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BOUNDEDNESS OF SUBLINEAR OPERATORS ON THE HOMOGENEOUS HERZ SPACES

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Abstract

Some boundedness results are established for sublinear operators on the homogeneous Herz spaces. As applications, some new theorems about the boundedness on homogeneous Herz spaces for commutators of singular integral operators are obtained.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Let $K(x)$ be a function on $\mathbb{R}^n \backslash \{0\}$ which satisfies the size condition

$$|K(x)| \leq C|x|^{-n}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

A celebrated result of Stein [**15**] tells us that if the operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)\, dy$$

is bounded on $L^q(\mathbb{R}^n)$ for some $q > 1$, then T is also bounded on $L^q(\mathbb{R}^n, |x|^\alpha\, dx)$ provided that $-n < \alpha < n(q-1)$, where $L^q(\mathbb{R}^n, |x|^\alpha\, dx)$ denotes the weighted Lebesgue space defined by

$$L^q(\mathbb{R}^n, |x|^\alpha\, dx) = \left\{ f \text{ is measurable on } \mathbb{R}^n \text{ and } \|f\|_{L^q(\mathbb{R}^n, |x|^\alpha\, dx)} = \int_{\mathbb{R}^n} |f(x)|^q |x|^\alpha\, dx < \infty \right\}.$$

Soria and Weiss [**14**] gave some beautiful generalizations of Stein's result. In particular, they obtained the result of Stein in the case $q = 1$.

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Yang and some other authors (see [**10**], [**11**] and [**12**]) considered the boundedness of sublinear operators on homogeneous Herz spaces. Since the homogeneous Herz spaces are some kind of generalization of the weighted Lebesgue spaces with power weights, these results are of great interest. Lu and Yang [**12**], Hu, Lu and Yang [**10**] considered the boundedness on homogeneous Herz spaces for commutators and obtained many useful theorems. But in some interesting cases, the methods and results in [**12**] and [**10**] break down. Let us consider the commutator of singular integral operator defined by

$$T_{b,m}f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy, \tag{1}$$

where Ω is homogeneous of degree zero and has mean value zero on the unit sphere S^{n-1}, m is a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If $\Omega \in L^q(S^{n-1})$ for some $q > 1$, we can deduce the boundedness on Herz space for $T_{b,m}$ by employing the results of [**12**] and [**10**]. Although it was proved in [**9**] that $\Omega \in L(\log L)^{m+1}(S^{n-1})$ (i.e. $\int_{S^{n-1}} |\Omega(x)| \log^{m+1}(2 + |\Omega(x)|)\, dx < \infty$) is a sufficient condition such that the operator $T_{b,m}$ defined by (1) is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n,m,p)\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $1 < p < \infty$, we do not know the behaviour on homogeneous Herz spaces, even on weighted Lebesgue spaces with power weights, for $T_{b,m}$ when $\Omega \notin \cup_{q>1} L^q(S^{n-1})$. The main purpose of this paper is to establish some boundedness results on the homogeneous Herz spaces for sublinear operators which are particularly suitable for commutators of singular integral operators and some other important operators. To state our results, let us recall the definition of the homogeneous Herz spaces.

Let $C_k = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$ for $k \in \mathbb{Z}$. For a measurable set $E \subset \mathbb{R}^n$, denote by χ_E the characteristic function of E. Set $\chi_k = \chi_{C_k}$.

Definition. Let $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$. The homogeneous Herz space $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_q(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \right\}^{1/p}$$

with the usual modification made when $p = \infty$ or $q = \infty$.

For the properties and applications of the space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, and the boundedness of some classical operators on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, we refer to the references [**2**], [**3**], [**6**], [**10**], [**11**] and [**12**]. It is obvious that $\dot{K}_q^{\alpha/q,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^\alpha\, dx)$. Our main results in this paper can be stated as follows.

Theorem 1. *Let $1 < q < \infty$, $0 < p \le \infty$, $-n < \beta_1 < \beta_2 < n(q-1)$, T be a sublinear integral operator which is bounded on $L^q(\mathbb{R}^n)$ and satisfies*

$$|Tf(x)| \le \int_{\mathbb{R}^n} |b(x) - b(y)|^m |K(x,y) f(y)|\, dy, \tag{2}$$

where m is a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, $K(x,y)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,y) : x \ne y\}$. For $s \ge 1$, set $\Phi_s(t) = t \log^s(2+t)$. Suppose that for any $0 < r < \infty$ and $\beta_1 < \beta < \beta_2$, the operator

$$U_{m,r} f(x) = \int_{r<|x-y|\le 2r} \Phi_m(r^n |K(x,y)|) |f(y)|\, dy \tag{3}$$

is bounded on $L^q(\mathbb{R}^n, |x|^\beta\, dx)$ with bound Br^n and B is independent of r. Then the operator T is bounded on the Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ provided that $\beta_1/q < \alpha < \beta_2/q$.

