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BOUNDEDNESS OF THE WEYL FRACTIONAL INTEGRAL ON ONE-SIDED WEIGHTED LEBESGUE AND LIPSCHITZ SPACES

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Abstract

In this paper we introduce the one-sided weighted spaces $\mathcal{L}_w^-(\beta)$, $-1 < \beta < 1$. The purpose of this definition is to obtain an extension of the Weyl fractional integral operator I_α^+ from L_w^p into a suitable weighted space.

Under certain condition on the weight w, we have that $\mathcal{L}_w^-(0)$ coincides with the dual of the Hardy space $H_-^1(w)$. We prove for $0 < \beta < 1$, that $\mathcal{L}_w^-(\beta)$ consists of all functions satisfying a weighted Lipschitz condition. In order to give another characterization of $\mathcal{L}_w^-(\beta)$, $0 \leq \beta < 1$, we also prove a one-sided version of John-Nirenberg Inequality.

Finally, we obtain necessary and sufficient conditions on the weight w for the boundedness of an extension of I_α^+ from L_w^p into $\mathcal{L}_w^-(\beta)$, $-1 < \beta < 1$, and its extension to a bounded operator from $\mathcal{L}_w^-(0)$ into $\mathcal{L}_w^-(\alpha)$.

1. Notations, definitions and prerequisites

Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set. We shall denote its Lebesgue measure by $|E|$ and the characteristic function of E by χ_E.

As usual, a weight w is a measurable, non-negative and locally integrable function defined on $\mathbb{R}$.

Let w be a weight. Given a Lebesgue measurable set $E \subseteq \mathbb{R}$, its w-measure will be denote by $w(E) = \int_E w(t)\,dt$.

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Let $1 < p < \infty$. The weight w belongs to the class A_p^- if there exists a constant C such that

$$\sup_{h>0}\left[\frac{1}{h^p}\int_a^{a+h} w(x)\,dx\left(\int_{a-h}^{a} w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}\right]\le C,$$

for all real number a. In a similar way, w belongs to A_p^+ if

$$\sup_{h>0}\left[\frac{1}{h^p}\int_{a-h}^{a} w(x)\,dx\left(\int_{a}^{a+h} w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}\right]\le C,$$

for all real number a. The class A_1^- is defined by the condition

$$\sup_{h>0}\left[\frac{1}{h}\int_a^{a+h} w(x)\,dx\right]\le Cw(a),$$

for almost every real number a. The weight w belongs to A_1^+ if

$$\sup_{h>0}\left[\frac{1}{h}\int_{a-h}^{a} w(x)\,dx\right]\le Cw(a),$$

for almost every a. These classes A_p^- and A_p^+ were introduced by E. Sawyer in [**12**]. We recall three basic results on these weights.

(i) For $1 < p < \infty$, a weight w belongs to A_p^- if and only if $w^{1-p'}$ belongs to $A_{p'}^+$, where $\frac{1}{p}+\frac{1}{p'}=1$.

(ii) If $1 \le p < q < \infty$, then $A_p^- \subset A_q^-$.

(iii) If $1 < p < \infty$ and w belongs to A_p^-, then w belongs to $A_{p-\epsilon}^-$ for some $\epsilon > 0$.

The proof of (i) and (ii) are very simple and (iii) can be found in Proposition 3 in [**3**].

In the sequel, for each bounded interval $I = [a,b]$ we shall denote $I^- = [a-|I|, a]$ and $I^+ = [b, b+|I|]$.

Let $1 \le q < \infty$. A weight w satisfies the condition $RH^-(q)$ if there exists a constant C such that for every bounded interval I.

$$\left[\frac{1}{|I|}\int_I w(x)^q\,dx\right]^{1/q}\le C\frac{1}{|I|}\int_{I^-} w(x)\,dx.$$

We shall say that a weight w belongs to D^- if there exists a constant C such that for every bounded interval I,

$$w(I\cup I^+)\le Cw(I).$$

It is well known that if $w \in A_p^-$, $1 \le p < \infty$, then $w \in D^-$.

Let w be a weight, $1 \le p < \infty$ and f a measurable function. We shall say that f belongs to L_w^p if

$$\|f\|_{p,w}^p = \int_{-\infty}^{\infty} \left[\frac{|f(x)|}{w(x)}\right]^p dx$$

is finite. The function f belongs to $\widetilde{L_w^p}$ if

$$[f]_{p,w}^p = \sup_{t>0} t^p \left|\left\{x \in \mathbb{R} : \frac{|f(x)|}{w(x)} > t\right\}\right|$$

is finite.

Let $0 < \alpha < 1$. Given f a measurable function on $\mathbb{R}$, its Weyl fractional integral is defined by

$$I_\alpha^+ f(x) = \int_x^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}}\, dy,$$

whenever this integral is finite.

In the sequel, the letter C will denote a positive finite constant not necessarily the same at each occurrence. If $1 \le p \le \infty$ then p' will be its conjugate exponent, that is, $1/p + 1/p' = 1$.

Let w be a weight and $-1 < \beta < 1$.

Definition 1.1. We say that a locally integrable function f defined on $\mathbb{R}$ belongs to $\mathcal{L}_w(\beta)$, if there exists a constant C such that

$$\frac{1}{w(I)|I|^\beta} \int_I |f(y) - f_I|\, dy \le C,$$

for every bounded interval I, where $f_I = \frac{1}{|I|}\int_I f$. The least constant C will be denoted $\|f\|_{\mathcal{L}_w(\beta)}$.

The spaces $\mathcal{L}_w(\beta)$ were introduced by E. Harboure, O. Salinas and B. Viviani in [**1**]. They are a weighted version of the spaces $\mathcal{L}_{\lambda,p}$, for $p = 1$, defined by J. Peetre in [**8**]. If w belongs to A_q^-, $1 \le q < 2$, then $\mathcal{L}_w(0)$ is the dual space of the one-sided weighted Hardy space $H_-^1(w)$, see [**10**] and [**11**].

Definition 1.2. We say that a locally integrable function f defined on $\mathbb{R}$ belongs to $\mathcal{L}_w^-(\beta)$, if there exists a constant C such that

$$\frac{1}{w(I^-)|I|^\beta}\int_I |f(y)-f_I|\,dy \le C,$$

for every bounded interval I. The least constant C satisfying this inequality will be denoted $\|f\|_{\mathcal{L}_w^-(\beta)}$.

In the following definition, we consider a one-sided version of the classes $H(\alpha,p)$ defined in [**1**].

Definition 1.3. Let $0<\alpha<1$ and $1<p\le\infty$. We say that a weight w belongs to $H^-(\alpha,p)$ if there exists a constant C such that for every bounded interval $I=[a,b]$, the inequality

$$|I|^{\frac{1}{p}-\alpha+1}\left[\int_b^\infty \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}}\,dy\right]^{1/p'} \le C\frac{w(I)}{|I|},$$

holds.

2. Statement of the main results

Lemma 4.1(iii) shows that if w belongs to $H^-(\alpha,p)$, $1<p\le\infty$, then w belongs to D^- and therefore $\mathcal{L}_w(\beta)\subseteq\mathcal{L}_w^-(\beta)$ for every β: $-1<\beta<1$. The next theorem states that w belonging to D^- is a sufficient condition for the equality of these spaces, whenever $0\le\beta<1$.

Theorem 2.1. *Let $0\le\beta<1$ and let w belong to D^-. Then, the spaces $\mathcal{L}_w(\beta)$ and $\mathcal{L}_w^-(\beta)$ are equal, and their norms are equivalent.*

The next theorem gives us a characterization of the spaces $\mathcal{L}_w(\beta)$, $0\le\beta<1$, whenever w belongs to A_p^-. In the case $\beta=0$, we shall prove this result using Proposition 3.6, which states a one-sided weighted version of John-Nirenberg Inequality.

Theorem 2.2. *Let $0\le\beta<1$ and $1\le p<\infty$. Let w be a weight such that w belongs to A_p^-. Then, $f\in\mathcal{L}_w(\beta)$ if and only if there exists a constant C such that*

$$\int_{I^-} |f(x)-f_{I^+}|^q w(x)^{1-q}\,dx \le Cw(I^-)|I|^{\beta q}, \tag{2.1}$$

for all bounded interval I and every $q: 1\le q\le p'$, $q<\infty$.

The following two theorems state a sufficient and necessary condition on the weight w to obtain extensions of I_α^+ defined on certain spaces.

Theorem 2.3. *Let $0 < \alpha < 1$, $1 < p < \infty$ and $\beta = \alpha - 1/p$. The following statements are equivalent.*

(i) *The weight w belongs to $H^-(\alpha, p)$.*

(ii) *The operator I_α^+ can be extended to a linear bounded operator $\widetilde{I_\alpha^+}$ from $\widetilde{L_w^p}$ into $\mathcal{L}_w^-(\beta)$ by means of*

$$\widetilde{I_\alpha^+}(f)(x) = -\int_{x_0}^{x} \frac{f(y)\,dy}{|y-x|^{1-\alpha}} + \int_{x_0}^{\infty} \left[\frac{1}{|y-x|^{1-\alpha}} - \frac{1-\chi_{[x_0,x_0+1]}(y)}{(y-x_0)^{1-\alpha}}\right] f(y)\,dy, \tag{2.2}$$

for any $x_0 \in \mathbb{R}$.

