distributive.*

8.2. General extensions of braided sets.

Definition 8.5. Let (X, r_X) and (Y, r_Y) be disjoint quadratic sets. Let (Z,r) be a set with a bijection $r\colon Z \times Z \to Z \times Z$. We say that (Z,r) is *a (general) extension of* (X, r_X), (Y, r_Y) if $Z = X \cup Y$ as sets and r extends the maps r_X and r_Y, i.e. $r_{|X^2} = r_X$ and $r_{|Y^2} = r_Y$. Clearly, in this case X, Y are r-invariant subsets of Z. We have that (Z,r) is *a YB-extension of* (X, r_X) and (Y, r_Y) if r obeys the YBE.

Remark 8.6. In the assumption of the above definition, suppose (Z, r) is *a nondegenerate* extension of (X, r_X), (Y, r_Y). Then the equalities $r(x, y) = ({}^x y, x^y)$, $r(y, x) = ({}^y x, y^x)$ and the nondegeneracy of r, r_X, r_Y imply that

$$ {}^y x, x^y \in X \text{ and } {}^x y, y^x \in Y, \quad \text{for all } x \in X,\ y \in Y. $$

Therefore, r induces bijective maps

$$ \rho\colon Y \times X \longrightarrow X \times Y \quad \text{and} \quad \sigma\colon X \times Y \longrightarrow Y \times X, $$

and left and right "actions"

$$ {}^Y\bullet\colon Y \times X \longrightarrow X, \quad \bullet^X\colon Y \times X \longrightarrow Y, \text{ projected from } \rho, \tag{8.1} $$

$$ {}^X\bullet\colon X \times Y \longrightarrow Y, \quad \bullet^Y\colon X \times Y \longrightarrow X, \text{ projected from } \sigma. \tag{8.2} $$

Clearly, the 4-tuple of maps (r_X, r_Y, ρ, σ) uniquely determine the extension r. The map r is also uniquely determined by r_X, r_Y, and the maps (8.1), (8.2).

We call the actions (8.1) and (8.2) projected from $r_{|Y\times X}$ and $r_{|X\times Y}$ *the associated ground actions.*

Lemma 8.7. *Suppose (Z, r) is a nondegenerate braided set which splits as a disjoint union $Z = X \cup Y$ of two r-invariant subsets X and Y. Denote by (X, r_1) and (Y, r_2) the induced sub-solutions. The following conditions hold:*

(1) *The assignment $\alpha \to {}^\alpha\bullet = \mathcal{L}_{\alpha|X}$ extends to a left action of the associated monoid S_Y on X and induces a left action of G_Y on X. The assignment $\alpha \to \bullet^\alpha = \mathcal{R}_{\alpha|X}$ extends to a right action of the associated monoid S_Y on X and induces a right action of G_Y on X.*
(2) *The assignment $x \to {}^x\bullet = \mathcal{L}_{x|Y}$ extends to a left action of the associated monoid S_X on Y and induces a left action of G_X on Y. The assignment $x \to \bullet^x = \mathcal{R}_{x|Y}$ extends to a right action of the associated monoid S_X on Y and induces a right action of S_X on Y.*
(3) *Moreover, if the braided set (Z, r) is injective (that is, the natural map $Z \to G_Z$ is an embedding), then each of the assignments in part (1) extends to an action of G_Y on X, and each of the assignments in part (2) extends to an action of G_X on Y.*

Recall that in [**22**] a (general) extension (Z, r) of (X, r_X), (Y, r_Y) is called *a regular extension* of (X, r_X) and (Y, r_Y) if r is bijective, and the restrictions $r_{|Y\times X}$ and $r_{|X\times Y}$ satisfy

$$ (r \circ r)_{|Y\times X} = \mathrm{id}_{|Y\times X}, \quad (r \circ r)_{|X\times Y} = \mathrm{id}_{|X\times Y}, $$

but r is not necessarily involutive on $X \times X$, neither on $Y \times Y$. Regular extensions of arbitrary braided sets were introduced and studied in [**22**], where the theory of matched pairs of monoids was applied to characterize regular extensions and their monoids. A regular extension (Z, r) of two involutive solutions is also involutive. The extensions constructed in Section 7 are not regular.