Theorem 2. *Let $1 < q < \infty$, $0 < p \le \infty$, $-n < \beta_1 < \beta_2 < n(q-1)$, T be a sublinear integral operator which is bounded on $L^q(\mathbb{R}^n)$ and satisfies*

$$|Tf(x)| \le \int_{\mathbb{R}^n} |K(x,y)| \frac{|R_{m+1}(A; x, y)|}{|x-y|^{n+m}} |f(y)|\, dy, \tag{4}$$

where $K(x,y)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,y) : x \ne y\}$, m is a positive integer, A has derivatives of order m in $\mathrm{BMO}(\mathbb{R}^n)$, R_{m+1} is the $(m+1)$-th order Taylor series remainder of A at x expanded about y, that is,

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\mu| \le m} D^\mu A(y)(x-y)^\mu.$$

Suppose that for any $0 < r < \infty$ and $\beta_1 < \beta < \beta_2$, the operator $U_{1,r}$ defined by (3) is bounded on $L^q(\mathbb{R}^n, |x|^\beta\, dx)$ with bound Br^n and B is independent of r. Then the operator T is bounded on the Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ provided that $\beta_1/q < \alpha < \beta_2/q$.

To prove Theorem 1 and Theorem 2, we will use

Theorem 3. *Let* $1 < q < \infty$*,* $0 < p \le \infty$*,* $\beta_1, \beta_2 \in \mathbb{R}$ *and* $\beta_1 < \beta_2$*,* T *be a sublinear integral operator which is bounded on* $L^q(\mathbb{R}^n)$ *and satisfies*

$$|Tf(x)| \le \int_{\mathbb{R}^n} |K(x,y)f(y)|\, dy.$$

Suppose that for any $0 < r < \infty$ *and* $\beta_1 < \beta < \beta_2$*, the operator*

$$T_r f(x) = \int_{r<|x-y|\le 2r} |K(x,y)f(y)|\, dy$$

is bounded on $L^q(\mathbb{R}^n, |x|^\beta\, dx)$ *with bound independent of* r*. Then* T *is bounded on the Herz space* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *provided that* $\beta_1/q < \alpha < \beta_2/q$*.*

Remark. Theorem 3 has independent interest and is more general and suitable for many operators in harmonic analysis, but it should be pointed out that the main idea in the proof of this theorem comes from the paper [**10**].

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose value may differ from line to line, A_q denotes the weight function class of Muckenhoupt (see [**16**, Chapter V] for definition and the properties of A_q). For a cube I, let $I^* = 4nI$. For a locally integrable function f, a real number $s \ge 1$, a cube I and a nonnegative weighted function w, define

$$\|f\|_{L(\log L)^s;I,w} = \inf\left\{\lambda > 0 : \frac{1}{w(I)}\int_I \frac{|f(y)|}{\lambda}\log^s\left(2 + \frac{|f(y)|}{\lambda}\right) w(y)\, dy \le 1\right\}$$

and

$$\|f\|_{\exp(L)^{1/s};I,w} = \inf\left\{\lambda > 0 : \frac{1}{w(I)}\int_I \exp\left(\frac{|f(y)|}{\lambda}\right)^{1/s} w(y)\, dy \le 2\right\},$$

where $w(I) = \int_I w(y)\, dy$. Since that $\Phi_s(t) = t\log^s(2+t)$ is a Young function on $[0,\infty)$ and its complementary Young function is $\Psi_s(t) \approx \exp t^{1/s}$, the generalized Hölder inequality

$$\frac{1}{w(I)}\int_I |f(y)h(y)|\, dy \le C\|f\|_{L(\log L)^s;I,w}\|h\|_{\exp(L)^{1/s};I,w} \tag{5}$$

holds for locally integrable functions f and h, see [**1**, Chapter 8] or [**13**, p. 168] for details.

2. Proof of Theorems

We begin with some preliminary lemmas.