(iii) *The operator I_α^+ can be extended to a linear bounded operator $\widetilde{I_\alpha^+}$ from L_w^p into $\mathcal{L}_w^-(\beta)$, where $\widetilde{I_\alpha^+}$ is defined as in (2.2).*

Theorem 2.4. *Let w a weight and $0 < \alpha < 1$. The following statements are equivalent.*

(i) *The weight w belongs to $H^-(\alpha, \infty)$.*

(ii) *The operator I_α^+ can be extended to a linear bounded operator $\widetilde{I_\alpha^+} : \mathcal{L}_w(0) \to \mathcal{L}_w(\alpha)$ by means of*

$$\widetilde{I_\alpha^+}(f)(x) = \int_{-\infty}^{\infty} \left[\frac{\chi_{[x_0,\infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x,\infty)}(y)}{|y-x|^{1-\alpha}}\right] f(y)\,dy,$$

for an appropriate choice of $x_0 \in \mathbb{R}$.

Remark 2.5. Let $1 < p < \frac{1}{\alpha}$ and $\beta = \alpha - 1/p < 0$.

(i) It is easy to see that if w belongs to $RH^-(\frac{1}{1+\beta})$, then $L_w^{-1/\beta} \subseteq \mathcal{L}_w^-(\beta)$.

(ii) By Lemma 4.4 in [**9**], if $w^{p'}$ belongs to $A^-_{-\beta p'+1}$ then w satisfies the condition $RH^-(p')$, and taking into account that $\frac{1}{1+\beta} < p'$, it follows that w belongs to $RH^-(\frac{1}{1+\beta})$.

(iii) Theorem 6 in [**4**] states the fact that $w^{p'}$ belongs to $A^-_{-\beta p'+1}$ is a necessary and sufficient condition for the boundedness of I_α^+ from L_w^p into $L_w^{-1/\beta} \subseteq \mathcal{L}_w^-(\beta)$.

(iv) If $w^{p'}$ belongs to $A^-_{-\beta p'+1}$, since $w^{p'} \in A^-_{p'+1}$, we have that w belongs to $H^-(\alpha, p)$. However, there exist weights w belonging to $H^-(\alpha, p)$ such that $w^{p'}$ does not belong to $A^-_{p'+1}$, for example, $w(x) = |x|^\gamma$ for $-\beta \le \gamma < 1 - \beta$, see Remark 4.3.

In consequence, if $-1 < \beta < 0$ and $w^{p'}$ belongs to $A^-_{-\beta p'+1}$, the extension of I^+_α in Theorem 2.3 can be obtained from Theorem 6 in [**4**]. But, (iv) shows that Theorem 2.3 can be applied to a larger class of weights.

Remark 2.6. Let w be a weight. We shall say that a locally integrable function f defined on $\mathbb{R}$, belongs to $MW^-(w)$ if there exists a constant C such that

$$\frac{1}{|I|}\frac{1}{\operatorname{ess\,inf}_{I^-} w}\int_I |f(y)-f_I|\,dy \le C,$$

for every bounded interval I.

(i) By Definition 1.2, it follows that $MW^-(w) \subseteq \mathcal{L}^-_w(0)$. Moreover, if w belongs to A^-_1 then $\mathcal{L}_w(0) \subseteq MW^-(w)$, and as a consequence of Theorem 2.1, $\mathcal{L}^-_w(0) = MW^-(w)$.

(ii) Following the same lines of Theorem 7 in [**7**], it can be seen that, in the case $\alpha = 1/p$, the weight $w^{p'}$ belongs to A^-_1 if and only if the operator I^+_α is bounded from L^p_w into $MW^-(w)$. Also see [**2**].

(iii) If $w^{p'}$ belongs to A^-_1 then, by Remark 4.3, w belongs to $H^-(\alpha,p)$.

In consequence, the fact that $w^{p'}$ belongs to A^-_1 implies the boundedness of I^+_α from L^p_w into $MW^-(w)$, is contained in Theorem 2.3.

3. The spaces $\mathcal{L}_w(\beta)$ and $\mathcal{L}^-_w(\beta)$

The next lemma will be used in the proof of Theorem 2.1.

Lemma 3.1. *Let $-1 < \beta < 1$, f a locally integrable function defined on $\mathbb{R}$, and $w \in D^-$. The following statements are equivalent.*

(i) $f \in \mathcal{L}^-_w(\beta)$.

(ii) *There exists a constant C such that for every $a \in \mathbb{R}$ and $h > 0$,*

$$\frac{1}{w([a-h/2,a])h^\beta}\int_a^{a+h} |f(y)-f_{[a+h/2,a+h]}|\,dy \le C.$$

(iii) *There exists a constant C such that for every $a \in \mathbb{R}$ and $h > 0$,*

$$\frac{1}{w([a-h/2,a])h^\beta}\int_a^{a+h} |f(y)-f_{[a+h,a+3h]}|\,dy \le C.$$

The constants C in (ii) and (iii) are equivalent to $\|f\|_{\mathcal{L}^-_w(\beta)}$.

Proof: (i) $\Rightarrow$ (ii). Using (i) and taking into account that $w \in D^-$, we have

$$\begin{aligned}
&\int_a^{a+h/2} |f(y) - f_{[a+h/2,a+h]}|\,dy \\
&\quad\leq \int_a^{a+h/2} |f(y)-f_{[a+h/4,a+h/2]}|\,dy + 2\int_{a+h/4}^{a+h} |f(y)-f_{[a+h/2,a+h]}|\,dy \\
&\quad\leq 3\int_a^{a+h/2} |f(y) - f_{[a,a+h/2]}|\,dy + 5\int_{a+h/4}^{a+h} |f(y) - f_{[a+h/4,a+h]}|\,dy \\
&\quad\leq C\|f\|_{\mathcal{L}_w^-(\beta)} w([a-h/2,a])h^\beta + C\|f\|_{\mathcal{L}_w^-(\beta)} w([a-h/2,a+h/4])h^\beta \\
&\quad\leq C\|f\|_{\mathcal{L}_w^-(\beta)} w([a-h/2,a])h^\beta.
\end{aligned}$$

From these inequalities and using (i) again, we have the estimate

$$\begin{aligned}
&\int_a^{a+h} |f(y) - f_{[a+h/2,a+h]}|\,dy \\
&\quad= \int_a^{a+h/2} |f(y) - f_{[a+h/2,a+h]}|\,dy + \int_{a+h/2}^{a+h} |f(y) - f_{[a+h/2,a+h]}|\,dy \\
&\quad\leq C\|f\|_{\mathcal{L}_w^-(\beta)} w([a-h/2,a])h^\beta + C\|f\|_{\mathcal{L}_w^-(\beta)} w([a,a+h/2])h^\beta \\
&\quad\leq C\|f\|_{\mathcal{L}_w^-(\beta)} w([a-h/2,a])h^\beta,
\end{aligned}$$

which shows that (ii) holds. In a similar way it can be proved that (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i). □

As we have already mencioned if w belongs to D^- then, for every $-1 < \beta < 1$ we have the inclusion $\mathcal{L}_w(\beta) \subseteq \mathcal{L}_w^-(\beta)$. In order to prove Theorem 2.1, it will be sufficient to show that $\mathcal{L}_w^-(\beta) \subseteq \mathcal{L}_w(\beta)$.

Proof of Theorem 2.1: We suppose that $f \in \mathcal{L}_w^-(\beta)$. Let $a \in \mathbb{R}$ and $h > 0$. For each $j \geq 0$ we define $a_j = a + h/2^j$. Then,

$$\begin{aligned} &\int_a^{a+h/2} |f(y) - f_{[a+h/2,a+h]}|\,dy \\ &\quad = \sum_{j=1}^{\infty} \int_{a_{j+1}}^{a_j} |f(y) - f_{[a+h/2,a+h]}|\,dy \\ &\quad \leq \sum_{j=1}^{\infty} \int_{a_{j+1}}^{a_j} |f(y) - f_{[a_j,a_{j-1}]}|\,dy + \sum_{j=2}^{\infty} \frac{h}{2^{j+1}} |f_{[a_j,a_{j-1}]} - f_{[a_1,a_0]}| \\ &\quad = I + II. \end{aligned} \tag{3.1}$$

Taking into account that for each $j \geq 2$,

$$|f_{[a_j,a_{j-1}]} - f_{[a_1,a_0]}| \leq \frac{2^j}{h} \int_{a_j}^{a_{j-1}} |f - f_{[a+h/2,a+h]}|$$

it follows that,

$$II \leq \sum_{j=2}^{\infty} \frac{1}{2} \int_{a_j}^{a_{j-1}} |f - f_{[a+h/2,a+h]}| = \frac{1}{2} \int_a^{a+h/2} |f(y) - f_{[a+h/2,a+h]}|\,dy.$$

Then, by (3.1)

$$\int_a^{a+h/2} |f(y) - f_{[a+h/2,a+h]}|\,dy \leq 2I. \tag{3.2}$$

Now, using (iii) of Lemma 3.1 and keeping in mind that $\beta \geq 0$ we have that,

$$I \leq C \sum_{j=1}^{\infty} \left(\frac{h}{2^j}\right)^{\beta} w([a_{j+2}, a_{j+1}]) \leq C h^{\beta} w([a, a+h/4]). \tag{3.3}$$

From (3.2) and (3.3), and taking into account that $f \in \mathcal{L}_w^-(\beta)$, we get

$$\begin{aligned}
&\int_a^{a+h} |f(y) - f_{[a+h/2,a+h]}|\, dy \\
&\quad = \int_a^{a+h/2} |f(y) - f_{[a+h/2,a+h]}|\, dy + \int_{a+h/2}^{a+h} |f(y) - f_{[a+h/2,a+h]}|\, dy \\
&\quad \leq Ch^\beta w([a, a+h/4]) + Ch^\beta w([a, a+h/2]) \\
&\quad \leq Ch^\beta w([a, a+h]).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_a^{a+h} |f(y) - f_{[a,a+h]}|\, dy \\
&\qquad\qquad \leq 3\int_a^{a+h} |f(y) - f_{[a+h/2,a+h]}|\, dy \leq Ch^\beta w([a, a+h]),
\end{aligned}$$

which shows that $f \in \mathcal{L}_w(\beta)$. □

Remark 3.2. Let $-1 < \beta < 0$ and $w(t) = e^{-t}$. The weight w belongs to A_1^- however, we only have the strict inclusion $\mathcal{L}_w(\beta) \subset \mathcal{L}_w^-(\beta)$. For example, given $a > 1$ we consider the function

$$f(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 1, & t < 0. \end{cases}$$

We observe, using Remark 2.5(i), that $f \in \mathcal{L}_w^-(\beta)$. On the other hand,

$$\begin{aligned}
\frac{1}{h^\beta w([0,h])}\int_0^h |f - f_{[h,2h]}| &= \frac{1}{h^\beta(1-e^{-h})}\left[\frac{1-e^{-ah}}{a} - \frac{e^{-ah}}{a}(1-e^{-ah})\right] \\
&= \frac{(1-e^{-ah})^2}{h^\beta(1-e^{-h})a},
\end{aligned}$$

which tends to infinite whenever h tends to infinite. This implies that $f \notin \mathcal{L}_w(\beta)$.