In this paper we have a particular interest in *noninvolutive* nondegenerate braided sets (Z, r), and it is natural to search for methods proposing constructions of *new* solutions using already known braided sets. We have shown in Section 7 (see Theorem 7.2) that one can construct new noninvolutive solutions (Z, r) with a prescribed orders $|Z|$ and $|r|$ using general (nonregular) extensions of well-known involutive solutions. So it is natural to study general extensions (Z, r), possibly *not regular* (in the sense of [**22**]). In notation and assumptions as above, let (Z, r) be a nondegenerate braided set which is an extension of the disjoint braided sets (X, r_X), (Y, r_Y). Denote $S = S(X, r_X)$, $T = S(Y, r_Y)$, $U = S(Z, r)$. It follows from Fact 8.3 that $U = S(Z, r)$ has the structure of a graded braided monoid (U, r_U) with a braiding operator r_U extending r. Moreover, (U, r_U) is an extension of the disjoint braided monoids (S, r_S) and (T, r_T), and one can apply the theory of matched pairs of monoids to give more detailed description of the behaviour of the matched pairs (S, T), (T, S), (U, U), etc, in the spirit of the results in [**22**]. We propose an explicit construction: *generalized strong twisted unions of braided sets.*

8.3. Generalized strong twisted unions of nondegenerate braided sets. Theorem 7.2 gives a method to construct a new type of extensions of braided sets. The properties of these extensions motivate our Definition 8.8 of *generalized strong twisted unions of solutions* which is a generalization of the notion of a strong twisted union of solutions; see [**22**, Definition 5.1]. According the *old* definition, the notion of a strong twisted union is restricted only to *regular extensions.* Note that a strong twisted union (Z, r) of solutions (X, r_X) and (Y, r_Y) does not necessarily obey the YBE, but if $(Z, r) = X \natural Y$ is (a regular) extension of symmetric sets and obeys the YBE, then (Z, r) is also a symmetric set ($r^2 = 1$).

In our new setting, if (X, r_X) and (Y, r_Y) are symmetric (or braided) sets with $|X| > 2$, $|Y| > 2$, we construct extensions (Z, r) which are braided sets (satisfy the YBE), but the solution r may have order > 2; see for example Section 7 and the results therein.

Definition 8.8. Suppose (X, r_X) and (Y, r_Y) are disjoint quadratic sets. We call an extension (Z, r) *a generalized strong twisted union* of (X, r_X) and (Y, r_Y), and write $Z = X \natural^* Y$, if the ground actions satisfy

$$\text{(8.3)} \qquad \begin{array}{ll} \text{stu1}: {}^{\alpha^y}x = {}^{\alpha}x, & \text{stu2}: x^{{}^y\alpha} = x^{\alpha}, \\ \text{stu3}: {}^{x^{\beta}}\alpha = {}^{x}\alpha, & \text{stu4}: \alpha^{{}^{\beta}x} = \alpha^{x}, \end{array}$$

for all $x, y \in X$, $\alpha, \beta \in Y$.

We define a generalized strong twisted union of more than two quadratic sets analogously to [**21**, Definition 3.5]. Let (Z, r) be a nondegenerate quadratic set of arbitrary cardinality, let X_i, $i \in I$, be a set of pairwise disjoint r-invariant proper subsets of Z, where I is a set of indices, $|I| \geq 2$. We say that (Z, r) is *a generalized strong twisted union of* X_i*,* $i \in I$, and write $Z = \natural^*_{i\in I} X_i$, if $Z = \bigcup_{i\in I} X_i$ and for each pair $i, j \in I$, $i \neq j$, the r-invariant subset $X_{ij} = X_i \cup X_j$ is a generalized strong twisted union, $X_{ij} = X_i \natural^* X_j$. In the particular case when I is a finite set, say $I = \{1 \leq i \leq m\}$, we write $X = X_1 \natural^* X_2 \natural^* \cdots \natural^* X_m$.

Lemma 8.9. *Suppose (X, r) is an SD nondegenerate braided set, i.e. $r(x, y) = ({}^x y, x)$, $\forall x, y \in X$ (so $(X, \triangleright)$ is a rack). If (X, r) decomposes as a union of disjoint r-invariant subsets $X = \bigcup_{1\leq i\leq m} X_i$, then X is a generalized strong twisted union of racks, $X = X_1 \natural^* X_2 \natural^* \cdots \natural^* X_m$, where for $1 \leq i \leq m$, (X_i, r_i) is the corresponding subsolution.*

Lemma 8.10. *Let (Z, r) be a nondegenerate quadratic set which splits as a disjoint union $Z = X \cup Y$ of its r-invariant subsets (X, r_X), (Y, r_Y), so Z is an extension of X and Y. Suppose $x, y \in X$, $\alpha \in Y$. Each two of the following conditions imply the third:*

(1) *l1*(α, y, x) : ${}^{\alpha}({}^{y}x) = {}^{{}^{\alpha}y}({}^{\alpha^{y}}x)$,
(2) *laut*(α, y, x) : ${}^{\alpha}({}^{y}x) = {}^{{}^{\alpha}y}({}^{\alpha}x)$,
(3) *stu1*(α, y, x) : ${}^{\alpha^{y}}x = {}^{\alpha}x$.