Lemma 1 (see [**4**]). *Let $A(x)$ be a function on $\mathbb{R}^n$ with derivatives of order m in $L^s(\mathbb{R}^n)$ for some $n < s \le \infty$. Then*

$$|R_m(A;x,y)| \le C_{m,n}|x-y|^m \sum_{|\mu|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\mu A(z)|^s \, dz \right)^{1/s},$$

where $\widetilde{Q}(x,y)$ is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

Lemma 2. *Let m be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < r < \infty$ and $1 < q < \infty$, $K(x,y)$ be defined on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that for some $w \in A_q$, the operator*

$$Vf(x) = \int_{\mathbb{R}^n} \Phi_m(r^n|K(x,y)|) f(y) \, dy$$

is bounded on $L^q(\mathbb{R}^n, w(x)\,dx)$ with bound Cr^n. Then there exists some constant $C = C(n,m,q)$ such that for each integer l with $0 \le l \le m$ and each cube Q with side length r, the operator

$$S_l f(x) = \int_{\mathbb{R}^n} \Phi_l(r^n|K(x,y)|) f(y) \, dy$$

satisfies

$$\|(S_l f)^q\|_{L(\log L)^{(m-l)q}, Q^*, w} \le C r^{nq} w(Q)^{-1} \|f\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)}$$

provided that supp $f \subset Q$.

Proof: Without loss of generality, we may assume that $\|f\|_1 = 1$. Note that

$$\inf\left\{\lambda > 0; \frac{1}{w(Q^*)} \int_{Q^*} \Phi_{(m-l)q}\left(\frac{|S_l f(x)|^q}{\lambda}\right) w(x)\,dx \le 1\right\}$$

$$\le \left(\inf\left\{\lambda > 0; \frac{q^{mq}}{w(Q^*)} \int_{Q^*} \left(\Phi_{m-l}\left(\frac{|S_l f(x)|}{\lambda}\right)\right)^q w(x)\,dx \le 1\right\}\right)^q$$

$$\le q^{qm} \left(\inf\left\{\lambda > 0; \frac{1}{w(Q^*)} \int_{Q^*} \left(\Phi_{m-l}\left(\frac{|S_l f(x)|}{\lambda}\right)\right)^q w(x)\,dx \le 1\right\}\right)^q.$$

Obviously, there exists a positive constant $C(m)$ such that for any $t > 0$, $\Phi_{m-l}(\Phi_l(t)) \le C(m)\Phi_m(t)$. By the Jensen inequality, we have

$$\Phi_{m-l}(|S_l f(x)|) \le \int_Q \Phi_{m-l}(\Phi_l(r^n|K(x,y)|))|f(y)|\,dy \le CV(|f|)(x).$$

Thus,

$$\int_{\mathbb{R}^n} (\Phi_{m-l}(|S_l f(x)|))^q w(x)\,dx \le Cr^{nq}\|f\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)}.$$

Let $q' = q/(q-1)$ and $\widetilde{w}(x) = w(x)^{-q'/q}$. Recall that $\|f\|_1 = 1$, it follows that

$$1 = \|f\|_1^q \le \|f\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)} \widetilde{w}(Q)^{q/q'},$$

which together with the fact $w \in A_q$ implies

$$\frac{r^{nq}\|f\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)}}{w(Q)} \ge \frac{r^{nq}}{w(Q)\widetilde{w}(Q)^{q/q'}} \ge C.$$

Choose $\lambda_0 = r^n\|f\|_{L^q(\mathbb{R}^n, w(x)\,dx)} w(Q)^{-1/q}$. Trivial computation gives that

$$\begin{aligned}\frac{1}{w(Q^*)}\int_{Q^*}&\left(\Phi_{m-l}\left(\frac{|S_l f(x)|}{\lambda_0}\right)\right)^q w(x)\,dx\\ &\le \frac{C}{w(Q^*)\lambda_0^q}\int_{Q^*}(\Phi_{m-l}(|S_l f(x)|))^q w(x)\,dx \le C,\end{aligned}$$

and so

$$\inf\left\{\lambda > 0; \frac{1}{w(Q^*)}\int_{Q^*}\left(\Phi_{m-l}\left(\frac{|S_l f(x)|}{\lambda}\right)\right)^q w(x)\,dx \le 1\right\} \le C\lambda_0.$$