The next proposition will be used in the proof of Theorem 2.2.

Proposition 3.3. *Let $0 < \beta < 1$ and let w belong to D^-. Then, $f \in \mathcal{L}_w(\beta)$ if and only if, there exists a constant C such that*

$$|f(x) - f(y)| \leq C\left[\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}}\,dz + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}}\,dz\right], \tag{3.4}$$

for almost every real numbers x and y.

Proof: We suppose that $f \in \mathcal{L}_w(\beta)$. We shall show that for every $h > 0$ and almost every x,

$$|f(x) - f_{[x+h/2,x+h]}| \leq C\int_x^{x+h/2} \frac{w(z)}{(z-x)^{1-\beta}}\,dz. \tag{3.5}$$

For each $i \geq 0$ let $x_i = x + h/2^i$. If x is a Lebesgue point of f we have that,

$$\begin{aligned} |f(x)-f_{[x+h/2,x+h]}| &\leq |f(x) - f_{[x_{i+1},x_i]}| + |f_{[x_{i+1},x_i]} - f_{[x_1,x_0]}| \\ &\leq |f(x) - f_{[x_{i+1},x_i]}| + \sum_{j=1}^{i} |f_{[x_{j+1},x_j]} - f_{[x_j,x_{j-1}]}| \\ &\leq \sum_{j=1}^{\infty} |f_{[x_{j+1},x_j]} - f_{[x_j,x_{j-1}]}|. \end{aligned} \tag{3.6}$$

For each $j \geq 1$, since $f \in \mathcal{L}_w(\beta)$ we obtain

$$|f_{[x_{j+1},x_j]} - f_{[x_j,x_{j-1}]}| \leq C\frac{1}{(x_{j+1} - x_{j-1})^{1-\beta}} w([x_{j+1}, x_{j-1}]).$$

From this inequality, (3.6) and taking into account that $w \in D^-$ we get,

$$\begin{aligned} |f(x) - f_{[x+h/2,x+h]}| &\leq C\sum_{j=1}^{\infty} \int_{x_{j+1}}^{x_{j-1}} \frac{w(z)}{(z-x)^{1-\beta}}\,dz \\ &= C\int_x^{x+h} \frac{w(z)}{(z-x)^{1-\beta}}\,dz \\ &\leq C\int_x^{x+h/2} \frac{w(z)}{(z-x)^{1-\beta}}\,dz, \end{aligned}$$

which shows that (3.5) holds. Let $x < y$ two Lebesgue points of f. By (3.5) we have that,

$$
\begin{aligned}
|f(x)-f(y)| &\leq |f(x) - f_{[\frac{x+y}{2},y]}| + |f(y) - f_{[y+\frac{y-x}{2},y+(y-x)]}| \\
&\quad + |f_{[\frac{x+y}{2},y]} - f_{[y+\frac{y-x}{2},y+(y-x)]}| \\
&\leq C\left[\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}}\,dz + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}}\,dz\right] \qquad (3.7) \\
&\quad + |f_{[\frac{x+y}{2},y]} - f_{[y+\frac{y-x}{2},y+(y-x)]}|.
\end{aligned}
$$

From the hypotheses $f \in \mathcal{L}_w(\beta)$ and $w \in D^-$, it follows that the third term on the right hand is bounded by

$$
\begin{aligned}
&\frac{C}{y-x}\int_{x+\frac{y-x}{2}}^{y+(y-x)} |f(t) - f_{[x+\frac{y-x}{2},y+(y-x)]}|\,dt \\
&\qquad \leq \frac{C}{(y-x)^{1-\beta}} w([x, y+\frac{y-x}{2}]) \leq C\int_x^{(x+y)/2} \frac{w(z)}{(z-x)^{1-\beta}}\,dz.
\end{aligned}
$$

Therefore, by (3.7) we have that (3.4) holds.

Conversely, given a real number a and $h > 0$, by (3.4)

$$
\begin{aligned}
&\int_a^{a+h} |f(x) - f_{[a,a+h]}|\,dx \qquad (3.8) \\
&\leq C\left[\int_a^{a+h}\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}}\,dz\,dx + \int_a^{a+h}\int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}}\,dz\,dy\right].
\end{aligned}
$$

Changing the order of integration and taking into account that $w \in D^-$, it follows that (3.8) is bounded by $Ch^{\beta}w([a, a+h])$. This completes the proof of the proposition. □

The next two lemmas will be needed in the proof of Proposition 3.6.

Lemma 3.4. *Let $w \in D^-$ and $f \in \mathcal{L}_w(0)$. Given two intervals $I \subseteq J$ the inequality*

$$
\frac{1}{w(J)}\int_J |f(y) - f_{I^+}|\chi_{I^-\cup I}(y)\,dy \leq C\|f\|_{\mathcal{L}_w(0)},
$$

holds with a constant C only depending on w.

Proof: Let $I = (a, b)$ and $J = (c, d)$. We consider $\alpha = \max\{a - |I|, c\}$ and $\beta = b + |I|$. Since $J \cap (I^- \cup I) \subseteq (\alpha, \beta)$ we have that,

$$
\begin{aligned}
&\frac{1}{w(J)} \int_J |f(y) - f_{I^+}|\chi_{I^- \cup I}(y)\,dy \le \frac{1}{w(J)} \int_\alpha^\beta |f(y) - f_{I^+}|\,dy \\
(3.9)\quad &\le \frac{1}{w(J)} \left[\int_\alpha^\beta |f(y) - f_{(\alpha,\beta)}|\,dy + \frac{(\beta - \alpha)}{|I^+|} \int_{I^+} |f(y) - f_{(\alpha,\beta)}|\,dy \right].
\end{aligned}
$$

We observe that $(\beta - \alpha) \le 3|I|$, which implies

$$(3.9) \le \frac{4}{w(J)} \int_\alpha^\beta |f(y) - f_{(\alpha,\beta)}|\,dy.$$

From the hypotheses $f \in \mathcal{L}_w(0)$ and $w \in D^-$, and taking into account that $(\alpha, \beta) \subseteq J \cup J^+$, (3.9) is bounded by

$$\frac{4}{w(J)} \|f\|_{\mathcal{L}_w(0)} w((\alpha, \beta)) \le C\|f\|_{\mathcal{L}_w(0)},$$

as we wanted to prove. □

Lemma 3.5. *Let $1 < p < \infty$ and $w \in A_p^-$. Then, there exists a constant C such that for every $\beta > 0$ the inequality*

$$w(\{x \in I^- : w(x) < \beta\}) \le C \left[\beta \frac{|I^+|}{w(I^+)} \right]^{p'} w(I^+), \tag{3.10}$$

holds.

Proof: This lemma is a simple variant of Lemma 3.1 in [**6**]. □

The following result is a one-sided weighted version of John-Nirenberg Inequality. For its proof we shall use the method employed in Theorem 3 in [**6**] and the techniques of Lemma 1 in [**5**].

Proposition 3.6. *Let f belong to $\mathcal{L}_w(0)$. Then,*

(i) *If $w \in A_1^-$ there exist positive constants C_1 and C_2 such that for every $\lambda > 0$,*

$$w(\{x \in I^- : |f(x) - f_{I^+}|w(x)^{-1} > \lambda\}) \le C_1 e^{-C_2\lambda/\|f\|_{\mathcal{L}_w(0)}} w(I^-)$$

holds for every bounded interval I.

(ii) *If $w \in A_p^-$, $1 < p < \infty$ there exists a positive constant C_3 such that for every $\lambda > 0$,*

$$w(\{x \in I^- : |f(x) - f_{I^+}|w(x)^{-1} > \lambda\}) \le C_3 \left(1 + \lambda/\|f\|_{\mathcal{L}_w(0)}\right)^{-p'} w(I^-)$$

holds for every bounded interval I.

Proof: Without loss of generality we can suppose that $\|f\|_{\mathcal{L}_w(0)} = 1$. For each $\lambda > 0$ and each bounded interval I, let

$$A(\lambda, I) = w(\{x \in I^- : |f(x) - f_{I^+}|w(x)^{-1} > \lambda\}),$$

and

$$\mathcal{A}(\lambda) = \sup \frac{A(\lambda, I)}{w(I^-)}, \tag{3.11}$$

where the supremum is taken over all $f : \|f\|_{\mathcal{L}_w(0)} = 1$, and all bounded interval I. Thus, for every $\lambda > 0$, we have that $\mathcal{A}(\lambda) \leq 1$.

By Lemma 3.4 there exists a constant μ satisfying

$$\frac{1}{w(J)} \int_J |f(y) - f_{I^+}|\chi_{I^- \cup I}(y)\, dy \leq \mu, \tag{3.12}$$

for every bounded intervals $I \subseteq J$ and every $f : \|f\|_{\mathcal{L}_w(0)} = 1$.

