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ON CANONICAL HOMOTOPY OPERATORS FOR $\bar\partial$ IN FOCK TYPE SPACES IN $\mathbb{C}^n$

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Abstract

We show that a certain solution operator for $\bar\partial$ in a space of forms square integrable against $e^{-|z|^2}$ is canonical, i.e., that it gives the minimal solution when applied to a $\bar\partial$-closed form, and gives zero when applied to a form orthogonal to $\operatorname{Ker}\bar\partial$.

As an application, we construct a canonical homotopy operator for $i\partial\bar\partial$.

0. Introduction

One way to solve $\bar\partial$ equations is to use integral formulas, and well-known methods have been developed to construct explicit solution operators. Properties of the solutions can be deduced, studying the integral kernels of these operators. Even though the well-known methods in a way seem natural, in some cases the operators obtained are in a certain sense incompatible with the geometry. To make this statement precise, first recall that a solution operator K is called canonical if $u = Kf$ is the minimal solution to $\bar\partial u = f$ when $f \in \operatorname{Ker}\bar\partial$ and $Kf = 0$ when f is orthogonal to $\operatorname{Ker}\bar\partial$. It turns out, that certain solution operators are not canonical with respect to the Euclidean metric; the solution operator in strictly pseudoconvex domains, obtained by Henkin, Skoda and others, see [**H**], [**S**], is *not* canonical with respect to the Euclidean metric. This statement has to be interpreted with some care. First we note, that the operator canonical with respect to the Euclidean metric, the Kohn operator K^{K} is known (see, e.g., [**Ha-P**]). The Henkin-Skoda operator yields *boundary values* of solutions, and that it not is canonical means that the boundary values that it produces do not coincide with the boundary values of the solutions produced by K^{K}.

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In [**ABO**], we studied canonical solution operators in strictly pseudoconvex domains. One major result was that, in the special case of the ball, the Henkin-Skoda operator is canonical with respect to the metric $\Omega = i\,(-\rho)\,\partial\bar{\partial}\log(1/-\rho)$, where $-\rho$ is the distance to the boundary. This means that the values given by the Henkin-Skoda operator coincide with the boundary values given by the canonical operator.

There were additional advantages of using the non-Kähler metric Ω instead of the Euclidean metric, for example the domain of the formal adjoint operator $\bar{\partial}^*$ contains all forms that are smooth up to the boundary, contrary to the Euclidean case. This suggests that the Ω metric in some sense is more natural than the Euclidean metric. (The metric Ω is also related to the natural metric on the boundary.)

In a general strictly pseudoconvex domain, the Henkin-Skoda operator is only approximately canonical with respect to Ω in a certain sense, see [**A-Boo**].

In this paper, we study a space of forms in all of $\mathbb{C}^n$ with growth of infinite order. Using a technique described in [**A-Be**], a solution operator is obtained. The main result of this paper is that the solution operator is canonical with respect to the Euclidean metric. As an application, following the lines in [**ABO**], we construct a canonical homotopy operator for $i\partial\bar{\partial}$.

The paper is organized like this: In Section 1 we construct the solution operator, and in Section 2 the operator is expressed in terms of the metric, and we can see that the operator is canonical. Finally, in Section 3, we obtain some simple regularity results and construct a homotopy operator for $i\partial\bar{\partial}$.

1. Construction of the operator

In this section, we construct a homotopy operator K for $\bar{\partial}$. The operator is essentially well known even in a much more general setting, see for instance [**A-Be**], but nevertheless we sketch the construction in our case.