This leads to the desired estimate. □

Proof of Theorem 3: We only consider the case of $0 < p < \infty$, the proof for the case $p = \infty$ is very similar. Set $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\beta_1/q < \alpha_1/q < \alpha < \alpha_2/q < \beta_2/q$. Write

$$f = \sum_{j=-\infty}^{\infty} f\chi_j = \sum_{j=-\infty}^{\infty} f_j.$$

The $L^q(\mathbb{R}^n)$ boundedness of T gives that

$$\begin{aligned}
\|Tf\|^p_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \chi_k T\left(\sum_{j=k-2}^{k+2} f_j\right)\right\|^p_{L^q(\mathbb{R}^n)} \\
&\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \chi_k T\left(\sum_{j=-\infty}^{k-3} f_j\right)\right\|^p_{L^q(\mathbb{R}^n)} \\
&\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \chi_k T\left(\sum_{j=k+3}^{\infty} f_j\right)\right\|^p_{L^q(\mathbb{R}^n)} \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=k-2}^{k+2} \|f_j\|^p_{L^q(\mathbb{R}^n)} \\
&\quad + \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2/q)p} \left(\sum_{j=-\infty}^{k-3} \|\chi_k T f_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_2}\,dx)}\right)^p \\
&\quad + \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_1/q)p} \left(\sum_{j=k+3}^{\infty} \|\chi_k T f_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_1}\,dx)}\right)^p \\
&\le C\|f\|^p_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2/q)p} \left(\sum_{j=-\infty}^{k-3} \|\chi_k T f_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_2}\,dx)}\right)^p \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_1/q)p} \left(\sum_{j=k+3}^{\infty} \|\chi_k T f_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_1}\,dx)}\right)^p.
\end{aligned}$$

Obviously, if $k \in \mathbb{Z}$ and $j \le k-3$, then

$$|\chi_k(x) T f_j(x)| \le \int_{2^{k-2}<|x-y|\le 2^{k+1}} |K(x-y) f_j(y)|\,dy.$$

Similary, for $k \in \mathbb{Z}$ and $j \ge k+3$,

$$|\chi_k(x) T f_j(x)| \le \int_{2^{j-2}<|x-y|\le 2^{j+1}} |K(x,y) f_j(y)|\,dy.$$

Our hypothesis now says that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2/q)p}\left(\sum_{j=-\infty}^{k-3}\|\chi_k Tf_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_2}\,dx)}\right)^p \\
&\quad+\sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_1/q)p}\left(\sum_{j=k+3}^{\infty}\|\chi_k Tf_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_1}\,dx)}\right)^p \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2/q)p}\left(\sum_{j=-\infty}^{k-3}\|f_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_2}\,dx)}\right)^p \\
&\quad+C\sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_1/q)p}\left(\sum_{j=k+3}^{\infty}\|f_j\|_{L^q(\mathbb{R}^n,|x|^{\alpha_1})\,dx}\right)^p \\
&\le C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3} 2^{j\alpha}\|f_j\|_{L^q(\mathbb{R}^n)}2^{(k-j)(\alpha-\alpha_2/q)}\right)^p \\
&\quad+C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+3}^{\infty} 2^{j\alpha}\|f_j\|_{L^q(\mathbb{R}^n)}2^{(k-j)(\alpha-\alpha_1/q)}\right)^p.
\end{aligned}$$

For the case of $0 < p \le 1$, we have

$$\begin{aligned}
\|Tf\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} &\le C\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \\
&\quad+C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-3} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)}2^{(k-j)(\alpha-\alpha_2/q)p} \\
&\quad+C\sum_{k=-\infty}^{\infty}\sum_{j=k+3}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)}2^{(k-j)(\alpha-\alpha_1/q)p} \\
&= C\|f\|^p_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \\
&\quad+\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)}\sum_{k=j+3}^{\infty} 2^{(k-j)(\alpha-\alpha_2/q)p} \\
&\quad+C\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)}\sum_{k=-\infty}^{j-3} 2^{(k-j)(\alpha-\alpha_2/q)p} \\
&= C\|f\|^p_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand, if $1 < p < \infty$, it follows from the Hölder inequality that

$$\begin{aligned}
\|Tf\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} &\le C\|f\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} \\
&\quad + C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-3} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)} 2^{(k-j)(\alpha-\alpha_2/q)p/2} \\
&\qquad \times \left(\sum_{j=-\infty}^{k-3} 2^{(k-j)(\alpha-\alpha_2/q)p'/2}\right)^{p/p'} \\
&\quad + C\sum_{k=-\infty}^{\infty}\sum_{j=k+3}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)} 2^{(k-j)(\alpha-\alpha_1/q)p/2} \\
&\qquad \times \left(\sum_{j=k+3}^{\infty} 2^{(k-j)(\alpha-\alpha_1/q)p'/2}\right)^{p/p'} \\
&\le C\|f\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} \\
&\quad + C\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)} \sum_{k=j+3}^{\infty} 2^{(k-j)(\alpha-\alpha_2/q)p/2} \\
&\quad + C\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)} \sum_{k=-\infty}^{j-3} 2^{(k-j)(\alpha-\alpha_1/q)p/2} \\
&\le C\|f\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof of Theorem 3. □

Theorem 1 is an easy consequence of Theorem 3 and the following Lemma 3.