Let (Z, r) be a nondegenerate braided set which splits as a disjoint union $Z = X \cup Y$ of two r-invariant subsets X and Y, and let $G_Z = G(Z, r)$. Denote by (X, r_X) and (Y, r_Y) the induced subsolutions. Due to the nondegeneracy of r, each of the sets X and Y is invariant under the left action of G_Z on Z. Similarly, X and Y are invariant under the right action of G_Z on Z. (We call such sets *G-invariant.*) Let $\alpha \in Y$ and let $\mathcal{L}_{\alpha}$ be the corresponding left action on Z. Denote by $\mathcal{L}_{\alpha|X}$, $\alpha \in Y$, the restriction of $\mathcal{L}_{\alpha}$ on X. The restrictions $\mathcal{R}_{\alpha|X}$, $\mathcal{L}_{x|Y}$, and $\mathcal{R}_{x|Y}$ are defined analogously for $x \in X$ and $\alpha \in Y$.

Proposition 8.11. *Suppose (Z, r) is a nondegenerate braided set which splits as a disjoint union $Z = X \cup Y$ of two r-invariant subsets X and Y. Denote by (X, r_1) and (Y, r_2) the induced subsolutions. The following conditions hold.*

(1) $\mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_1)$ *if and only if*

$$^{\alpha}({}^{y}x) = {}^{^{\alpha}y}({}^{\alpha}x) \text{ and } {}^{\alpha}(x^{y}) = ({}^{\alpha}x)^{({}^{\alpha}y)}, \quad \forall x, y \in X.$$

(2) $\mathcal{R}_{\alpha|X} \in \mathrm{Aut}(X, r_1)$ *if and only if*

$$({}^{y}x)^{\alpha} = {}^{(y^{\alpha})}(x^{\alpha}) \text{ and } (x^{y})^{\alpha} = (x^{\alpha})^{(y^{\alpha})}, \quad \forall x, y \in X.$$

(3) *The following implications hold*

$$\begin{array}{ll} & stu1: {}^{\alpha^{y}}x = {}^{\alpha}x,\ \forall \alpha \in Y,\ x, y \in X \iff \mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_1),\ \forall \alpha \in Y, \\ & stu2: x^{{}^{y}\alpha} = x^{\alpha},\ \forall \alpha \in Y,\ x, y \in X \iff \mathcal{R}_{\alpha|X} \in \mathrm{Aut}(X, r_1),\ \forall \alpha \in Y, \\ (8.4) & stu3: {}^{x^{\beta}}\alpha = {}^{x}\alpha,\ \forall x \in X,\ \alpha, \beta \in Y \iff \mathcal{L}_{x|Y} \in \mathrm{Aut}(Y, r_2),\ \ \forall x \in X, \\ & stu4: \alpha^{{}^{\beta}x} = \alpha^{x},\ \forall x \in X,\ \alpha, \beta \in Y \iff \mathcal{R}_{x|Y} \in \mathrm{Aut}(Y, r_2),\ \ \forall x \in X. \end{array}$$

Proof: (1) By definition, $\mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_X)$ *iff*

$$(\mathcal{L}_{\alpha|X} \times \mathcal{L}_{\alpha|X}) \circ r = r \circ (\mathcal{L}_{\alpha|X} \times \mathcal{L}_{\alpha|X}),$$

so part (1) follows straightforwardly from the equalities in X^2 given below:

$$\begin{aligned} &(\mathcal{L}_{\alpha|X} \times \mathcal{L}_{\alpha|X} \circ r)(x, y) = ({}^{\alpha}({}^{x}y), {}^{\alpha}(x^{y})), \\ &r \circ (\mathcal{L}_{\alpha|X} \times \mathcal{L}_{\alpha|X})(x, y) = ({}^{^{\alpha}x}({}^{\alpha}y), ({}^{\alpha}x)^{({}^{\alpha}y)}), \quad \alpha \in Y,\ x, y \in X. \end{aligned}$$

Part (2) is analogous.

(3) We shall prove the first implication

$$\text{(8.5)} \quad \text{stu1}: {}^{\alpha^{y}}x = {}^{\alpha}x,\ \forall \alpha \in Y,\ x, y \in X \iff \mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_1),\ \forall \alpha \in Y.$$

Recall first that the braided set (Z, r) satisfies conditions l1, lr3; see Remark 2.2.

stu1 $\Rightarrow \mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_1)$. Assume stu1 holds in Z. This is condition (3) of Lemma 8.10. Note that (Z, r) satisfies l1, and therefore condition (1) in Lemma 8.10 is also satisfied. Hence, by Lemma 8.10 the remaining condition (2) also holds. This gives

$${}^{\alpha}({}^{y}x) = {}^{^{\alpha}y}({}^{\alpha}x), \quad \forall \alpha \in Y,\ x, y \in X.$$