Fixed $I = [a, b]$, let $s > \mu$ and

$$\Omega_s = \{x \in \mathbb{R} : M_w^-(|f - f_{I^+}|\chi_{I^- \cup I} w^{-1})(x) > s\},$$

where M_w^- is the left sided maximal function with respect to the measure w defined as

$$M_w^-(g)(x) = \sup_{h>0} \frac{\int_{x-h}^x |g(y)|w(y)\, dy}{w([x - h, x])}.$$

Since Ω_s is an open set, we can write $\Omega_s = \cup_{i \geq 1} J_i$, where the J_i's are its connected components.

We observe that if $J_i \cap I^- \neq \emptyset$ then $J_i \cap I^+ = \emptyset$. In fact, suppose that $J_i \cap I^- \neq \emptyset$ and let $J_i = (\alpha, \beta)$. If $\beta \geq b$ a simple variant of Lemma 2.1 in [**12**], shows that

$$\mu < s \leq \frac{1}{w((\alpha, b))} \int_\alpha^b |f(y) - f_{I^+}|\chi_{I^- \cup I}(y)\, dy.$$

However, using (3.12) we have that

$$\frac{1}{w((\alpha, b))} \int_\alpha^b |f(y) - f_{I^+}|\chi_{I^- \cup I}(y)\, dy \leq \mu.$$

In consequence, $\beta < b$ and $J_i \cap I^+ = \emptyset$.

Let $\{J_i : J_i \cap I^- \neq \emptyset\} = \{H_i\}_{i \geq 1}$, where $H_i = (a_i, b_i)$. For each i, since $M_w^-(|f - f_{I^+}|\chi_{I^- \cup I} w^{-1})(b_i) \leq s$ we have that,

$$H_i \subseteq I^- \cup I \quad \text{and} \quad \frac{1}{w(H_i)} \int_{H_i} |f(y) - f_{I^+}|\, dy = s. \tag{3.13}$$

By Lebesgue's Differentiation Theorem with respect to w for almost every $x \in I^- \setminus \cup_{i\geq 1} H_i$,

$$|f(x) - f_{I^+}|w(x)^{-1} \leq s.$$

Using (3.13), (3.12) and keeping in mind that $w \in D^-$, we obtained that

$$\begin{aligned}\sum_{i\geq 1} w(H_i) &= \frac{1}{s}\sum_{i\geq 1}\int_{H_i} |f(y) - f_{I^+}|\,dy \\ &\leq \frac{1}{s}\int_{I^-\cup I} |f(y) - f_{I^+}|\,dy \leq \frac{1}{s}\mu w(I^-\cup I) \leq \frac{1}{s}\mu C_w w(I^-).\end{aligned} \tag{3.14}$$

Fixed $H_i = (a_i, b_i)$ we define the sequences $(x_k)_{k\geq 1}$ and $(y_k)_{k\geq 1}$ by $b_i - x_k = 2(b_i - y_k) = (2/3)^k|H_i|$, and the intervals $H_{i,k} = (x_k, y_k)$. Therefore,

$$H_i = \bigcup_{k\geq 1} H_{i,k}^-, \quad \frac{1}{w(H_{i,k}^+)}\int_{H_{i,k}^+} |f(y) - f_{I^+}|\,dy \leq s, \tag{3.15}$$

and

$$|f(x) - f_{I^+}|w(x)^{-1} \leq \lambda \quad \text{a.e.} \quad x \in I^- \setminus \bigcup_{k,i} H_{i,k}^-.$$

Then,

$$A(\lambda, I) \leq \sum_{i,k} w(\{x \in H_{i,k}^- : |f(x) - f_{I^+}|w(x)^{-1} > \lambda\}).$$

If $\mu < s \leq \lambda$ and $0 < \gamma < \lambda$, we have that

$$\begin{aligned}A(\lambda, I) &\leq \sum_{i,k} w(\{x \in H_{i,k}^- : |f(x) - f_{H_{i,k}^+}|w(x)^{-1} > \lambda - \gamma\}) \\ &\quad + \sum_{i,k} w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I^+}|w(x)^{-1} > \gamma\}) = I + II.\end{aligned} \tag{3.16}$$

From (3.11), (3.15) and (3.14) we obtain the estimate

$$\begin{aligned}I &\leq \sum_{i,k} \mathcal{A}(\lambda-\gamma) w(H_{i,k}^-) = \mathcal{A}(\lambda-\gamma)\sum_i w(H_i) \\ &\leq \frac{C_w \mu}{s}\mathcal{A}(\lambda-\gamma) w(I^-).\end{aligned} \tag{3.17}$$

On the other hand, (3.15) implies that

$$|f_{H_{i,k}^+} - f_{I^+}| \leq \frac{1}{|H_{i,k}^+|}\int_{H_{i,k}^+} |f(y) - f_{I^+}|\,dy \leq s\frac{w(H_{i,k}^+)}{|H_{i,k}^+|}. \tag{3.18}$$

If $w \in A_1^-$ there exists $\rho > 1$ such that for every i, k and almost every $x \in H_{i,k}^-$,

$$\frac{w(H_{i,k}^+)}{|H_{i,k}^+|} \leq \rho w(x).$$

Then, using (3.18) we have

$$|f_{H_{i,k}^+} - f_{I^+}| \leq \rho\, s \operatorname{ess\,inf}_{x \in H_{i,k}^-} w(x).$$

In consequence,

$$\begin{aligned} & w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I^+}|w(x)^{-1} > \gamma\}) \\ & \leq w\left(\left\{x \in H_{i,k}^- : w(x) < \frac{\rho\, s}{\gamma} \operatorname{ess\,inf}_{x \in H_{i,k}^-} w(x)\right\}\right). \end{aligned}$$

Choosing $s = 2\mu C_w$ and $\gamma = \rho\, s$, if $\lambda > \gamma$ we have $\mu < s < \lambda$ and $II = 0$. Then, from (3.16) and (3.17) we obtain that

$$A(\lambda, I) \leq \frac{1}{2}\mathcal{A}(\lambda - \gamma) w(I^-),$$

that is, if $\lambda > \gamma$,

$$\mathcal{A}(\lambda) \leq \frac{1}{2}\mathcal{A}(\lambda - \gamma).$$

Now, proceeding as in Theorem 3 of [**6**], it can be obtained (i) of this proposition.

In order to prove (ii), we suppose that $w \in A_p^-$, $1 < p < \infty$. Using (3.18), Lemma 3.5 and taking into account that $w \in D^-$

$$\begin{aligned} & w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I^+}|w(x)^{-1} > \gamma\}) \\ & \leq w\left(\left\{x \in H_{i,k}^- : w(x) < \frac{s}{\gamma}\frac{w(H_{i,k}^+)}{|H_{i,k}^+|}\right\}\right) \\ & \leq C\left[\frac{s}{\gamma}\frac{w(H_{i,k}^+)}{|H_{i,k}^+|}\frac{|H_{i,k}|}{w(H_{i,k})}\right]^{p'} w(H_{i,k}) \\ & \leq C\left(\frac{s}{\gamma}\right)^{p'} w(H_{i,k}^-). \end{aligned}$$

By (3.15) and (3.14), we have

$$II \le C\left(\frac{s}{\gamma}\right)^{p'} \sum_{i,k} w(H^-_{i,k}) = C\left(\frac{s}{\gamma}\right)^{p'} \sum_i w(H_i) \le C\mu \frac{s^{p'-1}}{\gamma^{p'}} w(I^-).$$

Then, (3.16) and (3.17) imply that

$$A(\lambda, I) \le C\mu\left[\frac{A(\lambda-\gamma)}{s} + \frac{s^{p'-1}}{\gamma^{p'}}\right] w(I^-).$$

From this inequality, (ii) follows as in Theorem 3 of [**6**]. □

Proposition 3.7. *Let $0<\beta<1$ and $1<p<\infty$. Let w be a weight such that $w^{1+\frac{\beta}{1-\beta}p}$ belongs to A^-_p. Then, $f \in \mathcal{L}_w(\beta)$ if and only if there exists a constant C such that (2.1) holds for all bounded interval I and every $q: 1 \le q \le p'/(1-\beta)$.*

Proof: Suppose that (2.1) holds for every $q: 1\le q \le p'/(1-\beta)$. Taking $q=1$ it is easy to show that $f \in \mathcal{L}_w(\beta)$. Conversely, let f belong to $\mathcal{L}_w(\beta)$. We observe that it will be sufficient to consider $q = p'/(1-\beta)$, because from this case and applying Hölder's inequality we obtain (2.1) for every $1 \le q < p'/(1-\beta)$. Given a bounded interval I and using Proposition 3.3, we have that

$$\begin{aligned}
&\int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q}\,dx \\
&\quad\le \int_{I^-}\left[\frac{1}{|I^+|}\int_{I^+}|f(x)-f(y)|\,dy\right]^q w(x)^{1-q}\,dx \\
&\quad\le C\int_{I^-} w(x)^{1-q}\left[\frac{1}{|I^+|}\int_{I^+}\left(\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}}\,dz\right.\right. \\
&\qquad\qquad\left.\left. + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}}\,dz\right)\,dy\right]^q\,dx \qquad (3.19) \\
&\quad\le C\int_{I^-} w(x)^{1-q}\left(\int_x^{x+\frac{3|I|}{2}} \frac{w(z)}{(z-x)^{1-\beta}}\,dz\right)^q\,dx \\
&\qquad + \frac{C}{|I^+|^q}\int_{I^-} w(x)^{1-q}\left(\int_{I^+}\int_y^{y+\frac{3|I|}{2}} \frac{w(z)}{(z-y)^{1-\beta}}\,dz\,dy\right)^q\,dx \\
&= A + B.
\end{aligned}$$