We start with a general process of constructing homotopy operators. Let $\eta = \zeta - z$. Let Q and S be mappings from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}^n$. Define forms q and s by $q = \sum Q_j d\eta_j$ and $s = \sum S_j d\eta_j$. For $t \geq 0$ we let

$$P_t\,(\zeta, z) = C_n e^{(Q+tS)\cdot\eta}\,(d\,(q + ts))^n\,,$$

where $C_n^{-1} = (-1)^n n!\,(2\pi i)^n$, and $S \cdot \eta$ is defined by $S \cdot \eta\,(\zeta, z) = \sum S_j\,(\zeta, z)\,\eta_j\,(\zeta, z)$, and so on. Define the kernel K by

$$K\,(\zeta, z) = \int_{t=0}^{\infty} P_t\,(\zeta, z)\,.$$

Note that $d\,(q+ts) = dq+tds-s\wedge dt$, so $(d\,(q+ts))^n = A - n\,(dq+tds)^{n-1}\wedge s\wedge dt$, where A contains no differentials with respect to t. Hence

$$K\,(\zeta, z) = -C_n n \int_0^{\infty} e^{(Q+tS)\cdot\eta} s \wedge (dq+tds)^{n-1}\,dt.$$

Put

$$I_k\,(\zeta, z) = C_n \int_0^{\infty} (-1)^{k+1} \frac{n!}{(n-k-1)!} e^{(Q+tS)\cdot\eta} \frac{s\wedge(dq+tds)^{n-k-1}\wedge(ds)^k}{(S\cdot\eta)^k}\,dt$$

and

$$T_k\,(\zeta, z) = C_n\,(-1)^{k+1} \frac{n!}{(n-k)!} e^{Q\cdot\eta} \frac{s\wedge(dq)^{n-k}\wedge(ds)^{k-1}}{(S\cdot\eta)^k}.$$

By formally integrating by parts, we see that if $1 \le k \le n-1$, then $K\,(\zeta, z) = T_1\,(\zeta, z) + \cdots + T_k\,(\zeta, z) + I_k\,(\zeta, z)$. If we note that $I_{n-1} = T_n$, we get the formula

$$K\,(\zeta, z) = \sum_{k=1}^{n} T_k\,(\zeta, z)\,.$$

Change the summation variable and, to let the operator fit into our situation, choose $Q\,(\zeta, z) = -\bar\zeta$ and $S\,(\zeta, z) = \bar\eta$. Then:

$$K\,(\zeta, z) = C_n \sum_{k=0}^{n-1} \frac{n!}{k!} e^{z\cdot\bar\zeta - |\zeta|^2} \times \frac{(\bar\zeta - \bar z)\cdot(d\zeta - dz)\wedge\left((d\zeta - dz)\cdot d\bar\zeta\right)^k \wedge\left((d\zeta - dz)\cdot\left(d\bar\zeta - d\bar z\right)\right)^{n-k-1}}{|\zeta - z|^{2n-2k}}. \tag{1.1}$$

The kernel K is of total bidegree $(n, n-1)$. Denote by K_q the component of K which is of bidegree $(0, q)$ in z, and hence $(n, n-q-1)$ in ζ. We find K_q by expanding

$$\left(d\zeta\cdot\left(d\bar\zeta - d\bar z\right)\right)^{n-k-1} = \sum_{q=0}^{n-k-1} \binom{n-k-1}{q} \left(d\zeta\cdot d\bar\zeta\right)^{n-k-q-1} \wedge (-d\zeta\cdot d\bar z)^q\,.$$

This gives the formula

$$K_q(\zeta,z) = C_n \sum_{k=0}^{n-q-1} \frac{n!}{k!}\binom{n-k-1}{q}(-1)^q e^{z\cdot\bar{\zeta}-|\zeta|^2} \times \frac{\left(\bar{\zeta}-\bar{z}\right)\cdot d\zeta \wedge \left(d\zeta \cdot d\bar{\zeta}\right)^{n-q-1} \wedge (d\zeta \cdot d\bar{z})^q}{|\zeta - z|^{2n-2k}}.$$

(In this formula, k only occurs in the constant and in the exponent of the denominator.) Change the definition of K by letting $K(\zeta,z) = \sum_{q=0}^{n-1} K_q(\zeta,z)$. This is motivated by the fact that we will integrate K against $(0,q)$-forms in ζ; we simply ignore the irrelevant parts of K.