We shall prove

$$\text{(8.6)} \qquad {}^{\alpha}(x^{y}) = ({}^{\alpha}x)^{({}^{\alpha}y)}, \quad \forall \alpha \in Y,\ x, y \in X.$$

We use lr3 and stu1 to deduce the following equalities:

$$\begin{aligned} ({}^{\alpha}x)^{({}^{\alpha^x}y)} &= {}^{(\alpha^{x}y)}(x^y) &&: \text{by lr3},\\ {}^{(\alpha^{x}y)}(x^y) &= {}^{\alpha}(x^y) &&: \text{by stu1},\\ ({}^{\alpha}x)^{({}^{\alpha^x}y)} &= ({}^{\alpha}x)^{({}^{\alpha}y)} &&: \text{by stu1}, \end{aligned}$$

which imply (8.6). Hence $\mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_1)$, $\forall \alpha \in Y$.

$\mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_1) \Rightarrow$ stu1. Suppose $\mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_1)$, so by part (1) of our proposition

$$ {}^{\alpha}({}^{y}x) = {}^{{}^{\alpha}y}({}^{\alpha}x), \quad \forall x, y \in X, $$

which is exactly condition (2) of Lemma 8.10. Condition (1) of Lemma 8.10 holds (this is l1), and therefore the remaining condition (3) of Lemma 8.10 is also satisfied, but this is exactly stu1. We have proven the equivalence (8.5). An analogous argument proves the remaining three equivalences in (8.4). □

Lemma 8.7 and Proposition 8.11 imply straightforwardly the following.

Corollary 8.12. *Suppose (Z, r) is a nondegenerate injective braided set which splits as a disjoint union $Z = X \cup Y$ of its r-invariant subsets X and Y. Let (X, r_1) and (Y, r_2) be the induced subsolutions (so (X, r_1) and (Y, r_2) are also injective). Then $(Z, r) = X \natural^* Y$ is a generalized strong twisted union if and only if the following four conditions hold:*

(1) *The assignment $x \mapsto \mathcal{L}_{x|Y}$ extends to a group homomorphism*

$$\mathcal{L}_{X|Y} \colon G_X \longrightarrow \mathrm{Aut}(Y, r_Y).$$

(2) *The assignment $x \to \mathcal{R}_{x|Y}$ extends to a group homomorphism*

$$\mathcal{R}_{X|Y} \colon G_X \longrightarrow \mathrm{Aut}(Y, r_Y).$$

(3) *The assignment $\alpha \to \mathcal{L}_{\alpha|X}$ extends to a group homomorphism*

$$\mathcal{L}_{Y|X} \colon G_Y \longrightarrow \mathrm{Aut}(X, r).$$

(4) *The assignment $\alpha \to \mathcal{R}_{\alpha|X}$ extends to a group homomorphism*

$$\mathcal{R}_{Y|X} \colon G_Y \longrightarrow \mathrm{Aut}(X, r).$$

Theorem 8.13. *Suppose (Z, r) is a nondegenerate 2-cancellative braided set which splits as a generalized strong twisted union $Z = X \natural^* Y$ of its r-invariant subsets X and Y. Let (X, r_X) and (Y, r_Y) be the induced subsolutions, $S = S(X, r_X)$, $T = S(Y, r_Y)$, $U = S(Z, r)$ in usual notation. Let (S, r_S), (T, r_T), (U, r_U) be the corresponding braided monoids; see Fact 8.3. Then the following conditions hold:*

(1) *The braided monoid* (U, r_U) *has a canonical structure of a generalized strong twisted union*

$$(U, r_U) = (S, r_S) \natural^* (T, r_T),$$

extending the ground actions of the generalized strong twisted union $Z = X \natural^* Y$.

(2) *Let* (G_Z, r_{G_Z}) *be the associated braided group. Suppose furthermore that* (Z, r) *is injective, so* X *and* Y *are also embedded in* G_Z*, and let* G_1 *and* G_2 *be the subgroups of* G_Z *generated by* X *and* Y*, respectively. Then* G_1 *and* G_2 *are* r_{G_Z}*-invariant and the braided group* (G_Z, r_{G_Z}) *has a canonical structure of a generalized strong twisted union*

$$(G_Z, r_{G_Z}) = (G_1, r_1) \natural^* (G_2, r_2),$$

where r_1 *is the restriction of* r_{G_Z} *on* $G_1 \times G_1$ *and* r_2 *is the restriction of* r_{G_Z} *on* $G_2 \times G_2$.

