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BOUNDING THE ORDERS OF FINITE SUBGROUPS

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Abstract

We give homological conditions on groups such that whenever the conditions hold for a group G, there is a bound on the orders of finite subgroups of G. This extends a result of P. H. Kropholler. We also suggest a weaker condition under which the same conclusion might hold.

1. Introduction

Let R be a non-trivial unital ring. An R-module M is said to be of type FP_n if there is a projective resolution

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0$$

of M over R in which $P_0, \ldots, P_n$ are finitely generated. M is said to be of type FP_∞ if M is FP_n for each n. Similarly, M is said to be of type FP (resp. FL) over R if there is a resolution of M of finite length in which each term is a finitely generated projective (resp. free) module. For any discrete group G and commutative ring R, the augmentation homomorphism $RG \to R$ gives R the structure of a module for the group algebra RG. The group G is said to be FP_n (resp. FP_∞, FP, FL) over R if the RG-module R is FP_n (resp. FP_∞, FP, FL) in the above sense. The cohomological dimension of G over R, denoted by $\mathrm{cd}_R(G)$, is the projective dimension of R as an RG-module. For further information concerning these definitions, see [**2**] or Chapter VIII of [**3**]. As usual, let $\mathbb{Q}$ and $\mathbb{Z}$ denote the rational numbers and the integers respectively. We prove the following.

Proposition 1. *Let G be a group with $\mathrm{cd}_{\mathbb{Q}}(G) = n < \infty$ and suppose that G is of type FP_n over $\mathbb{Z}$. Then there is a bound on the orders of finite subgroups of G.*

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A similar result was proved by P. H. Kropholler in Section 5 of [**6**], under the extra hypothesis that G should be FP_∞ over $\mathbb{Z}$. His proof made use of the complete cohomology introduced by D. Benson, J. Carlson, G. Mislin and F. Vogel [**1**], [**9**] as will ours. (Complete cohomology can be viewed as a generalization of Tate cohomology.)

The conclusion does not hold for all groups of type FP_{n-1} over $\mathbb{Z}$. K. S. Brown has shown [**4**] that for each $n > 0$, the Houghton groups [**5**] afford an example of a group $G = G(n)$ such that:

(a) G contains the infinite, finitary symmetric group;
(b) $\mathrm{cd}_{\mathbb{Q}}(G) = n$;
(c) G is FP_{n-1} over $\mathbb{Z}$.

The authors have recently constructed groups G of type FP_∞ over $\mathbb{Z}$ with $\mathrm{cd}_{\mathbb{Q}}\, G$ finite that contain infinitely many conjugacy classes of finite subgroups [**7**], and it was these examples that led to the authors' interest in Proposition 1. It is not known whether there is a bound on the orders of finite subgroups for every G of type FP over $\mathbb{Q}$. Some remarks concerning this question will be made at the end of the paper.

2. Proofs

Before starting, we recall a basic property of FP_n-modules. Suppose that M is an R-module of type FP_n, and that

$$P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is a partial projective resolution of M in which each P_i is finitely generated. Then K_{n-1}, defined as the kernel of the map from P_{n-1} to P_{n-2}, is finitely generated.

We shall also give a brief outline of Benson and Carlson's version of generalized Tate cohomology for arbitrary rings R [**1**]. For R-modules M and N let $P\operatorname{Hom}_R(M,N)$ be the group of all R-module homomorphisms which factor through a projective, and let

$$[M,N] = \operatorname{Hom}_R(M,N)/P\operatorname{Hom}_R(M,N).$$

For arbitrary R-modules M let FM be the free module on the set M and ΩM is the kernel of the canonical projection $FM \twoheadrightarrow M$. Let $\Omega^i M = \Omega(\Omega^{i-1}M)$. Then there is a well defined sequence of maps

$$[M,N] \to [\Omega M, \Omega N] \to [\Omega^2 M, \Omega^2 N] \to \cdots$$

and it is now possible to define the Tate cohomology group in degree zero as a direct limit as follows:

Definition.

$$\widehat{\operatorname{Ext}}^0_R(M,N) = \varinjlim [\Omega^i M, \Omega^i N].$$

From now on we shall concentrate on projective resolutions $P_* \twoheadrightarrow \mathbb{Z}$ of the trivial module $\mathbb{Z}$ over the group-ring $\mathbb{Z}G$. Let K_i be the kernel of the map $P_i \to P_{i-1}$ for $i \geq 1$ and $K_0 = \ker(P_0 \twoheadrightarrow \mathbb{Z})$.