The leading term in $K(\zeta,z)$, corresponding to $k=0$, equals $\phi(\zeta,z) = e^{z\cdot\bar{\zeta}-|\zeta|^2}$ times the Bochner-Martinelli kernel $B(\zeta,z)$, and $K(\zeta,z) = \phi(\zeta,z)B(\zeta,z) + K'(\zeta,z)$, where the kernel K' as well as $\bar{\partial}K'$ are integrable. It is well known that $\bar{\partial}B(\zeta,z) = [\Delta]$, where $\Delta = \{\zeta = z\}$ is the diagonal and $[\Delta]$ denotes the current of integration over Δ. Thus (since $\phi(z,z) = 1$ and $\bar{\partial}_z\phi(\zeta,z) = 0$)

$$\bar{\partial}K(\zeta,z) = \bar{\partial}_\zeta\phi(\zeta,z) \wedge B(\zeta,z) + \bar{\partial}K'(\zeta,z) + [\Delta].$$

Let $A' = n(dq + tds)^{n-1} \wedge s$, so that $P_t = C_n\phi A - C_n\phi A' \wedge dt = a - a' \wedge dt$ (with $A = (dq+tds)^n$ as before). Then, since P_t is a closed form, $0 = dP_t = da - d_{\zeta,z}a' \wedge dt$ and hence the formula

$$\bar{\partial}K = dK = \int_{t=0}^{\infty} d_{\zeta,z}a'dt = \int_0^\infty da = -a|_{t=0} = -C_n\phi(\zeta,z)(dq)^n = -P_0(\zeta,z)$$

is valid off the diagonal. (Also note that $P_0(\zeta,z) = C_n\phi(\zeta,z)\left(\bar{\partial}q\right)^n$.) By this we will have that $\bar{\partial}K = [\Delta] - P_0$.

Let K and P also denote the operators associated to the kernels $K(\zeta,z)$ and $P_0(\zeta,z)$; $Kf(z) = \int K(\zeta,z) \wedge f(\zeta)$ and similarily for P. Since the kernel K is a form of total degree $2n-1$, we will have that

$$\bar{\partial}Kf = \bar{\partial}\int K(\zeta,z) \wedge f(\zeta) = \int \bar{\partial}K(\zeta,z) \wedge f(\zeta) - \int K(\zeta,z) \wedge \bar{\partial}f(\zeta) = \bar{\partial}K.f - K\bar{\partial}f = [\Delta].f - Pf - K\bar{\partial}f = f - Pf - K\bar{\partial}f.$$

Thus we have obtained the homotopy formula

$$\bar{\partial}K + K\bar{\partial} = I - P, \tag{1.2}$$

that a priori is valid only for, say, C^1-forms with compact support, but as we will see in Section 2, by completeness (1.2) stays valid for all forms square integrable against $e^{-|z|^2}$.

2. Expressing the operator in the metric

Let $\beta = i\partial\bar{\partial}\,|z|^2/2$. Denote by $\langle\cdot,\cdot\rangle$ the pointwise Euclidean metric (for forms) generated by β, and let $\beta_k = \beta^k/k!$. The Lebesgue volume form equals the form $dV = \beta_n$. If f and g are $(0,q)$-forms, then

$$\langle f, g\rangle\, dV = c_q f \wedge \bar{g} \wedge \beta_{n-q}, \tag{2.1}$$

where the constant c_q equals 1 if q is even and $-i$ if q is odd. Further, we have that $d\zeta \cdot d\bar{\zeta} = -2i\beta$.

Let L^2_q be the set of all $(0,q)$-forms with finite norm with respect to the metric

$$(f,g) = c\int e^{-|z|^2}\langle f,g\rangle\, dV,$$

where $c = \pi^{-n}$, which gives the constant function 1 norm 1. Let $\mathcal{K}_q = L^2_q \cap \operatorname{Ker}\bar{\partial}$; in particular $\mathcal{K}_0 = F^2$ is the Fock space of entire functions, square integrable against $e^{|z|^2}$.