Proof: (1) It follows from Fact 8.3 that $U = S(Z, r_Z)$ has the structure of a graded braided monoid (U, r_U) with a braiding operator r_U extending r. Moreover (U, r_U) is a (general) extension of the disjoint braided monoids (S, r_S) and (T, r_T). We have to show that the four stu conditions are satisfied; see (8.3). We shall use induction on lengths of words to prove

$$\text{stu1}: \ {}^{u^b}a = {}^u a, \quad \forall u \in T,\, a, b \in S. \tag{8.7}$$

Step 1: First we prove (8.7) for all $a \in S$, $b = y \in X$, $u = \alpha \in Y$ by induction on the length $|a|$ of a. Condition stu1 on Z gives the base for the induction. Assume (8.7) is true for all $u \in Y$, $b \in X$, and all $a \in S$ with $|a| \leq n$. Suppose $a \in S$, $|a| = n+1$, $u = \alpha \in Y$, $b = y \in X$. Then $a = tc$, where $c \in S$, $|c| = n$, $t \in X$, and the following equalities hold in U:

$$\begin{aligned} {}^{\alpha^y}a = {}^{\alpha^y}(tc) &= ({}^{\alpha^y}t)^{({}^{\alpha^y})^t}c : \text{ by ML2} \\ &= ({}^{\alpha}t)^{({}^{\alpha^y})}c \quad : \text{ by stu1 and IH} \\ &= ({}^{\alpha}t)({}^{\alpha}c) \quad : \text{ by stu1 and IH,} \end{aligned} \tag{8.8}$$

where IH is the inductive assumption. Also:

$$\begin{aligned} {}^{\alpha}a = {}^{\alpha}(tc) &= ({}^{\alpha}t)^{({}^{\alpha^t})}c : \text{ by ML2} \\ &= ({}^{\alpha}t)({}^{\alpha}c) : \text{ by stu1 and IH.} \end{aligned} \tag{8.9}$$

Equalities (8.8) and (8.9) imply ${}^{\alpha^y}a = {}^{\alpha}a$, and therefore

$${}^{\alpha^y}a = {}^{\alpha}a, \quad \forall a \in S,\ \forall y \in X,\ \alpha \in Y. \tag{8.10}$$

Step 2: We use induction on the length $|u|$ of $u \in T$ to prove

$$ {}^{u^y}a = {}^u a, \quad \forall a \in S,\ u \in T,\ y \in X. \tag{8.11} $$

Condition (8.10) gives the base for the induction. Assume (8.11) holds for all $a \in S$, $y \in X$, and all $u \in T$ with $|u| \leq n$. Let $a \in S$, $y \in X$, and $u \in T$, $|u| = n + 1$. Then $u = \alpha v$, $v \in T$, $|v| = n$, $\alpha \in Y$, and the following equalities hold in U:

$$
\begin{aligned}
{}^{u^y}a = {}^{(\alpha v)^y}a &= {}^{(\alpha^{{}^v y})(v^y)}a && : \text{ by MR2} \\
&= {}^{(\alpha^{{}^v y})}\big({}^{(v^y)}a\big) && \\
&= {}^{\alpha}({}^{v}a) && : \text{ by stu1 and IH} \\
&= {}^{(\alpha v)}a = {}^{u}a. &&
\end{aligned}
$$

This proves (8.11).

Step 3: Finally, we prove (8.7) for all $a, b \in S$, $u \in T$, by induction on the length $|b|$ of b. The base of the induction is given by (8.11). Assume (8.7) holds for all $b \in S$ with $|b| \leq n$. Let $b = cy$, $c \in S$, $|c| = n$, $y \in X$. We have:

$$
\begin{aligned}
{}^{u^b}a = {}^{(u^{cy})}a &= {}^{(u^c)^y}a && \\
&= {}^{(u^c)}a && : \text{ since } u^c \in T \text{ and by IH} \\
&= {}^{u}a && : \text{ by IH.}
\end{aligned}
$$

The remaining stu conditions (see (8.3)) are proven by a similar argument. We have proven part (1).

Each of the parts (1) and (2) should be proved separately, although we use similar arguments since, in general, the braided monoids U, S, and T are not embedded in the corresponding braided groups.

