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ON COMMUTING POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

Cinzia Bisi*

Abstract

We characterize the commuting polynomial automorphisms of $\mathbb{C}^2$, using their meromorphic extension to $\mathbb{P}^2$ and looking at their dynamics on the line at infinity.

1. Introduction

The group of polynomial automorphisms of $\mathbb{C}^2$, $\mathrm{Aut}(\mathbb{C}^2)$, consists of bijective maps:

$$f\colon (z,w)\in\mathbb{C}^2 \to (f_1(z,w), f_2(z,w))\in\mathbb{C}^2$$

where $f_1, f_2 \in \mathbb{C}[z,w]$.

When f is polynomial and bijective, then the inverse f^{-1} is polynomial.

Following [**4**], we introduce two subgroups of $\mathrm{Aut}(\mathbb{C}^2)$, the group $\mathcal{E}$ of *elementary* maps

$$\mathcal{E} = \{(z,w)\to(\alpha z+p(w), \beta w+\gamma) : \alpha,\beta,\gamma\in\mathbb{C},\ \alpha\beta\neq 0,\ p\in\mathbb{C}[w]\}$$

and the group $\mathcal{A}$ of *affine* maps

$$\mathcal{A}=\{(z,w)\to(a_1z+b_1w+c_1, a_2z+b_2w+c_2) : a_i,b_i,c_i\in\mathbb{C},\ a_1b_2-a_2b_1\neq 0\}.$$

An elementary map preserve the horizontal foliation $dw=0$.

We denote by $\mathcal{AT}=\mathcal{A}\cap\mathcal{E}$ the group of the automorphisms *affine and triangular*, i.e.:

$$\mathcal{AT}=\{(z,w)\to(a_1z+b_1w+c_1, b_2w+c_2) : a_1,b_i,c_i\in\mathbb{C},\ a_1b_2\neq 0\}.$$

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We recall now a theorem on the structure of $\mathrm{Aut}(\mathbb{C}^2)$ which is known only in dimension 2. It is due to Jung, [**5**]; it was reproved in several different ways [**9**] and recently also in [**8**]. Jung's Theorem asserts that the group $\mathrm{Aut}(\mathbb{C}^2)$ is the amalgamated product of its subgroups $\mathcal{E}$ and $\mathcal{A}$ with respect to their intersection $\mathcal{AT}$. By this theorem, each automorphism $\varphi \in \big(\mathrm{Aut}(\mathbb{C}^2) - \mathcal{AT}\big)$ can be written as a composition of elementary and affine automorphisms which can start or finish indifferently with an affine or an elementary map.

A finite composition of maps of the form:

$$h_j(z,w) = (p_j(z) - a_j w, z) = (-a_j z + p_j(w), w) \circ (w,z) = e_j \circ a$$

(where $a_j \in \mathbb{C}^*$, p_j is a polynomial of degree $d_j \geq 2$, $e_j \in \mathcal{E}$, $a \in \mathcal{A}$ and it is the inversion of the coordinates) is called a *Hénon map*.

The set of Hénon maps is a semigroup and it is denoted by $\mathcal{H}$.

Proposition 1.1. [**4**] *A polynomial automorphism of $\mathbb{C}^2$ is conjugate, in the group of polynomial automorphisms, to an elementary map or to a map in $\mathcal{H}$.*

Let $f = (f_1, f_2)$ be a polynomial automorphism of $\mathbb{C}^2$ of algebraic degree $d \geq 2$. We will denote by $\overline{f}$ its meromorphic extension to $\mathbb{P}^2$.

The graph Γ of $\overline{f}$ is the closure in $\mathbb{P}^2$ of the graph of f. Let (z,w) be affine coordinates in $\mathbb{C}^2$ and let $[z:w:t]$ be corresponding homogeneous coordinates in $\mathbb{P}^2$, then the line at infinity L_∞ has equation $\{t=0\}$.

We will denote respectively I^+ and I^- the indeterminacy subsets of $\overline{f}$ and of $\overline{f^{-1}}$. These are two analytic subsets of codimension at least 2 in $\mathbb{P}^2$, contained in L_∞. It is known, [**11**, p. 106], that they are both composed by at most one point. If p is an indeterminacy point, we define $\overline{f}(p)$ as the analytic subset of Γ which projects on p, it coincides with $\cap_{\epsilon>0}\overline{f(B(p,\epsilon)-I)}$; we call $\overline{f}(p)$ the blow-up at p.