The operator K can be expressed as inner multiplication by the kernel $k(\zeta,z) = \sum_q k_q(\zeta,z)$, where

$$k_q(\zeta,z) = \sum_{k=0}^{n-q-1} C_{n,q,k}\, e^{z\cdot\bar{\zeta}}\, \frac{\left(\bar{\zeta}-\bar{z}\right)\cdot d\zeta \wedge (d\zeta\cdot d\bar{z})^q}{|\zeta - z|^{2n-2k}}$$

and

$$\begin{aligned} C_{n,q,k} &= \frac{C_n}{c_{q+1}}\frac{n!}{k!}\binom{n-k-1}{q}(-1)^{n-1}\,2^{n-q-1}i^{n-q-1}\,(n-q-1)! \\ &= \frac{(-1)^{n-1}}{2^{q+1}\pi^n i^{q+1} c_{q+1}}\,\frac{(n-k-1)!\,(n-q-1)!}{k!\,(n-k-q-1)!\,q!}, \end{aligned}$$

i.e. $Kf(z) = \left(f, \overline{k(\cdot,z)}\right)$. (Note, that the constant $C_{n,q,k}$ is real.)

Proposition 2.1. *The operator K is L^2-bounded.*

Proof: We have the estimate

$$|k(\zeta,z)| \lesssim \frac{e^{\operatorname{Re} z\cdot\bar{\zeta}}}{|\zeta - z|^{2n-1}}$$

for the kernel. Use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\|Kf\|^2 &= \left(\int e^{-|z|^2}\left|\left(f,\bar{k}\right)\right|^2 dV(z)\right)^2\\
&= \left(\int\left|\int e^{-\frac{|\zeta-z|^2}{2}} e^{-\operatorname{Re} z\cdot\bar{\zeta}} e^{-\frac{|\zeta|^2}{2}}\left\langle f(\zeta),\overline{k(\zeta,z)}\right\rangle dV(\zeta)\right|^2 dV(z)\right)^2\\
&\le \int I_1(z)\, I_2(z)\, dV(z),
\end{aligned}$$

where

$$I_1(z) = \int \frac{e^{-\frac{|\zeta-z|^2}{2}}}{|\zeta-z|^{2n-1}} dV(\zeta) = C < \infty$$

and

$$\begin{aligned}
I_2(z) &= \int e^{-\frac{|\zeta-z|^2}{2}} |\zeta-z|^{2n-1} e^{-2\operatorname{Re} z\cdot\bar{\zeta}} e^{-|\zeta|^2}\left|\left\langle f(\zeta),\overline{k(\zeta,z)}\right\rangle\right|^2 dV(\zeta)\\
&\le \int e^{-\frac{|\zeta-z|^2}{2}} |\zeta-z|^{2n-1} e^{-2\operatorname{Re} z\cdot\bar{\zeta}} e^{-|\zeta|^2} |f(\zeta)|^2 |k(\zeta,z)|^2 dV(\zeta)\\
&\lesssim \int e^{-\frac{|\zeta-z|^2}{2}} e^{-|\zeta|^2} |f(\zeta)|^2 \frac{1}{|\zeta-z|^{2n-1}} dV(\zeta).
\end{aligned}$$

Hence

$$\|Kf\|^2 \lesssim \int e^{-|\zeta|^2} |f(\zeta)|^2 \int \frac{e^{-\frac{|\zeta-z|^2}{2}}}{|\zeta-z|^{2n-1}} dV(z)\, dV(\zeta) \lesssim \|f\|^2. \qquad \square$$

Remark 1. In [**A-Boo**], we make extensive use of the fact that a certain homotopy operator is compact. In this situation, however, the operator K is *not* compact. This can be seen as follows.

Consider one complex variable. The set of all

$$f_k = \frac{z^k}{\sqrt{k!}} d\bar{z}$$

is an orthonormal set in K_1. Let $u_k = Kf_k$. Since the functions

$$u_k = \frac{|z|^2 - k}{\sqrt{k!}} z^{k-1}$$

also constitute an orthonormal set, we have an example of a bounded sequence (f_k) such that the image sequence (Kf_k) has no convergent subsequence. Hence K is not compact on L^2. $\square$

By considering the kernel of the operator P defined in Section 1, P is easily seen to be the orthogonal projection from L_0^2 onto F^2.

Since $\mathbb{C}^n$ equipped with the β-metric is a complete manifold, the smooth, compactly supported forms are dense in the graph norms (see e.g. [**B**]). We already know that the homotopy formula (1.2) is valid for smooth, compactly supported forms, hence the formula is valid for all forms in $\operatorname{Dom}\bar{\partial}$.