Lemma 2. *For every $i \geq 0$ the following groups are isomorphic:*

$$[K_i, K_i] \cong [\Omega^{i+1}\mathbb{Z}, \Omega^{i+1}\mathbb{Z}].$$

Proof: This follows from Shanuel's Lemma and an application of the fact that for arbitrary M, N and projective modules P and Q,

$$[M \oplus P, N] \cong [M,N] \cong [M, N \oplus Q]. \qquad \square$$

Proof of Proposition 1: Consider a partial projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ where all P_i, $i \leq n-1$, are finitely generated:

$$P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0,$$

and let K be the kernel of the map $P_{n-1} \to P_{n-2}$. As G is of type FP_n the kernel K is finitely generated. Since tensoring with $\mathbb{Q}$ is exact we obtain a projective resolution of $\mathbb{Q}$ over $\mathbb{Q}G$, which is of type FP:

$$0 \to K \otimes \mathbb{Q} \to P_{n-1} \otimes \mathbb{Q} \to \cdots \to P_0 \otimes \mathbb{Q} \to \mathbb{Q} \to 0.$$

Therefore $K \otimes \mathbb{Q}$ is a direct summand of a finite rank $\mathbb{Q}G$-free module F, freely generated by $\{f_1, \ldots, f_r\}$, say. Let F_0 be the free $\mathbb{Z}G$-module on these generators.

Claim. *There is an integer m, such that multiplication with m from K to K factors through F_0.*

Let $\pi\colon F \twoheadrightarrow K \otimes \mathbb{Q}$ be the projection onto $K \otimes \mathbb{Q}$ and $\tau\colon K \otimes \mathbb{Q} \hookrightarrow F$ be a splitting, i.e., a map such that $\pi\tau = \mathrm{id}_{K \otimes \mathbb{Q}}$. Denote by $\iota\colon K \hookrightarrow K \otimes \mathbb{Q}$ the inclusion defined by $\iota(k) = k \otimes 1$. Suppose $k_1, \ldots, k_s$ generate K. For each $1 \leq j \leq s$ there exist $\lambda_{ij} \in \mathbb{Q}G$ such that $\iota\tau(k_j) = \sum_{i=1}^r \lambda_{ij} f_i$. Now pick $m \in \mathbb{Z}$ such that each $m\lambda_{ij} \in \mathbb{Z}G$. Since τ is a split injection we can precompose the identity $\mathrm{id}_{K \otimes \mathbb{Q}} = \pi\tau$ with multiplication by m. Hence the map

$$K \xrightarrow{\iota} K \otimes \mathbb{Q} \xrightarrow{\times m} K \otimes \mathbb{Q}$$

factors through F_0 and has image in K thus proving the claim.

The claim together with Lemma 2 gives that

$$m[K,K] \cong m[\Omega^n\mathbb{Z}, \Omega^n\mathbb{Z}] = 0.$$

Complete cohomology agrees with ordinary Tate cohomology for finite groups and we can therefore take an arbitrary finite subgroup H of G and get that

$$m\widehat{H}^0(H,\mathbb{Z}) \cong m[K,K] \cong \varinjlim m[\Omega^i\mathbb{Z},\Omega^i\mathbb{Z}] = 0.$$

The direct limit vanishes since for every $\varphi \in \mathrm{Hom}(\Omega^i\mathbb{Z},\Omega^i\mathbb{Z})$, which factors through a projective, the induced maps $\Omega^j\varphi\colon \Omega^{i+j}\mathbb{Z} \to \Omega^{i+j}\mathbb{Z}$ also factor through projectives. (Note that $\Omega^i\mathbb{Z}$ here denotes the ith kernel in the Benson-Carlson construction for $\mathbb{Z}H$ and not $\mathbb{Z}G$ as earlier used. This does not change the outcome, though.) But also $\widehat{H}^0(H,\mathbb{Z}) \cong \mathbb{Z}/|H|\mathbb{Z}$ and therefore the group order is a divisor of m, thus bounded. □

3. FP-groups over $\mathbb{Q}$

Let us consider again the partial resolution of the R-module M of type FP_n, which was mentioned at the beginning of the previous section:

$$P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

There is such a partial resolution in which each P_i is finitely generated and free. If also M has projective dimension n, then M is FP. If M has projective dimension n and the P_i are finitely generated free modules, then M is FL if and only if K is stably free. These results can be found in [**3**, Sections VIII.4–VIII.6]. The following lemma is well-known, but we could not find a reference, so we briefly sketch a proof. A similar topological result appears in [**8**, Corollary 5.5].