Definition 1.2. [**11**] A polynomial automorphism is *regular* if $I^+(f) \neq I^-(f)$.

The Hénon maps are regular, whereas for elementary maps we have $I^+ = I^-$.

Observe that the notion depends on choice of coordinates.

We study, in this paper, the equation $f \circ g = g \circ f$ for polynomial automorphisms of $\mathbb{C}^2$. The first result that we will prove is the following Main Lemma:

Lemma 1.3. *Suppose that f, g are two commuting polynomial automorphisms of $\mathbb{C}^2$, not of affine type, then at least one of the two following cases occurs:*

(i) $I_f^+ = I_g^+$ *(which implies also $I_f^- = I_g^-$);*
(ii) $I_f^+ = I_g^-$ *(which implies also $I_f^- = I_g^+$).*

As a consequence of it, we have that a regular map cannot commute with a non affine elementary map. We get:

Proposition 1.4. *Let $C_{\mathcal{A}}(f)$ be the group of affine automorphisms of $\mathbb{C}^2$ which commute with f. If f is regular, then $C_{\mathcal{A}}(f)$ is a finite cyclic subgroup of $\mathcal{A}$.*

Theorem 1.5. *Let f, g be two regular automorphisms of $\mathbb{C}^2$ respectively of degree d_1 and d_2. Suppose that $f \circ g = g \circ f$. Then there exist $n_0, m_0 \in \mathbb{N}$ such that $d_1^{n_0} = d_2^{m_0}$ and there exists an affine automorphism h such that $f^{n_0} = g^{m_0} \circ h$.*

Proposition 1.4 and Theorem 1.5 were proved by Lamy, [**6**], [**7**] using the action of $\mathrm{Aut}(\mathbb{C}^2)$ on the tree whose vertices are the cosets of the subgroups $\mathcal{A}$ and $\mathcal{E}$; unfortunately this action can be defined only when the group is an amalgamated product, [**10**], hence Lamy's approach depends on Jung's structure theorem and it cannot be generalized to higher dimensions. Since the analogue of Jung's Theorem is not available in higher dimension, we have introduced a new approach. We think that the approach we follow here will give the centralizer of a regular polynomial automorphism in higher dimension.

About commuting elementary maps, first Wright, [**14**], proved that the group generated by two commuting elementary maps contains $\mathbb{Z} \oplus \mathbb{Z}$, then Lamy, [**7**], mentioned that this group is not countable.

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2. Characterization of commuting polynomial automorphisms of $\mathbb{C}^2$

We start recalling the following preliminary result:

Proposition 2.1. [**11**] *If f is a non-affine polynomial automorphism of $\mathbb{C}^2$, then*

$$\overline{f}(L_\infty - I^+) = I^-;$$

$$\overline{f^{-1}}(L_\infty - I^-) = I^+.$$

This is an immediate consequence of the following elementary property:

Suppose that F and F^{-1} are the lifts of $\overline{f}$ and $\overline{f^{-1}}$ to $\mathbb{C}^3$, then

$$F \circ F^{-1}(x, y, t) = F^{-1} \circ F(x, y, t) = t^{d^2-1}(x, y, t)$$

where $d = \deg(f) = \deg(f^{-1})$.

Proof of Lemma 1.3: We show first that if $I_f^+ = I_g^+$ then $I_f^- = I_g^-$.

Suppose by contraddiction, that $I_f^- \neq I_g^-$. Then

(i) either $I_f^- \neq I_g^+$;
(ii) or $I_g^- \neq I_f^+$.

In case (i), $I_f^- \neq I_g^+ = I_f^+$ hence f is regular. In case (ii), $I_g^- \neq I_f^+ = I_g^+$ hence g is regular. Hence up to change f and g, we can suppose that $I_f^- \neq I_g^+$ and that f is regular. We know that the closure of the set K_f^- intersects the line at infinity only in one point I_f^- which is different from I_g^-. Therefore it exists at least one point $z \in \mathbb{C}^2$ such that $z \in K_f^-$ but $g(z) \notin K_f^-$. Hence the sequence $\{f^{-n}(z)\}$ is bounded, and also the sequence $g \circ f^{-n}(z)$ is bounded; on the contrary the sequence $f^{-n} \circ g(z)$ is not bounded: this contradicts that $f^{-n} \circ g = g \circ f^{-n}$. Assume now that $I_f^+ \neq I_g^+$. Then we have:

$\forall\, q \in L_\infty - \{I_f^+, I_g^+, I_f^-, I_g^-\}$,

$$\overline{f^{-1}} \circ \overline{g^{-1}}(q) = \overline{f^{-1}}(I_g^+) = I_f^+ \tag{2.1}$$

unless

$$I_g^+ = I_f^-, \tag{2.2}$$

and

$$\overline{g^{-1}} \circ \overline{f^{-1}}(q) = \overline{g^{-1}}(I_f^+) = I_g^+ \tag{2.3}$$

unless

$$I_f^+ = I_g^-. \tag{2.4}$$

By the commutation property of f with g, we have:

$$\overline{f^{-1}} \circ \overline{g^{-1}}(q) = \overline{g^{-1}} \circ \overline{f^{-1}}(q)$$

hence, except for the two cases (2.2) and (2.4), we have $I_f^+ = I_g^+$, which is in contradiction with the assumption. Therefore we have $I_g^+ = I_f^-$ or $I_f^+ = I_g^-$. But it turns out that one of the relations implies the other one by an argument similar to the starting one. □

Lemma 1.3 allows us to assume in the rest of the paper that we are in case (i).

Corollary 2.2. *A non affine elementary map cannot commute with a regular one.*

The proof follows immediately from Lemma 1.3.

In Corollary 2.2 the system of coordinates is fixed, indeed one can give an example of a regular map that after conjugation it is no more regular, [**11**].

Example 2.3. If $f(z, w) = (z^2 + aw, z)$ with $a \neq \{0\}$, then f is regular. Let $h(z, w) = (w, z + w^2)$, then $g = h^{-1} \circ f \circ h$ is no more regular.

Corollary 2.4. *Suppose that f and g are two commuting polynomial automorphisms of $\mathbb{C}^2$, where f is not of affine type. If f is conjugate to a regular map, then the same holds for g, in the same coordinates, or g is affine.*

Proof: By hypothesis, there exists an automorphism ρ such that $\tilde{f} = \rho \circ f \circ \rho^{-1}$ is regular. Since $\tilde{g} = \rho \circ g \circ \rho^{-1}$ commutes with $\tilde{f}$ then $\tilde{g}$ is regular or affine. □

Proof of Proposition 1.4: Recall, [**11**, p. 132], that a regular biholomorphism f has infinitely many distinct periodic orbits (this follows from Bezout Theorem), no subvariety of dimension greater or equal than 1 is periodic.

First we want to prove that all the periodic points of f cannot lie on the same complex line. Suppose on the contrary that there exists a complex line L such that $\bigcup_{n\in\mathbb{Z}} \mathrm{Fix}(f^n) \subset L$ (indeed a periodic point for f is a periodic point also for f^{-1} of the same period). Of course $L \neq L_\infty$ and L is at the same time f-invariant and f^{-1}-invariant. Let $\{p\} = L \cap L_\infty$, then p has to be equal to I^-, because $\overline{f}(I^-) = I^-$, and it has also to be equal to I^+ because $\overline{f^{-1}}(I^+) = I^+$. Since $I^+ \neq I^-$, this is a contradiction.

If h is affine and $f \circ h = h \circ f$, then, for all $N \in \mathbb{N}$, h induces a permutation on $\mathrm{Fix}(f^N) = \{$periodic points of order N for $f\}$. So we have a group homeomorphism φ from $C_{\mathcal{A}}(f)$ into the group Σ_N of the permutations of the points of $\mathrm{Fix}(f^N)$.

$$\varphi \colon C_{\mathcal{A}}(f) \to \Sigma_N.$$

If N is large enough, the points of $\mathrm{Fix}(f^N)$ do not lie on the same line and hence φ is injective (an affine map cannot fix more than 5 points not on the same line). Hence $C_{\mathcal{A}}(f)$ is a finite group of a suitable order p.

To prove the cyclicity of $C_{\mathcal{A}}(f)$, we prove that:

(1) $C_{\mathcal{A}}(f)$ is abelian.
(2) The eigenvalues of the linear part of each affine automorphism $h \in C_{\mathcal{A}}(f)$ are roots of unity of the same order.
(3) For all $h_1, h_2 \in C_{\mathcal{A}}(f)$ of the same order q, there exist $n_0, m_0 \in \mathbb{N}$ such that $h_1^{n_0} = h_2$ and $h_2^{m_0} = h_1$.