If we denote $J = I^- \cup I \cup I^+$ then we have the estimate

$$A \le C \int_{I^-} w(x)^{1-q} I_\beta^+(w\chi_J)(x)^q \, dx.$$

Our hypothesis $w^{1+\frac{\beta}{1-\beta}p} \in A_p^-$ is equivalent to

$$w^{1-\frac{p'}{1-\beta}} \in A_{p'}^+, \tag{3.20}$$

where $p' = 1 + \frac{q}{s'}$ and $\frac{1}{s} = \frac{1}{q} + \beta$. Then, by Theorem 6 in [**4**] it follows that

$$A \le C \left(\int_{-\infty}^{\infty} w(x)^{-\frac{s}{q'}} |w\chi_J(x)|^s \, dx \right)^{q/s} = C \left(\int_J w(x)^{s/q} \, dx \right)^{q/s}.$$

Since $q/s = q\beta + 1 > 1$, applying Hölder's inequality and taking into account that $w \in D^-$ we obtain

$$A \le C \int_J w(x) \, dx \; |J|^{\frac{q}{s}-1} \le C w(I^-)|I|^{\beta q}. \tag{3.21}$$

Let us estimate B. If we set $J' = I^+ \cup I^{++} \cup I^{+++}$, then

$$B \le \frac{C}{|I^+|^q} \int_{I^-} w(x)^{1-q} \left(\int_{I^+} I_\beta^+(w\chi_{J'})(y) \, dy \right)^q \, dx.$$

Applying Hölder's inequality,

$$B \le \frac{C}{|I^+|^q} \left(\int_{I^-} w(x)^{1-q} \, dx \right) \left(\int_{I^+} w(y) \, dy \right)^{q/q'} \int_{I^+} w(y)^{1-q} I_\beta^+(w\chi_{J'})(y)^q \, dx.$$

From (3.20), it follows that $w^{1-q} \in A_q^+$ then, we have that

$$B \le C \int_{I^+} w(y)^{1-q} I_\beta^+(w\chi_{J'})(y)^q \, dx.$$

Proceeding as in the estimation of A and taking into account that $w \in D^-$ we obtain

$$B \le C w(I^-)|I|^{\beta q}. \tag{3.22}$$

As consequence of (3.19), (3.21) and (3.22) we get (2.1) and the proof of this proposition is complete. □

Proof of Theorem 2.2: We shall prove that f belonging to $\mathcal{L}_w(\beta)$ is a sufficient condition for (2.1) holds. The fact that (2.1) is a necessary condition follows as in the previous proposition. For that, we shall consider different cases.

First of all, we assume that $\beta = 0$ and $f \in \mathcal{L}_w(0)$. If $w \in A_1^-$ we have that (2.1) is an immediate consequence of Proposition 3.6(i). If

$w \in A_p^-$, $1 < p < \infty$, we have that $w \in A_{p-\epsilon}^-$ for some $\epsilon > 0$. Then, by Proposition 3.6(ii), and proceeding as in Theorem 4 of [**6**], we obtain that f satisfies (2.1).

Let $0 < \beta < 1$ and $1 < p < \infty$. Since the weight w belongs to A_p^- there exists $0 < \alpha < \beta$ such that $w^{1+\frac{\alpha}{1-\alpha}p}$ belongs to A_p^-. Proceeding as in (3.19), we have that

$$\begin{aligned}
&\int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q}\, dx \\
&\quad \le C \int_{I^-} w(x)^{1-q} \left[\frac{1}{|I^+|} \int_{I^+} \left(\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}}\, dz \right.\right. \\
&\qquad\qquad \left.\left. + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}}\, dz \right) dy \right]^q dx \\
&\quad \le C|I|^{(\beta-\alpha)q} \int_{I^-} w(x)^{1-q} \left(\int_x^{x+\frac{3|I|}{2}} \frac{w(z)}{(z-x)^{1-\alpha}}\, dz \right)^q dx \\
&\qquad + \frac{C}{|I|^{(\beta-\alpha-1)q}} \int_{I^-} w(x)^{1-q} \left(\int_{I^+} \int_y^{y+\frac{3|I|}{2}} \frac{w(z)}{(z-y)^{1-\alpha}}\, dz\, dy \right)^q dx \\
&\quad = |I|^{(\beta-\alpha)q}(A+B).
\end{aligned}$$

Substituting in the proof of the previous proposition α for β in the estimation of A and B we obtain this case.

Finally, we suppose that $0 < \beta < 1$ and $p = 1$. Since the weight w belongs to A_1^- it follows that w belongs to A_s^- for every $1 < s < \infty$. Then, by the previous case we obtain that (2.1) holds for every $1 \le q < \infty$. □

4. The classes $H^-(\alpha, p)$

The next lemma states necessary conditions for that a weight w belongs to $H^-(\alpha, p)$.

Lemma 4.1. *Let $1 < p \le \infty$. If $w \in H^-(\alpha, p)$ then,*

(i) *$w^{p'}$ belongs to $\in D^-$,*

(ii) *w belongs to $\in RH^-(p')$,*

(iii) *w belongs to $\in D^-$.*

Proof: The proof of (i) and (ii) are similar to ones of Lemma 3.7 and Lemma 3.8, in [**1**], respectively. Applying Hölder's inequality and (ii), we obtain (iii). □

Lemma 4.2. *Let w be a weight. The following conditions are equivalent.*

(a) $w \in H^-(\alpha, p)$.

(b) $w \in RH^-(p')$ *and there exist positive constants C and ϵ such that,*

$$w^{p'}([a, a+\theta t]) \leq C\theta^{(2-\alpha)p'-\epsilon} w^{p'}([a, a+t]),$$

for every $a \in \mathbb{R}$, $t > 0$ and $\theta \geq 1$.

(c) *There exist positive constants C and ϵ such that,*

$$\left(\frac{w^{p'}([a, a+\theta t])}{\theta t}\right)^{1/p'} \leq C\theta^{\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'}} \frac{w([a-t, a])}{t},$$

for every $a \in \mathbb{R}$, $t > 0$ and $\theta \geq 1$.

Proof: (a) ⇒ (b). By Lemma 4.1(ii) we have that $w \in RH^-(p')$.

Let $I = [a, a+t]$. Applying Hölder's inequality and keeping in mind that $w \in H^-(\alpha, p)$,

$$\begin{aligned} \frac{w^{p'}(I)}{|I|} &\geq \left(\frac{w(I)}{|I|}\right)^{p'} \geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \int_{a+t}^{\infty} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}}\, dy \\ &\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{k\geq 0} \frac{1}{(2^{k+1}t)^{(2-\alpha)p'}} \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'}\, dy. \end{aligned} \tag{4.1}$$

Since $\sum_{i\geq k} \left(\frac{1}{2^{(2-\alpha)p'}}\right)^i = C\left(\frac{1}{2^{(2-\alpha)p'}}\right)^k$, by (4.1) and applying Fubini's Theorem,

$$\begin{aligned} \frac{w^{p'}(I)}{|I|} &\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \frac{1}{t^{(2-\alpha)p'}} \sum_{k\geq 0} \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'}\, dy \sum_{i\geq k} \left(\frac{1}{2^{(2-\alpha)p'}}\right)^i \\ &= C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i\geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \sum_{k=0}^{i} \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'}\, dy \\ &= C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i\geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \int_{a+t}^{a+2^{i+1}t} w(y)^{p'}\, dy. \end{aligned}$$

Therefore,

$$\frac{w^{p'}(I)}{|I|} \geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i\geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \int_a^{a+2^{i+1}t} w(y)^{p'}\,dy$$

$$\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i\geq 0} \int_{2^i t}^{2^{i+1}t} \frac{w^{p'}([a,a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s}$$

$$= C|I|^{(\frac{1}{p}-\alpha+1)p'} \int_t^{\infty} \frac{w^{p'}([a,a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s}.$$

In consequence,

$$\int_t^{\infty} \frac{w^{p'}([a,a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s} \leq C \frac{w^{p'}([a,a+t])}{t^{(2-\alpha)p'}}.$$

Now, using Lemma 3.3 in [**1**] with $\varphi(s) = w^{p'}([a,a+s])$ and $r = (2-\alpha)p'$, there exist C and ϵ such that

$$\varphi(\theta t) \leq C\theta^{r-\epsilon}\varphi(t),$$

for every $t > 0$ and $\theta \geq 1$. That is,

$$w^{p'}([a,a+\theta t]) \leq C\theta^{(2-\alpha)p'-\epsilon} w^{p'}([a,a+t]),$$

for every $t > 0$ and $\theta \geq 1$, This completes the proof of (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a). Let $I = [a,a+t]$. If (b) holds, we have that

$$\left(\int_{a+t}^{\infty} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}}\,dy\right)^{1/p'}$$

$$= \left(\sum_{k=0}^{\infty} \int_{a+2^k t}^{a+2^{k+1}t} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}}\,dy\right)^{1/p'}$$

$$\leq \left(\sum_{k=0}^{\infty} \frac{1}{(2^k t)^{(2-\alpha)p'}} w^{p'}([a+t,a+t+2^{k+1}t])\right)^{1/p'} \tag{4.2}$$

$$\leq C\left(\sum_{k=0}^{\infty} \frac{(2^{k+1})^{(2-\alpha)p'-\epsilon}}{(2^k t)^{(2-\alpha)p'}} w^{p'}([a+t,a+2t])\right)^{1/p'}$$

$$\leq C\left(\frac{1}{t}\int_{a+t}^{a+2t} w(y)^{p'}\,dy\right)^{1/p'} t^{\frac{1}{p'}-2+\alpha}.$$

Using the hypothesis $w \in RH^-(p')$ we obtain that (4.2) is bounded by

$$C\frac{1}{t}\int_a^{a+t} w(y)\,dy\,t^{\frac{1}{p'}-2+\alpha} = C\frac{w([a,a+t])}{t^{\frac{1}{p}+2-\alpha}},$$

which shows that $w \in H^-(\alpha,p)$.