Lemma 3. *Let C denote an infinite cyclic group. For any R, if G is a group of type FP over R, then $G \times C$ is of type FL over R.*

Proof: There is a free resolution Q_* of R over RC of length one, with $Q_1 \cong Q_0 \cong RC$. Now suppose that

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

is a projective resolution of R over RG in which each P_i is finitely generated, and P_i is free for $i < n$. Let P' be such that $P_n \oplus P'$ is a finitely-generated free RG-module. Writing $\otimes$ for tensor products over R, the total complex T_* for the double complex $P_* \otimes Q_*$ is a projective resolution of $R \otimes R = R$ over $RG \otimes RC \cong R(G \times C)$, of length $n+1$. Each T_i is finitely generated and T_i is free for $i < n$. Let S_* be the exact chain complex consisting of one copy of $P' \otimes RC$ in degree $n+1$ and one copy in degree n, with the identity map as the boundary. Then $S_* \oplus T_*$ is a finite free resolution of R over $R(G \times C)$. □

Lemma 4. *Let $F_n \to \cdots \to F_0$ be a finite-length chain complex of free $\mathbb{Z}G$-modules, suppose that $H_0(F_*)$ is isomorphic to the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, and that for each $j > 0$, there exists an integer $m_j > 0$ such that multiplication by m_j annihilates $H_j(F_*)$. Then any finite subgroup of G has order dividing $\prod_{j=1}^{n} m_j$.*

Sketch-proof: The above bound is obtained by comparing the two spectral sequences arising from the double complex

$$E_0^{i,j} = \mathrm{Hom}_H(P_i, F_j),$$

where H is a finite subgroup of G and P_* is a complete resolution for H. □

These lemmas can be used to prove a slightly weaker version of Proposition 1 using only ordinary Tate cohomology for finite groups. Suppose that G is FP_n over $\mathbb{Z}$, FP over $\mathbb{Q}$, and $\mathrm{cd}_{\mathbb{Q}}(G) = n - 1$. By Lemma 3, $G' = G \times C$ is FP_n over $\mathbb{Z}$, FL over $\mathbb{Q}$, and $\mathrm{cd}_{\mathbb{Q}}(G') = n$. A sequence of free $\mathbb{Z}G'$-modules satisfying the conditions of Lemma 4 can then be constructed.

Let us now consider the problem of bounding the orders of finite subgroups of an arbitrary group of type FP over $\mathbb{Q}$. Such a G is finitely generated, and by Lemma 3, we may assume without loss of generality that G is FL over $\mathbb{Q}$. Let P_0 be a free $\mathbb{Q}G$-module of rank one with generator v, and let P_1 be $\mathbb{Q}G$-free on a set $e_1, \ldots, e_m$ bijective with a set $g_1, \ldots, g_m$ of generators for G. Define a map from P_0 to $\mathbb{Q}$ by $v \mapsto 1$ and a map from P_1 to P_0 by $e_i \mapsto (1 - g_i)v$. Finally, let

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Q} \to 0$$

be a finite free resolution of $\mathbb{Q}$ over $\mathbb{Q}G$ extending this partial resolution.

Now let F_0 (resp. F_1) be the $\mathbb{Z}G$-submodule of P_0 (resp. P_1) generated by v (resp. $e_1, \ldots, e_m$). For $i \geq 2$, if F_{i-1} has already been chosen, let F_i be a $\mathbb{Z}G$-lattice in P_i (i.e., a $\mathbb{Z}G$-free $\mathbb{Z}G$-submodule such that $\mathbb{Q} \otimes F_i = P_i$), such that the image of F_i in P_{i-1} is contained in F_{i-1}. This defines a finite chain complex F_* of finitely-generated free $\mathbb{Z}G$-modules such that $H_0(F_*) \cong \mathbb{Z}$ and $H_i(F_*)$ is torsion for $i > 0$. If one could bound the exponent of the torsion in $H_i(F_*)$, Lemma 4 could be applied to bound the orders of finite subgroups of G. Note that in general $H_i(F_*)$ will not be finitely generated as $\mathbb{Z}G$-module. For example, if G is not FP_2 over $\mathbb{Z}$, then $H_1(F_*)$ will not be finitely generated.

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