Sketch of proof of (2): Note that every element $a \in G$ can be presented as a monomial

$$ a = \zeta_1 \zeta_2 \cdots \zeta_n, \quad \zeta_i \in Z \cup Z^{-1}. \tag{8.12} $$

By convention we consider *a reduced form of* a, that is, a presentation (8.12) with minimal length n. Bearing this in mind, we prove (8.3) in G_Z using an argument similar to our argument for monoids, but at each step we use induction on the length n of *the reduced form* of the corresponding words a, u, b. □

Corollary 8.14. *Retaining the notation of Theorem 8.13, suppose* (Z, r) *is a 2-cancellative SD braided set (that is,* $(X, \triangleright)$ *is a rack), which decomposes as a union of disjoint* r*-invariant subsets* $Z = X \cup Y$*. Then* Z *is a generalized strong twisted union of racks* $Z = X \natural^* Y$*. Moreover,*

(1) *The braided monoids* (U, r_U), (S, r_S), (T, r_T) *are self distributive and* U *is a generalized strong twisted union*

$$(U, r_U) = (S, r_S) \natural^* (T, r_T).$$

(2) *Let* (G_Z, r_{G_Z}) *be the associated braided group and suppose* (Z, r) *is injective, so* X *and* Y *are also embedded in* G_Z*. Let* G_1 *and* G_2 *be the subgroups of* G_Z *generated by* X *and* Y*, respectively. Then* G_1 *and* G_2 *are* r_{G_Z}*-invariant and the braided group* (G_Z, r_{G_Z}) *has a canonical structure of a generalized strong twisted union*

$$(G_Z, r_{G_Z}) = (G_1, r_1) \natural^* (G_2, r_2),$$

where r_1 *is the restriction of* r_{G_Z} *on* $G_1 \times G_1$ *and* r_2 *is the restriction of* r_{G_Z} *on* $G_2 \times G_2$*.*

8.4. "Local" conditions sufficient for a generalized strong twisted unions of nondegenerate braided sets to be also a braided set.

Definition 8.15 ([**22**])**.** Given a quadratic set (X, r) we extend the actions ${}^x\bullet$ and $\bullet^x$ on X to left and right actions on $X \times X$ as follows. For $x, y, z \in X$ we define:

$${}^x(y, z) := ({}^xy, {}^{x^y}z) \quad \text{and} \quad (x, y)^z := (x^{{}^yz}, y^z).$$

The map r is called, respectively, *left and right invariant* if

$$\text{l2}: \ r({}^x(y, z)) = {}^x(r(y, z)), \quad \text{r2}: \ r((x, y)^z) = (r(x, y))^z$$

hold for all $x, y, z \in Z$.

Conditions l2 and r2 give a more compact way to express l1, r1, lr3, since the following implications hold:

$$\text{l2} \Longleftrightarrow \text{l1}, \text{lr3}; \quad \text{r2} \Longleftrightarrow \text{r1}, \text{lr3}.$$

Remark 8.16 ([**22**]). Let (X, r) be a quadratic set. Then the following three conditions are equivalent:

(a) (X, r) is a braided set.
(b) (X, r) satisfies l1 and r2.
(c) (X, r) satisfies r1 and l2.

Notation 8.17 ([**22**])**.** When we study extensions it is convenient to have a "local" notation for some of our conditions, in which the specific elements for which the condition is being imposed will be explicitly indicated in lexicographical order of first appearance. Thus for example $\text{l1}(x, y, z)$ means the condition as written in Remark 2.2 for the specific elements x, y, z. Similarly $\text{r2}(x, y, z)$ has the same meaning for the elements x, y, z exactly as appearing as in Definition 8.15.

In this section we consider triples in the set Z^3 such as, for example

$$\mathrm{l1}(x,\alpha,y):\ {}^{x}({}^{\alpha}y) = {}^{{}^{x}\alpha}({}^{x^{\alpha}}y), \quad \alpha,x,y\in Z.$$

Finally, we use this notation to specify the restrictions of any of our conditions to subsets of interest. For example

$$\mathrm{l1}(X,Y,X) := \{\mathrm{l1}(x,\alpha,y),\ \forall x,y\in X,\ \alpha\in Y\}.$$
$$\mathrm{r2}(X,Y,X) := (r(x,\alpha))^{y} = r((x,\alpha)^{y}),\ \forall x,y\in X,\ \alpha\in Y.$$

The following result gives a necessary and sufficient condition so that a (general) quadratic set which is a generalized strong twisted union $(Z,r) = (X,r_1)\natural^{*}(Y,r_2)$ of two disjoint braided sets is also a braided set.

Proposition 8.18. *Suppose a nondegenerate and injective quadratic set (Z,r) is a generalized strong twisted union of two disjoint 2-cancellative braided sets (X,r_X) and (Y,r_Y). Then (Z,r) obeys the YBE* iff *the following hold:*

(1) *Conditions (1) through (4) in Corollary 8.12 are satisfied.*
(2) *The actions satisfy the following four* mixed *conditions*

$$\text{(8.13)}\qquad \mathit{l1}(X,Y,X),\quad \mathit{r2}(X,Y,X),\quad \mathit{l1}(Y,X,Y),\quad \mathit{r2}(Y,X,Y).$$

Proof: The proof is routine and an experienced reader may skip it.