The proof of (b) $\Rightarrow$ (c) is very simple and we shall omit it.

(c) $\Rightarrow$ (b). Taking $\theta = 1$ in (c) we have that $w \in RH^-(p')$. Using (c) and Hölder's inequality,

$$\left(\frac{w^{p'}([a-t,a+\theta t])}{\theta t}\right)^{1/p'} = \left(\frac{w^{p'}([a-t,a])}{\theta t}+\frac{w^{p'}([a,a+\theta t])}{\theta t}\right)^{1/p'}$$

$$\leq \left(\frac{w^{p'}([a-t,a])}{\theta t}\right)^{1/p'} + C\theta^{\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'}}\left(\frac{w^{p'}([a-t,a])}{t}\right)^{1/p'}.$$

We can suppose that $\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'} > 0$, then taking into account that $\theta \geq 1$

$$\left(\frac{w^{p'}([a-t,a-t+\theta t])}{\theta t}\right)^{1/p'} \leq \left(\frac{w^{p'}([a-t,a+\theta t])}{\theta t}\right)^{1/p'}$$

$$\leq C\theta^{\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'}}\left(\frac{w^{p'}([a-t,a])}{t}\right)^{1/p'}.$$

From these inequalities with $a = b+t$ we obtain that

$$w^{p'}([b,b+\theta t]) \leq C\theta^{(2-\alpha)p'-\epsilon} w^{p'}([b,b+t]),$$

which completes the proof. □

Remark 4.3. It is easy to see that if $w^{p'}$ belongs to A_1^- then, $w \in H^-(\alpha,p)$. On the other hand, applying Lemma 4.2 (b) $\Rightarrow$ (a), it follows that if $w(x) = |x|^\gamma$ with $0 < \gamma < 1/p - \alpha + 1$, then w belongs to $H^-(\alpha,p)$, but w does not belong to A_1^-. For $0 < \alpha < 1/p$, as an immediate consequence of Lemma 4.2 (c) $\Rightarrow$ (a) it follows that if $w^{p'}$ belongs to $A_{p'+1}^-$ then, w belongs to $H^-(\alpha,p)$.

The next two lemmas show that if w belongs to $H^-(\alpha,p)$, $1 < p < \infty$, then there exists $\eta > 0$ such that w belongs to $H^-(\alpha,q)$ for every $q : p-\eta < q < p+\eta$.

Lemma 4.4. *Let $1 < p < \infty$ and $w \in H^-(\alpha, p)$. Then, there exists $\delta_0 \in (0,1)$ such that $w \in H^-(\alpha, (p'\delta)')$ for any $\delta : \delta_0 < \delta \leq 1$.*

Proof: It is a simple variant of Lemma 3.13 in [**1**]. □

Lemma 4.5. *Let $1 < p < \infty$ and $w \in H^-(\alpha, p)$. Then, there exists $\tau_0 > 1$ such that $w \in H^-(\alpha, (p'\tau)')$ for any $1 \leq \tau \leq \tau_0$.*

Proof: Since $w \in RH^-(p')$ applying Theorem 5.3 in [**9**], there exists $\tau_0 > 1$ such that for every $\tau : 1 \leq \tau \leq \tau_0$ there exists a constant C such that

$$\begin{aligned}\left(\frac{1}{c-b}\int_b^c w(y)^{p'\tau}\,dy\right)^{\frac{1}{p'\tau}} &\leq C\left(\frac{1}{b-a}\int_a^b w(y)\,dy\right)\\ &\leq C\left(\frac{1}{b-a}\int_a^b w(y)^{p'}\,dy\right)^{\frac{1}{p'}}\end{aligned} \tag{4.3}$$

for every $a < b < c$ with $c - b = 2(b-a)$. Let $I = [a, b]$. Using (4.3) we have that,

$$\begin{aligned}&\int_b^\infty \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}}\,dy\\ &= \sum_{k\geq 0}\int_{2^k|I|\leq y-a\leq 2^{k+1}|I|}\frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}}\,dy\\ &\leq \sum_{k\geq 0}\frac{1}{(2^k|I|)^{(2-\alpha)p'\tau}}\int_{2^k|I|\leq y-a\leq 2^{k+1}|I|} w(y)^{p'\tau}\,dy\\ &\leq C\sum_{k\geq 0}\frac{1}{(2^k|I|)^{(2-\alpha)p'\tau-1}}\left(\frac{1}{2^k|I|}\int_{2^{k-1}|I|\leq y-a\leq 2^k|I|} w(y)^{p'}\,dy\right)^{\tau}.\end{aligned} \tag{4.4}$$

Taking into account that $\tau > 1$, (4.4) is bounded by

$$\begin{aligned}C\sum_{k\geq 0}2^k|I|&\left(\frac{1}{2^k|I|}\int_{2^{k-1}|I|\leq y-a\leq 2^k|I|}\frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}}\,dy\right)^{\tau}\\ &\leq C|I|^{1-\tau}\left(\int_{\frac{|I|}{2}\leq y-a}\frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}}\,dy\right)^{\tau}.\end{aligned}$$

Keeping in mind that $w \in H^-(\alpha,p)$ we have,

$$\int_b^\infty \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}}\,dy \le C|I|^{1-\tau}\left(\frac{w([a,a+|I|/2])}{|I|}|I|^{-1/p+\alpha-1}\right)^{p'\tau}$$

$$= C\left(\frac{w(I)}{|I|}\frac{1}{|I|^{\frac{1}{(p'\tau)'}-\alpha+1}}\right)^{p'\tau},$$

which implies that $w \in H^-(\alpha,(p'\tau)')$. □

Lemma 4.6. *Let $1 < p_1 < p_2 < \infty$. Suppose that $w \in H^-(\alpha,p_i)$ for $i = 1,2$. Then $w \in H^-(\alpha,p)$ for every $p : p_1 < p < p_2$.*

Proof: This is an one-sided version of Lemma 3.15 in [**1**]. □

Lemma 4.7. *Let $1 < p < \infty$ and $w \in RH^-(p')$. There exists a constant C such that for every $f \in \widetilde{L^p_w}$ and every bounded interval $I = [a,b]$, if we denote $\widetilde{I}^- = [a - \frac{|I|}{2}, a]$ then,*

$$\int_I |f(x)|\,dx \le C\frac{w(\widetilde{I}^-)}{|I|^{1/p}}[f]_{p,w}.$$

Proof: Since $w \in RH^-(p')$ by Theorem 5.3 in [**9**], there exists $s > p'$ such that $w \in RH^-(s)$, that is, there exists a constant C such that for every bounded interval I,

$$\left(\frac{1}{|I|}\int_I w(x)^s\,dx\right)^{1/s} \le C\frac{w(\widetilde{I}^-)}{|I|}.$$

From this fact, the proof follows as in Lemma 4.1 of [**1**]. □

Lemma 4.8. *Let $1 < p < \infty$ and $w \in H^-(\alpha,p)$. Then there exists a constant C such that for every $f \in \widetilde{L^p_w}$ and every bounded interval $I = [a,b]$,*

$$\int_b^\infty \frac{|f(y)|}{(y-a)^{2-\alpha}}\,dy \le C\frac{w(I)}{|I|^{2+\frac{1}{p}-\alpha}}[f]_{p,w}.$$

Proof: Taking into account Lemma 4.4 and Lemma 4.5, the proof of this lemma is similar to one in Lemma 4.4 of [**1**]. □

Lemma 4.9. *Let $\alpha > 0$ and $\delta \geq 0$ such that $0 < \alpha+\delta < 1$. Let $w \in D^-$. For $a < b$, we denote $c = \frac{a+b}{2}$ and $I = [c,b]$. Then, for every $f \in \mathcal{L}_w(\delta)$, there exists a constant C such that,*

$$\text{(i)} \qquad \int_b^\infty \frac{|f(y)-f_I|}{(y-a)^{2-\alpha}}\,dy \leq C\|f\|_{\mathcal{L}_w(\delta)} \int_c^\infty \frac{w(y)}{(y-a)^{2-\alpha-\delta}}\,dy.$$

$$\text{(ii)} \qquad \int_a^b \frac{|f(y)-f_I|}{(y-a)^{1-\alpha}}\,dy \leq C\|f\|_{\mathcal{L}_w(\delta)} \int_a^c \frac{w(y)}{(y-a)^{1-\alpha-\delta}}\,dy.$$

Proof: The proof of (i) and (ii) are similar, then we only prove (i).