Lemma 3. *Let m be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < r < \infty$ and $1 < q < \infty$, $K(x,y)$ be defined on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that for some $w \in A_q$, the operator*

$$U_{m,r}f(x) = \int_{r<|x-y|\le 2r} \Phi_m(r^n|K(x,y)|)f(y)\,dy$$

is bounded on $L^q(\mathbb{R}^n, w(x)\,dx)$ with bound Br^n. Then the commutator

$$S_{r;b,m}f(x) = \int_{r<|x-y|\le 2r} |K(x,y)||b(x)-b(y)|^m|f(y)|\,dy$$

is bounded on $L^q(\mathbb{R}^n, w(x)\,dx)$ with bound $C(n,m,q,B)\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}$.

Proof: Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. Write $\mathbb{R}^n = \bigcup_{j\in\mathbb{Z}} Q_j$, where each Q_j is a cube with side length r, and these cubes have disjoint interiors. Let f_j be the restriction of f on Q_j. Then

$$f(x) = \sum_{j\in\mathbb{Z}} f_j(x), \text{ a.e. } x \in \mathbb{R}^n.$$

Observe that the support of $S_{r;b,m} f_j$ is contained in Q_j^*, and that the supports of various terms $S_{r;b,m} f_j$ have bounded overlaps. So we have

$$\|S_{r;b,m} f\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)} \le C \sum_{j\in\mathbb{Z}} \|S_{r;b,m} f_j\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)},$$

where C is a positive constant which is independent of f and j. Thus we may assume that $\operatorname{supp} f \subset Q$ for some cube Q with side length r. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$, φ is identically one on Q^* and vanishes outside Q^{**}. Let $\widetilde{b}(x) = (b(x) - m_{Q^*}(b))\varphi(x)$, where $m_{Q^*}(b)$ denotes the mean value of b on Q^*. Define the operator $\widetilde{S}_r$ by

$$\widetilde{S}_r h(x) = r^n \int_{r<|x-y|\le 2r} |K(x,y)h(y)|\,dy.$$

Write

$$S_{r;b,m} f(x) \le \sum_{l=0}^{m} C_m^l |\widetilde{b}(x)|^l \widetilde{S}_r(\widetilde{b}^{m-l} f)(x) r^{-n}.$$

It is enough to show that for each l, $0 \le l \le m$,

$$\|\widetilde{b}^l \widetilde{S}_r(\widetilde{b}^{m-l} f)\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)} \le C\|f\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)} r^{nq}. \tag{6}$$

By the generalized Hölder inequality and the well-known John-Nirenberg inequality,

$$\begin{aligned}
&\|\widetilde{b}^l \widetilde{S}_r(\widetilde{b}^{m-l} f)\|^q_{L^q(\mathbb{R}^n, w(x)\,dx)}\\
&\quad \le C w(Q^*) \|\widetilde{b}^{lq}\|_{\exp(L)^{1/(lq)}, Q^*, w} \|(\widetilde{S}_r(\widetilde{b}^{m-l} f))^q\|_{L(\log L)^{lq}, Q^*, w}\\
&\quad \le C w(Q^*) \|(\widetilde{S}_r(\widetilde{b}^{m-l} f))^q\|_{L(\log L)^{lq}, Q^*, w}.
\end{aligned}$$

As in the proof of Lemma 2, we have

$$\begin{aligned}
&\|(\widetilde{S}_r(\widetilde{b}^{m-l} f))^q\|_{L(\log L)^{lq}, Q^*, w}\\
&\le C \left(\inf\left\{ \lambda > 0; \frac{1}{w(Q^*)} \int_{Q^*} \left(\Phi_l\left(\frac{|\widetilde{S}_r(\widetilde{b}^{m-l} f)(x)|}{\lambda} \right) \right)^q w(x)\,dx \le 1 \right\} \right)^q.
\end{aligned}$$