In order to prove (1), we recall that if $h \circ f = f \circ h$, then $\overline{h}(I_f^-) = I_f^-$ and $\overline{h}(I_f^+) = I_f^+$. Then,up to conjugation, we can assume that $I_f^- = [1 : 0 : 0]$ and $I_f^+ = [0 : 1 : 0]$.

In these coordinates

$$\overline{h}([x : y : t]) = [\alpha x + \gamma t : \beta y + \delta t : t]. \tag{2.5}$$

Consider now the commutator $[h_1, h_2]$ of two maps $h_1, h_2 \in C_{\mathcal{A}}(f)$, then its linear part in $\mathbb{C}^2$ is the identity 2×2 matrix, because the linear part of each of them is diagonal, see (2.5). But $[h_1, h_2]$ cannot be a translation of $\mathbb{C}^2$ because $C_{\mathcal{A}}(f)$ is a finite group. Hence the only possibility is $[h_1, h_2] = \mathrm{Id}$.

Since $C_{\mathcal{A}}(f)$ is abelian, it follows that all the elements in $C_{\mathcal{A}}(f)$ have a common fixed point, hence, up to conjugation, we can suppose that they are all rotations fixing the origin, therefore they are of type $(\alpha z, \beta w)$.

In order to prove (2), we recall that, since the order of the group $C_{\mathcal{A}}(f)$ is p, then for all $h \in C_{\mathcal{A}}(f)$ there exists $k \in \mathbb{N}$ which divides p such that $h^k = \mathrm{Id}$. This means that $\alpha^k = \beta^k = 1$, and the eigenvalues of h are k-roots of unity. But suppose that they have different orders, then there exists a $n \in \mathbb{N}$ which divides k such that h^n is the identity in one component but not in the other one. Suppose that $\alpha^n = 1$ and $\beta^n \neq 1$. This means that all the points $(z, 0)$ are fixed by h^n. Since for all $m \in \mathbb{Z}$, f^m commutes with h^n, the line $\{w = 0\}$ is invariant for all f^m, with $m \in \mathbb{Z}$. For the invariance of the line $\{w = 0\}$ by f and by f^{-1}, it follows that the unique point $p = \{w = 0\} \cap L_\infty$ has to be equal to I_f^+ and at the same time to I_f^-, but this contradicts the regularity of f.

The assertion in (3) follows directly from (1) and (2): since the order q of the rotation is exactly the common order of its eigenvalues, there exist a $n_0 \in \mathbb{N}$ such that $h_1^{n_0} \circ h_2^{-1}$ has an eigenvalue equal to 1. But $h_1^{n_0} \circ h_2^{-1}$ is still an element in $C_{\mathcal{A}}(f)$ and hence its two eigenvalues have the same order; this implies also that the second eigenvalue has to be equal to 1 and $h_1^{n_0} = h_2$.

The cyclicity of the group $C_{\mathcal{A}}(f)$ follows from (1), (2), (3). If h_0 is one of the elements of $C_{\mathcal{A}}(f)$ of maximal order $s \leq p$, then $\langle h_0 \rangle = C_{\mathcal{A}}(f)$. Indeed for each $h \in C_{\mathcal{A}}(f)$, the order of h has to be a divisor of the maximal order s; hence there exists an element in $\langle h_0 \rangle$, h_0^r, which has the same order of h, but, by (3), h is a power of h_0^r and so $h \in \langle h_0 \rangle$. In conclusion $C_{\mathcal{A}}(f)$ is isomorphic to $\mathbb{Z}_p$. ☐

We recall two examples, see [**7**], to show that it is possible to construct either regular maps f such that some element in $C_{\mathcal{A}}(f)$ has two equal eigenvalues, or regular maps f such that some element in $C_{\mathcal{A}}(f)$ has two different eigenvalues, but of the same order.

Example 2.5. 1) Consider $f = (y, y^{n+1} + x)$. Let α be equal to β and $\alpha^n = 1$, then $h = (\alpha x, \beta y)$ commutes with f.

2) Consider $f = (y, y^p + x)$ and $g = (y, y^q + x)$. Let α be different from β but $\alpha^p = \beta$ and $\beta^q = \alpha$, then $h = (\alpha x, \beta y)$ commutes with $f \circ g$.