For every $j \geq 0$, let $I_j = [a+2^j|I|, a+2^{j+1}|I|]$. We observe that $I_0 = [a+|I|, a+2|I|] = [c,b] = I$. Since $f \in \mathcal{L}_w(\delta)$ we have that,

$$\begin{aligned}
\int_b^\infty \frac{|f(y)-f_I|}{(y-a)^{2-\alpha}}\,dy &= \sum_{j=1}^\infty \int_{a+2^j|I|}^{a+2^{j+1}|I|} \frac{|f(y)-f_I|}{(y-a)^{2-\alpha}}\,dy \\
&\leq \sum_{j=1}^\infty \frac{1}{(2^j|I|)^{2-\alpha}} \int_{a+2^j|I|}^{a+2^{j+1}|I|} |f(y)-f_{I_0}|\,dy \\
&\leq \sum_{j=1}^\infty \frac{1}{(2^j|I|)^{2-\alpha}} \Bigg[\int_{a+2^j|I|}^{a+2^{j+1}|I|} |f(y)-f_{I_j}|\,dy \\
&\qquad + 2^j|I| \sum_{k=1}^{j} |f_{I_k}-f_{I_{k-1}}|\Bigg] \qquad (4.5)\\
&\leq \sum_{j=1}^\infty \frac{1}{(2^j|I|)^{1-\alpha}} \Bigg[C\|f\|_{\mathcal{L}_w(\delta)} w(I_j)(2^j|I|)^{\delta-1} \\
&\qquad + \sum_{k=1}^{j} \frac{1}{|I_{k-1}|}\int_{I_{k-1}} |f(y)-f_{I_k}|\,dy\Bigg].
\end{aligned}$$

Using that $f \in \mathcal{L}_w(\delta)$ and $w \in D^-$ we obtain the estimate,

$$\frac{1}{|I_{k-1}|}\int_{I_{k-1}} |f(y)-f_{I_k}|\,dy \leq C\|f\|_{\mathcal{L}_w(\delta)} w(I_{k-1})(2^{k-1}|I|)^{\delta-1}.$$

Then applying Fubini's Theorem, (4.5) is bounded by

$$
\begin{aligned}
&C\|f\|_{\mathcal{L}_w(\delta)} \sum_{j=1}^{\infty} \frac{1}{(2^j|I|)^{1-\alpha}} \sum_{k=0}^{j} w(I_k)(2^k|I|)^{\delta-1} \\
&= C\|f\|_{\mathcal{L}_w(\delta)} \sum_{k=0}^{\infty} w(I_k)(2^k|I|)^{\delta-1} \sum_{j=k}^{\infty} \frac{1}{(2^j|I|)^{1-\alpha}} \\
&= C\|f\|_{\mathcal{L}_w(\delta)} \sum_{k=0}^{\infty} \frac{1}{(2^k|I|)^{2-\alpha-\delta}} \int_{a+2^k|I|}^{a+2^{k+1}|I|} w(y)\,dy \\
&\leq C\|f\|_{\mathcal{L}_w(\delta)} \int_c^{\infty} \frac{w(y)}{(y-a)^{2-\alpha-\delta}}\,dy,
\end{aligned}
$$

as we wanted to prove. □

5. Proof of Theorems 2.3 and 2.4

Proof of Theorem 2.3: (i) ⇒ (ii). Let $w \in H^-(\alpha,p)$ and $x_0 \in \mathbb{R}$. Given $f \in \widetilde{L^p_w}$ let $\widetilde{I^+_\alpha}(f)$ define as in (2.2). Choose a bounded interval $I = [a, a+h]$. We consider $I_0 = [a+2h, x_0]$ if $a+2h \leq x_0$ and $I_0 = \emptyset$ if $x_0 < a+2h$, and we also define $I_1 = [x_0, a+2h]$ if $x_0 < a+2h$ and $I_1 = \emptyset$ in the other case. We set

$$
a_I = \int_{I_0} \frac{f(y)}{(y-a)^{1-\alpha}}\,dy + \int_{x_0}^{\infty} \left[\frac{1-\chi_{I_1}(y)}{(y-a)^{1-\alpha}} - \frac{1-\chi_{[x_0,x_0+1]}(y)}{(y-x_0)^{1-\alpha}}\right] f(y)\,dy.
$$

We shall show that a_I is a finite constant.

Suppose that $x_0 < a+2h$. Let n be a positive integer such that $a+2^n h > x_0+1$ and $|a-x_0| \leq 2^{n-1}h$. Then,

$$
\begin{aligned}
a_I &= \left(\int_{x_0}^{a+2^n h} + \int_{a+2^n h}^{\infty}\right)\left[\frac{1-\chi_{[x_0,a+2h]}(y)}{(y-a)^{1-\alpha}} - \frac{1-\chi_{[x_0,x_0+1]}(y)}{(y-x_0)^{1-\alpha}}\right] f(y)\,dy \\
&= J_1 + J_2.
\end{aligned}
$$

For each $y \geq a+2^n h$, by Mean Value Theorem, there exists $\theta : 0 < \theta < 1$ such that,

$$
\left|\frac{1}{(y-a)^{1-\alpha}} - \frac{1}{(y-x_0)^{1-\alpha}}\right| \leq C\frac{|x_0-a|}{|y-\theta a-(1-\theta)x_0|^{2-\alpha}} \leq C\frac{|x_0-a|}{|y-a|^{2-\alpha}}.
$$

Then, applying Lemma 4.8, we have that

$$|J_2|\le C|x_0-a|\int_{a+2^nh}^{\infty}\frac{|f(y)|}{|y-a|^{2-\alpha}}\,dy\le C|x_0-a|\frac{w([a,a+2^nh])}{(2^nh)^{2+\frac{1}{p}-\alpha}}[f]_{p,w}<\infty.$$

On the other hand, since $f\in\widetilde{L^p_w}$ and using Lemma 4.7, we get

$$\begin{aligned}|J_1| &\le \int_{a+2h}^{a+2^nh}\frac{|f(y)|}{(y-a)^{1-\alpha}}\,dy+\int_{x_0+1}^{a+2^nh}\frac{|f(y)|}{(y-x_0)^{1-\alpha}}\,dy\\ &\le \frac{1}{(2h)^{1-\alpha}}\int_{a+2h}^{a+2^nh}|f(y)|\,dy+\int_{x_0+1}^{a+2^nh}|f(y)|\,dy<\infty.\end{aligned}$$

The case $x_0\ge a+2h$ can be proved in a similar way.

Now, let

$$\begin{aligned}A(x) &= \int_x^{a+2h}\frac{f(y)}{(y-x)^{1-\alpha}}\,dy\\ &\quad+\int_{a+2h}^{\infty}\left[\frac{1}{(y-x)^{1-\alpha}}-\frac{1}{(y-a)^{1-\alpha}}\right]f(y)\,dy \qquad (5.1)\\ &= A_1(x)+A_2(x).\end{aligned}$$

It follows that,

$$\widetilde{I^+_\alpha}(f)(x)=A(x)+a_I. \qquad (5.2)$$

We shall show that,

$$\int_I|\widetilde{I^+_\alpha}(f)(x)-a_I|\,dx\le C|I|^{\alpha-1/p}w(I^-)[f]_{p,w}.$$

We observe that taking into account (5.2) and (5.1) it is sufficient to prove that

$$\int_I|A_j(x)|\,dx\le C|I|^{\alpha-1/p}w(I^-)[f]_{p,w},$$

for $j=1,2$. Applying Mean Value Theorem, Lemma 4.8 and Lemma 4.1(iii) for every $x\in I=[a,a+h]$ we have that,

$$\begin{aligned}|A_2(x)| &\le \int_{a+2h}^{\infty}\left|\frac{1}{(y-x)^{1-\alpha}}-\frac{1}{(y-a)^{1-\alpha}}\right||f(y)|\,dy\\ &\le Ch\int_{a+2h}^{\infty}\frac{|f(y)|}{|y-a|^{2-\alpha}}\,dy\le Ch\frac{w([a,a+2h])}{(2h)^{2+\frac{1}{p}-\alpha}}[f]_{p,w}\\ &\le C\frac{w([a-h,a])}{h^{1+\frac{1}{p}-\alpha}}[f]_{p,w}.\end{aligned}$$

Therefore,

$$\int_I |A_2(x)|\,dx \le C|I|^{\alpha-1/p} w(I^-)[f]_{p,w}.$$

With respect to $A_1(x)$, changing the order of integration and applying Lemma 4.7,

$$\begin{aligned}\int_a^{a+h} |A_1(x)|\,dx &\le \int_a^{a+h}\int_x^{a+2h} \frac{|f(y)|}{(y-x)^{1-\alpha}}\,dy\,dx \\ &\le \int_a^{a+2h} |f(y)| \int_a^y \frac{dx}{(y-x)^{1-\alpha}}\,dy \\ &\le Ch^\alpha \int_a^{a+2h} |f(y)|\,dy \\ &\le Ch^{\alpha-1/p} w([a-h,a])[f]_{p,w},\end{aligned}$$

which completes the proof of (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let $a \in \mathbb{R}$ and $h > 0$. We consider $f \ge 0$ such that $\operatorname{sop}(f) \subseteq [a+4h, \infty)$. For each $x \in [a, a+h]$ we have that,

$$\begin{aligned}&|I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a+2h,a+3h]}| \\ &\qquad = \frac{1}{h}\int_{a+2h}^{a+3h}\int_{a+4h}^{\infty} f(y)\left[\frac{1}{(y-t)^{1-\alpha}} - \frac{1}{(y-x)^{1-\alpha}}\right] dy\,dt.\end{aligned}$$

Applying Mean Value Theorem, for each $y \ge a+4h$ we obtain,

$$\frac{1}{(y-t)^{1-\alpha}} - \frac{1}{(y-x)^{1-\alpha}} \ge C\frac{|x-t|}{(y-a)^{2-\alpha}} \ge C\frac{h}{(y-a)^{2-\alpha}}.$$

In consequence,

$$|I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a+2h,a+3h]}| \ge Ch\int_{a+4h}^{\infty} \frac{f(y)}{(y-a)^{2-\alpha}}\,dy.$$

Then, if $f \in L^p_w$, using (iii) we have that,

$$\begin{aligned}Ch^2\int_{a+4h}^{\infty} \frac{f(y)}{(y-a)^{2-\alpha}}\,dy &\le 2\int_a^{a+3h} |I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a,a+3h]}|\,dx \\ &\le C(3h)^\beta w([a-3h,a])\left[\int\left(\frac{f(y)}{w(y)}\right)^p dy\right]^{1/p}.\end{aligned}$$

Now, taking into account that $\beta = \alpha - 1/p$ it follows that,

$$h^{1/p-\alpha+1} \int_{a+4h}^{\infty} \frac{f(y)}{(y-a+3h)^{2-\alpha}}\,dy \leq C\frac{w([a-3h,a+4h])}{h}\left[\int_{a+4h}^{\infty}\left(\frac{f(y)}{w(y)}\right)^p dy\right]^{1/p}. \tag{5.3}$$