Remark 2. The L^2-boundedness of K helps to explain why (1.2) is valid for all $f \in \operatorname{Dom}\bar{\partial}$, in the following way: If f is in the domain of $\bar{\partial}$, and in particular f itself is in L^2, then the terms f, Pf and $K\bar{\partial}f$ all are in L^2. By approximation with smooth, compactly supported forms, we see that $\bar{\partial}Kf$ is in L^2 as well and that (1.2) stays valid in the limit. In particular, we conclude, that $Kf \in \operatorname{Dom}\bar{\partial}$ for all $f \in \operatorname{Dom}\bar{\partial}$, and that $\bar{\partial}K\colon \operatorname{Dom}\bar{\partial} \to \mathcal{K}$ is a projection. □

It is easily checked that $k(\zeta, z) = \partial_\zeta h(\zeta, z)$, where $h(\zeta, z) = \sum_{q=0}^{n-1} h_q(\zeta, z)$,

$$h_q(\zeta, z) = \sum_{k=0}^{n-q-1} -\frac{C_{n,q,k}}{n-k-1} e^{z\cdot\bar{\zeta}} \frac{(d\zeta \cdot d\bar{z})^q}{|\zeta - z|^{2n-2k-2}}$$

for $q > 0$ and

$$h_0(\zeta, z) = \sum_{k=0}^{n-2} -\frac{C_{n,0,k}}{n-k-1} \cdot \frac{e^{z\cdot\bar{\zeta}}}{|\zeta - z|^{2n-2k-2}} + C_{n,0,n-1} e^{z\cdot\bar{\zeta}} \log|\zeta - z|^2 .$$

Since h is hermitean, i.e. $h_q(z, \zeta) = (-1)^q \overline{h_q(\zeta, z)}$, the operator H defined by $Hf(z) = \left(f, \overline{h(\cdot, z)}\right)$ is self-adjoint. Thus we have seen that $K = H\bar{\partial}^*$, where $\bar{\partial}^*$ is the formal adjoint of $\bar{\partial}$ with respect to $(\cdot,\cdot)$. As an immediate consequence, we get that $\bar{\partial}K$ is selfadjoint, and hence is the orthogonal projection onto $\mathcal{K}$. Thus the homotopy formula (1.2) gives the orthogonal decomposition $L^2 = \mathcal{K} \oplus \mathcal{K}^\perp$.

Remark 3. By the method in the proof of Proposition 2.1, one can see that the operator H is bounded on L^2. □

Remark 4. By the Kähler identities for vector bundles (see for instance [**B**]), $\bar{\partial}^* = i[\partial, \beta\neg] + \partial|z|^2\neg = \frac{1}{2}[\partial, d\bar{z}\cdot dz\neg] + \bar{z}\cdot dz\neg$, where the brackets denote the commutator and $\neg$ denotes interior multiplication with respect to β. When we, as in this context, only let $\bar{\partial}^*$ act on $(0,q)$-forms, this expression reduces to $\bar{\partial}^* = \bar{z}\cdot dz\neg - \frac{1}{2}d\bar{z}\cdot dz\neg\partial$. □

We conclude:

Theorem 2.2. *K is the canonical operator with respect to $(\cdot,\cdot)$.*

Proof: If f is orthogonal to $\operatorname{Ker}\bar\partial$, then $\bar\partial^* f = 0$, so $Kf = 0$. If, on the other hand, $\bar\partial f = 0$, then $f = \bar\partial K f$, so $Kf = K\bar\partial K f = Kf - \bar\partial KKf$, and hence $\bar\partial K\,(Kf) = 0$. Since $\bar\partial K$ is the orthogonal projection onto the kernel, Kf is orthogonal to the kernel, and hence the minimal solution. □