We now prove Theorem 1.5. We recall that, [**11**], if f is a regular polynomial automorphism of $\mathbb{C}^2$, we can associate to it the sets:

$$K^+ = \{z \in \mathbb{C}^2 : \{f^n(z)\}_{n\in\mathbb{N}} \text{ is bounded}\},$$

$$K^- = \{z \in \mathbb{C}^2 : \{f^{-n}(z)\}_{n\in\mathbb{N}} \text{ is bounded}\},$$

$$K = K^+ \cap K^-,$$

$$U^+ = \mathbb{C}^2 - K^+,$$

$$U^- = \mathbb{C}^2 - K^-,$$

and the Green functions:

$$\left.\begin{aligned} G^+(z,w) &= \lim_{n\to+\infty} \frac{1}{d^n} \log^+ |f^n(z,w)|, \\ G^-(z,w) &= \lim_{n\to+\infty} \frac{1}{d^n} \log^+ |f^{-n}(z,w)|, \end{aligned}\right. \quad \text{where } d = \deg(f) = \deg(f^{-1}),$$

$$G_K(z,w) = \sup\left(G^+(z,w), G^-(z,w)\right).$$

Proposition 2.6. [**1**], [**2**] *If f is a regular polynomial automorphism of $\mathbb{C}^2$ of algebraic degree $d \geq 2$, then*

- *G^+ and G^- are continuous functions on $\mathbb{C}^2$ and*
$$K^+ = \{G^+ = 0\},$$
$$K^- = \{G^- = 0\}.$$
- *G^+ and G^- are pluriharmonic (p.h.) respectively on U^+ and U^-, and plurisubharmonic (p.s.h.) on $\mathbb{C}^2$.*
- *$G^+ \circ f = d \cdot G^+$ and $G^- \circ f^{-1} = d \cdot G^-$.*
- *The closure $\overline{K^+}$ and $\overline{K^-}$ of K^+ and K^- in $\mathbb{P}^2$ verify:*
$$\overline{K^+} = K^+ \cup I^+,$$
$$\overline{K^-} = K^- \cup I^-.$$
- *I^+ is an attractive point for f^{-1} and I^- is an attractive point for f.*
- *$K = K^+ \cap K^-$ is a compact subset of $\mathbb{C}^2$.*

Proof of Theorem 1.5: First of all we want to prove that:

(i) If $d_2^m \leq d_1^n$ with $n, m \in \mathbb{N}$, then d_2^m divides d_1^n.

Then we will prove that:

(ii) If for $m, n \in \mathbb{N}$, $d_2^m \leq d_1^n$ implies that d_2^m divides d_1^n, then there exist $n_0, m_0 \in \mathbb{N}$ such that $d_1^{n_0} = d_2^{m_0}$.

A first way to prove (i) is to prove that the Green functions of the two commuting regular automorphisms are equal. The Green function's approach extends to $\mathbb{C}^k$, $k \geq 3$.

Let G_f^+ and G_g^+ the Green functions associated to f and g. Consider the function:
$$H_1 = \frac{G_f^+ \circ g}{d_2}.$$
H_1 is a solution of the following equation, because f commutes with g:
$$H_1 \circ f = d_1 \circ H_1. \tag{2.6}$$
Hence H_1 and G_f^+ satisfy the same functional equation.

But from [**11**], G_f^+ is the largest solution of the equation (2.6) among the p.s.h. functions bounded by $\log^+ |z| + O(1)$ at infinity.

Hence
$$H_1 = \frac{G_f^+ \circ g}{d_2} \leq G_f^+$$

and also, $\forall\, n \in \mathbb{N}$,

$$H_n = \frac{G_f^+ \circ g^n}{d_2^n} \le G_f^+ \tag{2.7}$$

because $\frac{G_f^+ \circ g^n}{d_2^n}$ also solves the equation (2.6).

On the other hand, $\lim_{n\to+\infty} \frac{G_f^+(g^n)}{d_2^n} = G_g^+$.

Indeed, if $(z,w) \in K_g^+$, then $g^n(z,w)$ is bounded when $n \to +\infty$; by the continuity of G_f^+ we have that $G_f^+ \circ g^n(z,w)$ is also bounded when $n \to +\infty$, and hence

$$\lim_{n\to+\infty} \frac{G_f^+ \circ g^n}{d_2^n} = 0 \quad \text{on} \quad K_g^+.$$

Hence any limit function of $\frac{G_f^+(g^n)}{d_2^n}$ is equal to G_g^+ on $K_g^+ = \{G_g^+ = 0\}$.