Assume (Z,r) obeys the YBE. Then, by Remark 8.16, conditions l1 and r2 (and r1 and l2) are satisfied for any triple $(a,b,c)\in Z^3$. In particular, the mixed conditions (8.13) hold, which proves (2). By assumption the braided set (Z,r) is a strong twisted union $Z = X\natural^{*}Y$, so the hypothesis of Corollary 8.12 is satisfied, which implies (1).

Assume now that (1) and (2) are satisfied. We have to show that (Z,r) is a braided set. Recall that the YB-diagram starting with the triple $(a,b,c)\in Z^3$ shows that

$$\begin{aligned} r^{12}r^{23}r^{12}(a,b,c) = r^{23}r^{12}r^{23}(a,b,c) &\Longleftrightarrow \mathrm{r1}(a,b,c),\ \mathrm{l2}(a,b,c)\\ &\Longleftrightarrow \mathrm{l1}(a,b,c),\ \mathrm{r2}(a,b,c), \end{aligned}\quad \forall a,b,c\in Z.$$

There is nothing to prove if $(a,b,c)\in X^3$, or $(a,b,c)\in Y^3$, since by hypothesis (X,r_X) and (Y,r_Y) are braided sets.

Our argument uses the presentation of the set $Z^3\setminus(X^3\cup Y^3)$ as a union of six disjoint subsets

$$\begin{aligned} Z^3\setminus(X^3\cup Y^3) = &(X\times X\times Y)\cup(Y\times X\times X)\cup(X\times Y\times Y)\\ &\cup(Y\times Y\times X)\cup(X\times Y\times X)\cup(Y\times X\times Y). \end{aligned}$$

Clearly, (Z, r) obeys the YBE *iff* each of the sets on the right-hand side of the above equality satisfies simultaneously the *mixed* conditions l1 and r2 (or equivalently, r1 and l2). Analyzing with details each of the corresponding six cases we note that condition (1) implies

(a) $\mathrm{l1}(X, X, Y)$ and $\mathrm{r2}(X, X, Y)$;
(b) $\mathrm{l1}(Y, X, X)$ and $\mathrm{r2}(Y, X, X)$;
(c) $\mathrm{l1}(X, Y, Y)$ and $\mathrm{r2}(X, Y, X)$;
(d) $\mathrm{l1}(Y, Y, X)$ and $\mathrm{r2}(Y, Y, X)$.

(In fact (1) encodes exactly these eight (mixed) conditions.)

Condition (2) gives the missing *mixed* conditions (8.13) not encoded in (1). □

9. Questions

9.1. Some open questions.

Question 9.1. Let (X, r) be a square-free nondegenerate *quadratic set* of finite order $|X| = n$. Suppose its associated algebra $A = A(\mathbf{k}, X, r)$ is a *PBW* algebra. (We know that these assumptions imply that $r^2 = 1$ and (X, r) is 2-cancellative; see Section 3.)

(1) Is it true that the algebra A has polynomial growth?

An equivalent question is:

(2) Is it true that the algebra A has finite global dimension?

This is so for $|X| = 3$; see Lemma 3.14.

For each $n \geq 3$ an affirmative answer of (1) or (2) would imply that (X, r) is a solution of the YBE, and all conditions (1) through (8) in Theorem 3.16 are satisfied. A counterexample would also be interesting.

Question 9.2. Suppose (X, r) is a square-free 2-cancellative *quadratic set* of finite order $|X| \geq 3$.

(1) Is it true that, if (X, r) is self distributive and satisfies the minimality condition $\dim A_2 = 2|X| - 1$, then (X, r) is a braided set?
    Our assumptions imply that (X, r) is nondegenerate and $\mathcal{L}_x(y) \neq y$, $\forall x, y \in X$, $x \neq y$; see Lemma 6.7.
(2) In particular, is it true that if (X, r) is a self distributive quadratic set of prime order $|X| = p$ and satisfies the minimality condition $\dim A_2 = 2|X| - 1$, then $\mathcal{L}_x^2 = \mathrm{id}_X$, $\forall x \in X$?
(3) What can be said about a (general) square-free 2-cancellative *quadratic set* (X, r) if its Koszul dual algebra satisfies $A_3^! = 0$? In particular, study the braided sets (X, r) for which $A_3^! = 0$.

It follows from Corollary 6.22 and Lemma 6.8 that the answers to (1) and (2) are affirmative whenever $3 \leq |X| \leq 5$. In this case (up to isomorphism) there are two SD quadratic sets with 2-cancellation and satisfying the minimality condition, namely:

(a) (X, r) is the quadratic set corresponding to the dihedral quandle of order 3, and
(b) (X, r) is the quadratic set corresponding to the dihedral quandle of order 5.

Clearly, each of those is a braided set.