3. Application to solving $i\partial\bar\partial$ problems

Note, that the kernel $K(\zeta, z)$ from (1.1) almost is a convolution kernel; $K(\zeta, z) = \phi(\zeta, z-\zeta)\, A(z-\zeta)$, where $\phi(\zeta, z) = e^{z\cdot\bar\zeta}$ and $A(\eta)$ is an (integrable) convolution kernel whose coefficients roughly are $\bar\eta_k / |\eta|^*$, where $*$ denotes an exponent not higher than $2n$. By performing an appropriate change of variables in the integral defining Kf and differentiating under the integral sign, and then substituting back, we see that Kf has partial derivatives with respect to z_k and $\bar z_k$ if f has, and furthermore K commutes with the holomorphic derivatives in the sense that

$$\frac{\partial}{\partial z_k} Kf = K \frac{\partial f}{\partial \zeta_k} \tag{3.1}$$

(where we let the derivative act as a Lie derivative on forms, i.e. so that it only affects the coefficient functions). As a consequence, we have that K preserves regularity. By similar arguments, P preserves regularity and satisfies the same commuting rule (3.1) as K. In particular, the homotopy formula (1.2) gives a C^∞-smooth orthogonal decomposition of $L^2 \cap C^\infty$.

Remark 5. For antiholomorphic derivatives, there is no rule as simple as (3.1). Instead, we have the formula

$$\frac{\partial}{\partial \bar z_k} Kf = K \frac{\partial f}{\partial \bar\zeta_k} + z_k K f - K(\zeta_k f).$$

However, the main reason for using (3.1) is to prove Proposition 3.1 below. The corresponding commutation rule for $\bar\partial$ would be hard to prove using the above formula for the antiholomorphic derivatives, and anyway the rule for $\bar\partial$ is known; it is just the homotopy formula (1.2). □

Recall that we restricted the operator K to operate on $(0,q)$-forms only. Now we extend K to an operator operating on (p,q)-forms by demanding that the $(p,0)$ part should be ignored; more precisely we let $K\left(a_{I,\bar J} d\bar\zeta^J \wedge d\zeta^I\right) = K\left(a_{I,\bar J} d\bar\zeta^J\right) \wedge dz^I$. The homotopy formula (1.2) and the commutation rule (3.1) still hold for this extended K. If $\Phi(\zeta, z) =$

$\sum_I dz^I \wedge d\bar\zeta^I$, then the kernel for this extended K is $k(\zeta,z)\wedge\Phi(\zeta,z)$, and thus the kernel for the corresponding operator H such that $K = H\bar\partial^*$ is $h(\zeta,z)\wedge\Phi(\zeta,z)$. In particular, H is self-adjoint, which in turn implies that the operator $\bar\partial K$, acting on (p,q)-forms, is the orthogonal projection onto the kernel of $\bar\partial$.

The observation above concerning holomorphic derivatives yields the following proposition.

Proposition 3.1. *$\partial K = -K\partial$ and $\partial P = P\partial$.*

Proof: If f is a q-form, then $\partial f = (-1)^q \sum \frac{\partial}{\partial z_k} f \wedge dz_k$. Thus

$$\begin{aligned}
\partial K\left(\alpha d\bar\zeta^J \wedge d\zeta^I\right) &= \partial K\left(\alpha d\bar\zeta^J\right)\wedge dz^I\\
&= (-1)^{|J|-1}\sum \frac{\partial}{\partial z_k} K\left(\alpha d\bar\zeta^J\right)\wedge dz_k\wedge dz^I\\
&= (-1)^{|J|-1}\sum K\left(\frac{\partial\alpha}{\partial\zeta_k} d\bar\zeta^J\right)\wedge dz_k\wedge dz^I\\
&= (-1)^{|J|-1} K\left(\sum \frac{\partial\alpha}{\partial\zeta_k} d\bar\zeta^J\wedge d\zeta_k\wedge d\zeta^I\right)\\
&= -K\left(\partial\left(\alpha d\bar\zeta^J\wedge d\zeta^I\right)\right).
\end{aligned}$$

That proves the statement for K, and by using (1.2) twice we get that

$$\partial P f = \partial\left(f - K\bar\partial f\right) = \partial f + K\partial\bar\partial f = \partial f - K\bar\partial\partial f = P\partial f,$$

which proves the statement for P. □

Remark 6. This proposition and the homotopy formula (1.2) gives the corresponding homotopy formula $dK + Kd = I - P$ for d. □