On the other hand, if $(z,w) \in U_g^+ = \mathbb{C}^2 - K_g^+$ and (z,w) is in a neighborhood of I_g^+,

$$\log^+|z| + c_2 \le G_f^+(z,w) \le \log^+|(z,w)| + c_1$$

we get

$$\frac{\log^+|z \circ g^n(z,w)| + c_2}{d_2^n} \le \frac{G_f^+ \circ g^n}{d_2^n} \le \frac{\log^+|g^n(z,w)| + c_1}{d_2^n}.$$

The first and the last member of the sequence of inequalities tend to $G_g^+(z,w)$. Therefore $\lim_{n\to+\infty} \frac{G_f^+(g^n)}{d_2^n} = G_g^+$ everywhere on $\mathbb{C}^2$. Hence we have that $G_g^+ \le G_f^+$ and interchanging f and g we get that $G_g^+ = G_f^+ = G^+$.

Observe that it would be sufficient that $f^n \circ g^m = g^m \circ f^n$, for some $n, m > 1$, in order to have $G_f^+ = G_g^+$.

It follows that:

$$G^+(f^n \circ g^{-m}) = d_1^n G^+(g^{-m}) = \frac{d_1^n}{d_2^m} G^+. \tag{2.8}$$

Suppose that $d_2 \le d_1$ (if this is not the case, we have that $d_1 < d_2$ and we can use the same argument exchanging g with f) and consider $h := g^{-1} \circ f$.

The map h is a polynomial automorphism of $\mathbb{C}^2$ which commutes with a regular one (for example f or g), hence, by Corollary 2.4, h is affine or regular.

Suppose h is regular and let H be its Green function (i.e. $H = \lim_{n\to+\infty} \frac{1}{\delta^n} \log^+(|h^n(z,w)|)$). Then $H(h) = \delta H$.

On the other hand, by (2.8) with $n = m = 1$, we have:

$$G^+(h) = \frac{d_1}{d_2} G^+.$$

But h commutes with f which is regular, therefore they have the same indeterminacy point and the same Green function.

Hence $H = G^+$ and $\delta = \frac{d_1}{d_2}$.

But $\delta \in \mathbb{N}$ because $\delta = \deg(h)$, hence $\frac{d_1}{d_2} \in \mathbb{N}$ and d_2 divides d_1.

On the other hand, if h is affine then $\deg(h) = 1$ and there is no curve mapped by h into I_g^+. Hence, from $f = g \circ h$, it follows that $\deg(f) = \deg(g \circ h) = \deg(g) \cdot \deg(h)$; then $\deg(f) = \deg(g)$ and $d_1 = d_2$.

Furthermore, repeating the same argument with $h := g^{-m} \circ f^n$, we have that, if $d_2^m \leq d_1^n$, then $\frac{d_1^n}{d_2^m} \in \mathbb{N}$. Hence, if $d_2^m \leq d_1^n$, then d_2^m divides d_1^n (d_2^m is exactly d_1^n if h is affine).

There is another way to prove (i) which avoids the Green functions.

Since f commutes with g, we can assume, by Lemma 1.3, that $I_f^+ = I_g^+$.

Consider then $h := f^n \circ g^{-m}$, with arbitrary $n, m \in \mathbb{N}$. The map h commutes with a regular one (i.e. f or g) hence, by Corollary 2.4, we have two possibilities:

(1) h is affine;
(2) h is regular.

In case (1), $f^n = h \circ g^m$ and $\deg(f^n) = (\deg h) \times \deg(g^m)$, hence $d_1^n = (\deg h) \times (d_2)^m$ but $\deg(h) = 1$ and therefore $d_1^n = d_2^m$.

In case (2), $f^n = h \circ g^m$ and $g^m = h^{-1} \circ f^n$.