Thus, the proof of the estimate (6) can be reduced to proving that

$$\inf\left\{\lambda>0;\frac{1}{w(Q^*)}\int_{Q^*}\left(\Phi_l\left(\frac{|\widetilde{S}_r(\widetilde{b}^{m-l}f)(x)|}{\lambda}\right)\right)^q w(x)\,dx\le 1\right\}$$
$$\le Cw(Q)^{-1/q}\|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}|Q|.$$

We claim that

$$\frac{1}{w(Q^*)}\int_{Q^*}\left(\Phi_l\left(\frac{|\widetilde{S}_r(\widetilde{b}^{m-l}f)(x)|}{\|\widetilde{b}^{m-l}f\|_1}\right)\right)^q w(x)\,dx$$
$$\le Cw(Q)^{-1}|Q|^q\|f\|^q_{L^q(\mathbb{R}^n,w(x)\,dx)}\|\widetilde{b}^{m-l}f\|_1^{-q}.$$

In fact, by the Jensen inequality,

$$\Phi_l\left(\frac{|\widetilde{S}_r(\widetilde{b}^{m-l}f)(x)|}{\|\widetilde{b}^{m-l}f\|_1}\right)\le\int_{r<|x-y|\le 2r}\Phi_l(r^n|K(x,y)|)\frac{|\widetilde{b}^{m-l}(y)f(y)|}{\|\widetilde{b}^{m-l}f\|_1}\,dy. \tag{7}$$

Let

$$U_lh(x)=\int_{r<|x-y|\le 2r}\Phi_l(r^n|K(x,y|)h(y)\,dy.$$

Denote by U_l^* the dual operator of U_l, that is

$$U_l^*h(y)=\int_{r<|x-y|\le 2r}\Phi_l(r^n|K(x,y)|)h(x)\,dx.$$

For each fixed $g\in L^{q'}(\mathbb{R}^n,\widetilde{w}(x)\,dx)$, $\operatorname{supp} g\subset Q^*$ and $\|g\|_{L^{q'}(\mathbb{R}^n,\tilde{w}(x)\,dx)}\le 1$, the inequality (7) says that

$$\int_{\mathbb{R}^n}g(x)\Phi_l\left(\frac{|\widetilde{S}_r(\widetilde{b}^{m-l}f)(x)|}{\|\widetilde{b}^{m-l}f\|_1}\right)$$
$$\le\int_{\mathbb{R}^n}\frac{|f(y)||\widetilde{b}^{m-l}(y)|}{\|\widetilde{b}^{m-l}f\|_1}\int_{r<|x-y|\le 2r}\Phi_l(r^n|K(x,y)|)g(x)\,dx\,dy$$
$$\le\|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}\|\widetilde{b}^{m-l}U_l^*g\|_{L^{q'}(\mathbb{R}^n,\tilde{w}(x)\,dx)}\|\widetilde{b}^{m-l}f\|_1^{-1}.$$

Applying the generalized Hölder inequality and the John-Nirenberg inequality again, we have

$$\|\widetilde{b}^{m-l}U_l^*g\|^{q'}_{q',\tilde{w}}\le C\widetilde{w}(Q)\|(U_l^*g)^{q'}\|_{L(\log L)^{(m-l)q'},Q^{**},\tilde{w}}.$$

A standard duality arguement states that U_l^* is bounded on $L^{q'}(\mathbb{R}^n, \widetilde{w}(x)\,dx)$. This via Lemma 2 yields

$$\|(U_l^* g)^{q'}\|_{L(\log L)^{(m-l)q'},Q^{**},\tilde{w}} \le C\widetilde{w}(Q)^{-1}|Q|^{q'}\|g\|^{q'}_{L^{q'}(\mathbb{R}^n,\tilde{w}(x)\,dx)}.$$

We thus obtain that

$$\begin{aligned}
&\|\Phi_l(\widetilde{S}_r(\widetilde{b}^{m-l}f/\|\widetilde{b}^{m-l}f\|_1))\|_{L^q(\mathbb{R}^n,w(x)\,dx)}\\
&\quad= \sup_{\operatorname{supp} g\subset Q^*,\|g\|_{L^{q'}(\mathbb{R}^n,\tilde{w}(x)\,dx)}\le 1} \left|\int_{\mathbb{R}^n} g(x)\Phi_l\left(\frac{\widetilde{S}_r(\widetilde{b}^{m-l}f)(x)}{\|\widetilde{b}^{m-l}f\|_1}\right)dx\right|\\
&\quad\le C\|\widetilde{b}^{m-l}f\|_1^{-1}\|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}|Q|,
\end{aligned}$$

which leads to our claim.