For each $m > 2$ we put,

$$f_m(y) = \frac{w(y)^{p'}}{(y-a+3h)^{\frac{2-\alpha}{p-1}}}\chi_{[a+4h,a+2^m h]}(y)\chi_{\{0\leq w\leq m\}}(y).$$

It is easy to check that $f_m \in L^p_w$. Using (5.3) with f_m and taking the limit, we obtain that

$$h^{1/p-\alpha+1}\left(\int_{a+4h}^{\infty}\frac{w(y)^{p'}}{(y-a+3h)^{(2-\alpha)p'}}\,dy\right)^{1/p'} \leq C\frac{w([a-3h,a+4h])}{h},$$

which shows that $w \in H^-(\alpha, p)$. □

Remark 5.1. By Theorem 2.1, if $0 \leq \beta < 1$, we can substitute in Theorem 2.3, $\mathcal{L}_w(\beta)$ for $\mathcal{L}_w^-(\beta)$. That is not possible for $-1 < \beta < 0$. In fact, if w and f are defined as in Remark 3.2(ii), then

$$I_\alpha^+(f)(x) = \begin{cases} \frac{\Gamma(\alpha)}{a^\alpha}e^{-ax}, & x \geq 0 \\ \frac{|x|^\alpha}{\alpha} + \frac{e^{-ax}}{a^\alpha}\int_{a|x|}^{\infty} e^{-u}u^{\alpha-1}\,du, & x < 0. \end{cases}$$

Therefore, the same arguments used in Remark 3.2 imply that $I_\alpha^+(f)$ does not belong to $\mathcal{L}_w(\beta)$.

Proof of Theorem 2.4: (i) ⇒ (ii). Let N be a positive integer. For any integer a applying Fubini's Theorem and taking into account that w is a locally integrable function, we have that

$$\int_a^{a+1}\int_x^{x+N}\frac{w(y)}{(y-x)^{1-\alpha}}\,dy\,dx < \infty.$$

In consequence, for almost every x and every positive integer N

$$\int_x^{x+N}\frac{w(y)}{(y-x)^{1-\alpha}}\,dy < \infty. \tag{5.4}$$

Let x_0 satisfying (5.4). We consider

$$\widetilde{I_\alpha^+}(f)(x) = \int_{-\infty}^{\infty}\left[\frac{\chi_{[x_0,\infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x,\infty)}(y)}{|y-x|^{1-\alpha}}\right]f(y)\,dy. \tag{5.5}$$

We shall show that if $f \in \mathcal{L}_w(0)$ then $\widetilde{I_\alpha^+}(f)$, defined as in (5.5), is finite for every x satisfying (5.4). Fix x satisfying (5.4). Suppose that $x_0 < x$ and let $R \in \mathbb{Q} : x_0 < x \le x_0 + R/4$. We consider the interval $I = [x_0 + R/2, x_0 + R]$. Taking into account that the function $g(y) = \frac{\chi_{[x_0,\infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x,\infty)}(y)}{|y-x|^{1-\alpha}}$ is integrable and $\int_{-\infty}^{\infty} g(y)\,dy = 0$ we can write,

$$\begin{aligned}\widetilde{I_\alpha^+}(f)(x) &= \int_{-\infty}^{\infty} \left[\frac{\chi_{[x_0,\infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x,\infty)}(y)}{|y-x|^{1-\alpha}}\right][f(y) - f_I]\,dy \\ &= I_1(x) + I_2(x),\end{aligned}$$

where,

$$I_1(x) = \int_{x_0}^{x_0+R} \quad \text{and} \quad I_2(x) = \int_{x_0+R}^{\infty}.$$

We shall prove that

$$|\widetilde{I_\alpha^+}(f)(x)| \le C\|f\|_{\mathcal{L}_w(0)}\left[\int_{x_0}^{x_0+5R/4} \frac{w(y)}{(y-x_0)^{1-\alpha}}\,dy + \int_{x}^{x+5R/4} \frac{w(y)}{(y-x)^{1-\alpha}}\,dy\right]. \tag{5.6}$$

We observe that,

$$|I_1(x)| \le \int_{x_0}^{x_0+R} \frac{|f(y)-f_I|}{|y-x_0|^{1-\alpha}}\,dy + \int_{x}^{x_0+R} \frac{|f(y)-f_I|}{|y-x|^{1-\alpha}}\,dy.$$

Let $J = [x + R/2, x + R]$. Applying Lemma 4.9(ii) we have that

$$\begin{aligned}|I_1(x)| &\le \int_{x_0}^{x_0+R} \frac{|f(y)-f_I|}{|y-x_0|^{1-\alpha}}\,dy \\ &\quad + \int_{x}^{x+R} \frac{|f(y)-f_J|}{|y-x|^{1-\alpha}} + |f_I - f_J|\int_{x}^{x+R} \frac{dy}{|y-x|^{1-\alpha}} \\ &\le C\|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+R/2} \frac{w(y)}{(y-x_0)^{1-\alpha}}\,dy \qquad (5.7) \\ &\quad + C\|f\|_{\mathcal{L}_w(0)} \int_{x}^{x+R/2} \frac{w(y)}{(y-x)^{1-\alpha}}\,dy \\ &\quad + \frac{R^\alpha}{\alpha}|f_I - f_J|.\end{aligned}$$

Since $x_0 < x < x_0 + R/4$ and $f \in \mathcal{L}_w(0)$ we have,

$$R^\alpha |f_I - f_J| \leq C\|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+5/4R} \frac{w(y)}{(y-x_0)^{1-\alpha}}\, dy.$$

Then, by (5.7)

$$|I_1(x)| \leq C\|f\|_{\mathcal{L}_w(0)} \left[\int_{x_0}^{x_0+5R/4} \frac{w(y)}{(y-x_0)^{1-\alpha}}\, dy + \int_{x}^{x+5R/4} \frac{w(y)}{(y-x)^{1-\alpha}}\, dy \right].$$

Now, let us estimate I_2. Applying Mean Value Theorem,

$$\begin{aligned} |I_2(x)| &\leq \int_{x_0+R}^{\infty} \left| \frac{1}{|y-x_0|^{1-\alpha}} - \frac{1}{|y-x|^{1-\alpha}} \right| |f(y) - f_I|\, dy \\ &\leq C|x_0 - x| \int_{x_0+R}^{\infty} \frac{|f(y) - f_{[x_0+R/2, x_0+R]}|}{(y-x_0)^{2-\alpha}}\, dy. \end{aligned}$$

Using Lemma 4.9(i) and taking into account that $w \in H^-(\alpha, \infty)$ we get,

$$\begin{aligned} |I_2(x)| &\leq CR\|f\|_{\mathcal{L}_w(0)} \int_{x_0+R/2}^{\infty} \frac{w(y)}{(y-x_0)^{2-\alpha}}\, dy \\ &\leq CR\|f\|_{\mathcal{L}_w(0)} \frac{w([x_0, x_0+R/2])}{R^{2-\alpha}} \\ &\leq C\|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+R/2} \frac{w(y)}{(y-x_0)^{1-\alpha}}\, dy. \end{aligned}$$

Then, if $x_0 < x < x_0 + R/4$ or in the case $x_0 - R/4 < x < x_0$, we have that (5.6) holds. Since $\mathbb{R} = \cup_{R \in \mathbb{Q}>0} [x_0 - R/4, x_0 + R/4]$, it follows that $\widetilde{I_\alpha^+}(f)(x)$ is finite for almost every x.

Let us show that $\widetilde{I_\alpha^+}(f) \in \mathcal{L}_w(\alpha)$. For almost every $x_1 < x_2$, if we define $R = 4|x_1 - x_2|$, we have that $x_1 < x_2 \leq x_1 + R/4$ and using (5.6) we get

$$
\begin{aligned}
&|\widetilde{I_\alpha^+}(f)(x_1) - \widetilde{I_\alpha^+}(f)(x_2)| \\
&\qquad \le \int_{-\infty}^{\infty} \left| \frac{\chi_{[x_1,\infty)}(y)}{(y-x_1)^{1-\alpha}} - \frac{\chi_{[x_2,\infty)}(y)}{(y-x_2)^{1-\alpha}} \right| |f(y) - f_{[x_1+R/2,x_1+R]}|\, dy \\
&\qquad \le C\|f\|_{\mathcal{L}_w(0)} \left[\int_{x_1}^{x_1+5|x_1-x_2|} \frac{w(y)}{(y-x_1)^{1-\alpha}}\, dy \right. \\
&\qquad\qquad\qquad \left. + \int_{x_2}^{x_2+5|x_1-x_2|} \frac{w(y)}{(y-x_2)^{1-\alpha}}\, dy \right].
\end{aligned}
$$

Taking into account that $w \in D^-$ and using Proposition 3.3 it follows that $\widetilde{I_\alpha^+}(f) \in \mathcal{L}_w(\alpha)$.

(ii) $\Rightarrow$ (i). This implication is similar to (iii) $\Rightarrow$ (i) in Theorem 2.3. □

Corollary 5.2. *Let $\alpha, \delta \in \mathbb{R}^+$ such that $0 < \alpha + \delta < 1$. The following statements are equivalent.*

(a) *$w \in H^-(\delta, \infty)$ and the operator I_α can be extended to a linear bounded operator $\widetilde{I_\alpha^+} : \mathcal{L}_w(\delta) \to \mathcal{L}_w(\alpha+\delta)$.*

(b) *$w \in H^-(\alpha+\delta, \infty)$.*

Proof: The proof is a simple variant of Corollary 2.12 in [**1**]. □

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