Then we use the fact, [**3**], that for meromorphic maps in $\mathbb{P}^2$, $\overline{f}$, $\overline{g}$, $\deg(\overline{f} \circ \overline{g}) = \deg(\overline{f}) \cdot \deg(\overline{g})$ if and only if there is no curve mapped by $\overline{g}$ into the indeterminacy set of $\overline{f}$. Therefore

(a) If $I_g^- \neq I_h^+$, then $\deg(f^n) = d_1^n = \deg(h \circ g^n) = (\deg h) \times d_2^m$ where $\deg(h) > 1$. Hence $d_2^m < d_1^n$ and d_2^m divides d_1^n.
(b) If $I_f^- \neq I_h^-$ then $\deg(g^m) = d_2^m = \deg(h^{-1} \circ f^n) = (\deg h^{-1}) \times \deg(f^n) = (\deg h^{-1}) \times d_1^n$ with $\deg(h^{-1}) > 1$. Hence $d_1^n < d_2^m$ and d_1^n divides d_2^m.

But h is regular, hence $I_h^+ \neq I_h^-$ and $I_g^- = I_f^-$.

So exactly one of the two cases (a) and (b) occurs.

Now we want to prove (ii): if for $n, m \in \mathbb{N}$, $d_2^m \leq d_1^n$ implies that d_2^m divides d_1^n, then there exist $n_0, m_0 \in \mathbb{N}$ such that $d_1^{n_0} = d_2^{m_0}$.

Suppose, on the contrary, that for all $n, m \in \mathbb{N}$, $d_1^n \neq d_2^m$.

Then $\frac{d_1^n}{d_2^m} \neq 1$, or equivalently, $\log\left(\frac{d_1^n}{d_2^m}\right) \neq 0$, which means that $\alpha = \frac{\log d_1}{\log d_2}$ is irrational. So for every $\epsilon > 0$ there is $n, m \in \mathbb{N}$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{\epsilon}{n}.$$

Multiplying both the sides of the inequality by $\frac{n}{m}$, we obtain:

$$\left|\frac{\log d_1^n}{\log d_2^m} - 1\right| < \frac{\epsilon}{m},$$

which is equivalent to the following:

$$|\log d_1^n - \log d_2^m| = \left|\log\left(\frac{d_1^n}{d_2^m}\right)\right| < \frac{\epsilon}{m} \cdot m \cdot \log(d_2) = \epsilon \log(d_2).$$

But if $d_1^n \neq d_2^m$ and d_2^m divides d_1^n, then $\frac{d_1^n}{d_2^m}$ is an integer greater or equal to 2, hence:

$$\log(2) \leq \left|\log\left(\frac{d_1^n}{d_2^m}\right)\right| < \epsilon \log(d_2),$$

a contradiction, if ϵ is sufficiently small.

Consider now:

$h := f^{n_0} \circ g^{-m_0}$, then h is affine or regular. If it were regular, by $f^{n_0} = h \circ g^{m_0}$ we would obtain $\deg(f^{n_0}) = \deg(h) \times \deg(g^{m_0})$, which means $d_1^{n_0} = \deg(h) \times d_2^{m_0} = \deg(h) \times d_1^{n_0}$; this implies $\deg(h) = 1$ and hence h is affine. $\square$

After Proposition 1.4 and Theorem 1.5 we have that the complete centralizer of a regular map is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_p$, as proved in [**7**, Proposition 4.8].

Note that this is not in general a direct product; for example let $g(z, w) = (w, z + w^2)$ and $f(z, w) = (jz, j^2 w)$ with $j^3 = 1$, then we have $f \circ g \neq g \circ f$, so $\text{cent}(g^2)$ is not the direct product of $\langle g \rangle$ and $\langle f \rangle$, but we have $\text{cent}(g^2) = \mathbb{Z} \rtimes \mathbb{Z}_3$.

On the other hand, the centralizer of an elementary map always contains the abelian free group $\mathbb{Z} \oplus \mathbb{Z}$; this fact can be deduced from a work of D. Wright [**14**], as noticed by Veselov [**12**], [**13**]. In fact this centralizer is not countable, as noticed by Lamy [**7**].

All these studies of the subgroups of the group $\mathrm{Aut}(\mathbb{C}^2)$ (the abelian ones by Wright, and the solvable ones by Lamy) rely on the particular structure of amalgamated product of $\mathrm{Aut}(\mathbb{C}^2)$; we hope that the method exposed in this paper could be a first step towards the study of the higher dimensional case.

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Dipartimento di Matematica U. Dini
Universitá di Firenze
Viale Morgagni 67/A
50134 Firenze
Italy
E-mail address: bisi@math.unifi.it

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