We can now conclude the proof of the inequality (6). Note that $\widetilde{w} \in A_{q'}$, we can choose $r > 1$ such that for any cube I,

$$\int_I \widetilde{w}(x)^r\,dx \le C|I|^{1-r}\left(\int_I \widetilde{w}(x)\,dx\right)^r.$$

By the Hölder inequality, it follows that

$$\begin{aligned}
\|\widetilde{b}^{m-l}f\|_1 &\le \|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}\|\widetilde{b}^{(m-l)}\|_{L^{q'}(\mathbb{R}^n,\tilde{w}(x)\,dx)}\\
&\le \|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}\left(\int_{Q^{**}}\widetilde{b}(x)^{(m-l)q'r'}\,dx\right)^{1/(q'r')}\\
&\qquad\times\left(\int_{Q^{**}}\widetilde{w}(x)^r\,dx\right)^{1/(rq')}\\
&\le C\|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}\widetilde{w}(Q)^{1/q'}.
\end{aligned}$$

Therefore,

$$\|\widetilde{b}^{m-l}f\|_1^{-q}\|f\|^q_{L^q(\mathbb{R}^n,w(x)\,dx)}|Q|^q w(Q)^{-1} \ge C|Q|^q w(Q)^{-1}\widetilde{w}(Q)^{-q/q'} \ge C.$$

Our claim together with the last estimate implies that

$$\frac{1}{w(Q^*)}\int_{Q^*}\left(\Phi_l\left(\frac{\widetilde{S}_r(\widetilde{b}^{m-l}f)(x)/\|\widetilde{b}^{m-l}f\|_1)}{\|\tilde{b}^{m-l}f\|_1^{-1}w(Q)^{-1/q}|Q|\|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}}\right)\right)^q w(x)\,dx \le C,$$

and

$$\inf\left\{\lambda>0;\frac{1}{w(Q^*)}\int_{Q^*}\left(\Phi_l\left(\frac{\widetilde{S}_r(\widetilde{b}^{m-l}f)(x))}{\lambda}\right)\right)^q w(x)\,dx\le 1\right\}$$

$$\le Cw(Q)^{-1/q}\|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}|Q|.$$

This finishes the proof of Lemma 3. □

Now we turn our attention to the proof of Theorem 2, which can be obtained from Theorem 3 and the following Lemma 4 directly.

Lemma 4. *Let m be a positive integer and $1<q<\infty$, A be a function on $\mathbb{R}^n$ with derivatives of order m in $\mathrm{BMO}(\mathbb{R}^n)$, $K(x,y)$ be defined on $\mathbb{R}^n\times\mathbb{R}^n$. Suppose that for some $w\in A_q$, the operator*

$$U_{1,r}f(x)=\int_{r<|x-y|\le 2r}\Phi_1(r^n|K(x,y)|)f(y)\,dy$$

is bounded on $L^q(\mathbb{R}^n,w(x)\,dx)$ with bound Br^n. Then the operator

$$V_{A;r}f(x)=\int_{r<|x-y|\le 2r}|K(x,y)|\frac{|R_{m+1}(A;x,y)|}{|x-y|^m}|f(y)|\,dy$$

is bounded on $L^q(\mathbb{R}^n,w(x)\,dx)$ with bound
$C(n,m,q,B)\sum_{|\mu|=m}\|D^\mu A\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

Proof: As in the proof of Lemma 3, we may assume that $\operatorname{supp} f\subset Q$ for some cube Q with side length r. Without loss of generality, we may assume that $\sum_{|\mu|=m}\|D^\mu A\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. Set

$$A^Q(y)=A(y)-\sum_{|\mu|=m}\frac{1}{\mu!}m_Q(D^\mu A)y^\mu.$$

Note that for $x,y\in\mathbb{R}^n$ with $r<|x-y|\le 2r$ and $n<s<\infty$, it follows from Lemma 1 that

$$|R_m(A^Q;x,y)|$$

$$\le C|x-y|^m\sum_{|\mu|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\mu A(z)-m_Q(D^\mu A)|^s\,dz\right)^{1/s}$$

$$\le C|x-y|^m.$$

Therefore,

$$\begin{aligned} V_{A;r}f(x) &= \int_{r<|x-y|\le 2r} |K(x,y)|\frac{|R_{m+1}(A^Q;x,y)|}{|x-y|^m}|f(y)|\,dy \\ &\le C\int_{r<|x-y|\le 2r} |K(x-y)f(y)|\,dy \\ &\quad +C\sum_{|\mu|=m}\int_{r<|x-y|\le 2r} |K(x-y)||D^\mu A(y)-m_Q(D^\mu A)||f(y)|\,dy. \end{aligned}$$