Problem 9.3. Consider the following data: (a) A set X of odd cardinality $n = 2k + 1$; (b) a cyclic permutation $r_0 \in \mathrm{Sym}(X^2 \setminus \Delta_2)$ of order n

$$\mathcal{O}\colon a_1b_1 \longrightarrow_{r_0} a_2b_2 \longrightarrow_{r_0} \cdots \longrightarrow_{r_0} a_nb_n \longrightarrow_{r_0} a_1b_1,$$

where $a_i \neq b_i$, $1 \leq i \leq n$, $a_i \neq a_j$, $b_i \neq b_j$, whenever $i \neq j$, $1 \leq i, j \leq n$.

Find an extension $r\colon X \times X \to X \times X$ of r_0 (equivalently, *find all maps $\mathcal{L}_x$, $x \in X$, explicitly*), so that

(1) (X, r) is a 2-cancellative square-free SD *quadratic set* (*we do not assume that (X, r) is a solution*);
(2) $\mathcal{L}_x^2 = \mathrm{id}$, $\forall x \in X$.

Analyze the obtained quadratic set. In particular, decide (a) whether this data determines an SD solution of the YBE and (b) if moreover, $n = p$ is a prime number and the quadratic set (X, r) satisfies the minimality condition $\mathbf{M}$, whether this implies that (X, r) is a braided set.

9.2. Questions posed in a previous version of this work which have been recently answered. Various questions on braided sets posed in [**20**] were recently answered in [**9**]. We give an account of some of our previous questions.

Question 9.4 ([**20**, Question 5.8])**.** (1) For which integers n this lower bound is attainable, that is, there exists a braided set (X, r), $|X| = n$, satisfying the minimality condition $\mathbf{M}$?
(2) Classify the square-free solutions (X, r) satisfying the minimality condition $\mathbf{M}$.

A complete answer is given in [**9**].

Conjecture 9.5 ([**20**, Conjecture 5.10])**.** Let (X, r) be an arbitrary finite nondegenerate braided set with 2-cancellation. Then the monoid $S(X, r)$ is cancellative if and only if r is involutive.

Theorem 5.5 of [**20**] (which is Theorem 5.5 of the current paper) confirms this conjecture in the case when (X, r) is an arbitrary square-free nondegenerate braided set of order $|X| = n$. It was shown in [**27**, Theorem 4.5] that the conjecture is true for arbitrary finite nondegenerate set-theoretic solution (X, r) of the Yang–Baxter equation.

Questions 9.6 ([**20**, Questions 6.3.1])**.** The following questions refer to finite square-free solutions (X, r) which are 2-cancellative.

(1) Is it true that if a dihedral quandle (X, r) satisfies the minimality condition **M**, then its order $|X|$ is a prime number? - *Confirmed in* [**9**].
(2) Suppose (X, r) is an indecomposable quandle such that the corresponding solution (X, r) satisfies the minimality condition **M**. Does this imply that the quandle (X, r) is simple? - *Yes, see* [**9**].
(3) Which of the known simple quandles satisfy the minimality condition **M**? - *Answer: the dihedral quandles of prime order p; see* [**9**].
(4) Study general square-free noninvolutive, braided sets (X, r) which are not self distributive. - *This is an ongoing project.*

    Our results in Section 7 (see Theorem 7.2) and Corollary 7.3 give a method for constructions of new noninvolutive solutions (Z, r) with prescribed orders $|Z|$ and $|r|$. In this case (Z, r) is a generalized strong twisted union $Z = X \natural^* Y$ of involutive (or noninvolutive) disjoint solutions (X, r_X), (Y, r_Y).
(5) Classify the square-free noninvolutive, braided sets (X, r) whose quadratic algebra satisfy $\operatorname{GK}\dim A(k, X, r) = 1$. Some answers are given in [**9**, Example 5.1].
(6) Classify the square-free, noninvolutive braided sets of small orders. In particular, classify the square-free, noninvolutive, and not SD braided sets (X, r) of small order.
(7) Find examples of indecomposable (not SD) finite square-free solutions.
(8) Find examples of indecomposable (not SD) square-free solutions which satisfy the minimality condition **M**.

    A complete classification of (general) square-free nondegenerate solutions (X, r) satisfying the minimality condition **M** is given by Cedó, Jespers, and Okniński; see [**9**, Theorem 5.5 and Corollary 5.6]. The classification is made in terms of the so called *derived solution* (X, r').

Remark 9.7. We have shown that if (X, r) is a finite nondegenerate square-free braided set, where r is not involutive, then the monoid $S = S(X, r)$ is not cancellative (even if (X, r) is 2-cancellative). This gives a negative answer to Open Question 3.24 in [**22**]: *Is it true that if (X, r) is a 2-cancellative braided set, then the associated monoid $S(X, r)$ is cancellative?*

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