In addition, we define operators $\bar K$ and $\bar P$ by $\bar K f = \overline{K\bar f}$ and analogously for $\bar P$. The operator $\bar K$ obviously takes $(p+1,q)$-forms into (p,q)-forms. Note that the operator $K\bar K$ solves the $\partial\bar\partial$ equation: If f is a d-closed (q,q)-form, then f is both ∂- and $\bar\partial$-closed. Hence $v = \bar K f$ satisfies $\partial v = f$ and (by Proposition 3.1) $\bar\partial v = 0$. Let $u = Kv$. Then $\bar\partial u = v$, hence $\partial\bar\partial K\bar K f = \partial\bar\partial u = \partial v = f$. Thus, we can easily find an operator that solves $\partial\bar\partial$ equations. However, to get a homotopy operator, we need a little extra effort. (Also note, that the solutions were minimal in each step, but that the resulting solution will not necessarily be minimal.)

Definition 1. Define operators M, D and Π acting on smooth (q,q)-forms by: Let $M = \frac{i}{2}\left(\bar{K}K - K\bar{K}\right)$, $D = \frac{1}{2}\left(\partial\bar{K}\bar{\partial}K + \bar{\partial}K\partial\bar{K}\right)$ if $q > 0$, and $D = P\bar{P}$ on functions. Finally, let $\Pi = \bar{\partial}K + \partial\bar{K} - D$ if $q > 0$ and $\Pi = P + \bar{P} - D$ on functions. □

Note that M lowers the degree by $(1,1)$, while D and Π preserve degrees. All three operators map real forms to real forms. Some geometrical interpretations of these operators are listed in the following theorem.

Theorem 3.2. *Consider D, Π and M as operators between L^2-spaces of smooth (q,q)-forms.*

1. *D is the orthogonal projection onto* $\operatorname{Ker} d$.
2. *Π is the orthogonal projection onto* $\operatorname{Ker} i\partial\bar{\partial}$.
3. *M is a canonical homotopy operator for $i\partial\bar{\partial}$ in the sense that*

$$Mi\partial\bar{\partial}u = u - \Pi u$$

and

$$i\partial\bar{\partial}Mu = Du.$$

Proof: We begin with a proof of part 3 concerning M. Since, for $q \geq 2$,

$$K\bar{K}\partial\bar{\partial} = -K\bar{K}\bar{\partial}\partial = K\bar{\partial}\bar{K}\partial = \left(I - \bar{\partial}K\right)\left(I - \partial\bar{K}\right) = I - \partial\bar{K} - \bar{\partial}K + \bar{\partial}K\partial\bar{K}$$

and in the same way $\bar{K}K\bar{\partial}\partial = I - \bar{\partial}K - \partial\bar{K} + \partial\bar{K}\bar{\partial}K$, we have that

$$\frac{1}{2}\left(K\bar{K}\partial\bar{\partial} + \bar{K}K\bar{\partial}\partial\right) = I + \frac{1}{2}\left(\bar{\partial}K\partial\bar{K} + \partial\bar{K}\bar{\partial}K\right) - \left(\partial\bar{K} + \bar{\partial}K\right),$$

which is the first assertion of 3 for $q \geq 1$; the case $q = 0$ is handled in the same way. The second assertion of 3 is immediate from the definitions and Proposition 3.1.

Part 1 follows from the observation that if $du = 0$, then $\partial u = \bar{\partial}u = 0$ and hence

$$-\partial\bar{\partial}\bar{K}Ku = \partial\bar{K}\bar{\partial}Ku = \partial\bar{K}\left(I - K\bar{\partial}\right)u = \partial\bar{K}u = u - \bar{K}\partial u = u;$$

in a similar way $\partial\bar{\partial}K\bar{K}u = u$, and the claim that D is a projection is proved ($dD = 0$). That it is orthogonal follows because it is self-adjoint.

That Π is a projection onto $\operatorname{Ker} i\partial\bar{\partial}$ is obvious from part 3, and it follows immediately that it in fact is the orthogonal projection. □

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