By the estimate (6) (with $m=1$ and $l=0$) in the proof of Lemma 3, we finally obtain

$$\|V_{A,r}f\|_{L^q(\mathbb{R}^n,w(x)\,dx)} \le CB\|f\|_{L^q(\mathbb{R}^n,w(x)\,dx)}. \qquad \square$$

3. Some applications

This section is devoted to some applications of our Theorem 1 and Theorem 2. We begin with the commutator of homogeneous singular integral operator. Note that $\Omega \in L(\log L)^m(S^{n-1})$ is equivalent to that $\Phi_m(|\Omega|) \in L^1(S^{n-1})$ and in this case the operator

$$W_{m,r}f(x) = r^{-n}\int_{r<|x-y|\le 2r} \Phi_m(|\Omega(x-y)|)f(y)\,dy$$

is bounded on $L^q(\mathbb{R}^n,|x|^\alpha\,dx)$ for all $1<q<\infty$ and $-1<\alpha<q-1$ (see [**5**, p. 874]). This together with Theorem 2 tells us that

Corollary 1. *Let $1<q<\infty$, $0<p\le\infty$, T be a sublinear operator which satisfies the size condition*

$$|Tf(x)| \le \int_{\mathbb{R}^n} |b(x)-b(y)|^m\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\,dy,$$

where m is a positive integer and $b\in \mathrm{BMO}(\mathbb{R}^n)$, Ω is homogeneous of degree zero. If T is bounded on $L^q(\mathbb{R}^n)$ and $\Omega\in L(\log L)^m(S^{n-1})$, then T is also bounded on $\dot K_q^{\alpha,p}$ with bound $C(n,m,p,\alpha)$ provided that $-1/q<\alpha<(q-1)/q$.

For the operator $T_{b,m}$ defined by (1) in Section 1, we have

Corollary 2. *Let $1<q<\infty$, $0<p\le\infty$, Ω be homogeneous of degree zero and have mean value zero. If $\Omega\in L(\log L)^{m+1}(S^{n-1})$ for some positive integer m, then the operator $T_{b,m}$ defined by (1) is bounded on $\dot K_q^{\alpha,p}$ with bound $C(n,m,p,\alpha)$ provided that $-1/q<\alpha<(q-1)/q$.*

Corollary 1 and Corollary 2 are new even for the the special case of $p = q$, i.e., the weighted Lebesgue spaces with power weights.

Now let us consider another class of nonstandard Calderón-Zygmund operators. We can easily obtain a general version similar to Corollary 1, but for brevity we only consider the operator defined by

$$T_A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y)\, dy, \tag{8}$$

where Ω is homogeneous of degree zero and has vanishing moment of order m, that is

$$\int_{S^{n-1}} \Omega(\theta)\theta^\mu \, d\theta = 0, \text{ for any multi-index } \mu \text{ with } |\mu| = m,$$

m is a positive integer, A has derivatives of order m in $\mathrm{BMO}(\mathbb{R}^n)$ and

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\mu| \le m} D^\mu A(y)(x-y)^\mu.$$

We have proved in [**7**, pp. 68–69] that if $\Omega \in L(\log L)^\gamma(S^{n-1})$ for some $\gamma > 1$, then $K_j(x) = \frac{\Omega(x)}{|x|^n}\chi_{\{2^j < |x| \le 2^{j+1}\}}(x)$ satisfies the Fourier transform estimate

$$|\widehat{K_j}(\xi)| \le C \min\{1, \log^{-\gamma}(2 + |2^j \xi|)\}.$$

This together with Theorem 1 in [**8**] says that if $\Omega \in L(\log L)^\gamma(S^{n-1})$ for some $\gamma > 3$, then the operator T_A is bounded on $L^2(\mathbb{R}^n)$. Thus by Theorem 3, we can obtain

Corollary 3. *Let* $0 < p \le \infty$, Ω *be homogeneous of degree zero and have vanishing moment of order* m, A *have derivetives of order* m *in* $\mathrm{BMO}(\mathbb{R}^n)$. *Suppose that* $\Omega \in L(\log L)^\gamma(S^{n-1})$ *for some* $\gamma > 3$, *then the operator defined by* (8) *is bounded on* $\dot{K}_2^{\alpha,p}$ *provided that* $-1/2 < \alpha < 1/2$.

Also, even for the special case of $p = 2$, Corollary 3 is new.

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