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HAUSDORFF MEASURES AND THE MORSE-SARD THEOREM

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Abstract

Let $F\colon U\subset\mathbb{R}^n\to\mathbb{R}^m$ be a differentiable function and $p<m$ an integer. If $k\geq 1$ is an integer, $\alpha\in[0,1]$ and $F\in C^{k+(\alpha)}$, if we set $C_p(F)=\{x\in U\mid \operatorname{rank}(Df(x))\leq p\}$ then the Hausdorff measure of dimension $(p+\frac{n-p}{k+\alpha})$ of $F(C_p(F))$ is zero.

1. Introduction

The Morse-Sard theorem is a fundamental theorem in analysis that is in the basis of transversality theory and differential topology. The classical Morse-Sard theorem states that the image of the set of critical points of a function $F\colon\mathbb{R}^n\to\mathbb{R}^m$ of class C^{n-m+1} has zero Lebesgue measure in $\mathbb{R}^m$. It was proved by Morse ([**M**]) in the case $m=1$ and by Sard ([**S1**]) in the general case.

Due to its theoretical importance, the Morse-Sard theorem was generalized in many directions. Many of these generalizations are related with Hausdorff measures and Hausdorff dimensions.

Given a metric space X and a positive real number α, we define the Hausdorff measure of dimension α associated to a covering $\mathcal{U}=(U_\lambda)_{\lambda\in L}$ of X by bounded sets U_λ by $m_\alpha(\mathcal{U})=\sum_{\lambda\in L}(\operatorname{diam}U_\lambda)^\alpha$, where $\operatorname{diam}U_\lambda$ denotes the diameter of U_λ, and, if we define the norm of a covering $\mathcal{U}$ by $||\mathcal{U}||=\sup_{U\in\mathcal{U}}(\operatorname{diam}U)$, then the Hausdorff measure of dimension α of X is $m_\alpha(X)=\liminf\limits_{\substack{\mathcal{U}\text{ covering of }X\\ ||\mathcal{U}||\to 0}} m_\alpha(\mathcal{U})$.

It is not difficult to see that there is a unique $d\in[0,+\infty]$ such that if $\alpha>d$ then $m_\alpha(X)=0$ and if $\alpha<d$ then $m_\alpha(X)=+\infty$. This number d is called the *Hausdorff dimension* of X. It is easy to see that if $X\subset\mathbb{R}^n$ then its Hausdorff dimension $d=:HD(X)$ belongs to $[0,n]$.

Sard himself proved that if $C_p(F) = \{x \in \mathbb{R}^n \mid \operatorname{rank}(DF(x)) \le p\}$ then for any $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that if F is C^k then $F(C_p(F))$ has zero Hausdorff measure of dimension $p + \varepsilon$ ([**S2**]). This result was made more precise by Federer ([**F**]), who proved that if $k \in \mathbb{N}$ then the Hausdorff measure of dimension $p + \frac{n-p}{k}$ of $F(C_p(F))$ is zero. We should also mention the works of Church ([**Ch1**], [**Ch2**]), which gave more results about the structure of the set of critical values of differentiable maps. Later, Yomdin ([**Y**]) proved that the Hausdorff dimension of $F(C_p(F))$ is at most $p + \frac{n-p}{k+\alpha}$, provided that $F \in C^{k+\alpha}$, where $k \in \mathbb{N}$ and $0 \le \alpha < 1$. More recently, Bates ([**B2**]) proved that if $F \in C^{k+\alpha}$ with $k \in \mathbb{N}$, $0 < \alpha \le 1$ and $p + \frac{n-p}{k+\alpha} = m$ then $F(C_p(F))$ has zero Lebesgue measure in $\mathbb{R}^m$ (this in particular improves the hypothesis of the classical Morse-Sard theorem from $F \in C^{n-m+1}$ to $F \in C^{n-m+\text{Lips.}}$, i.e., $F \in C^{n-m}$ and $D^{n-m}F$ Lipschitz).

The aim of this work is to generalize the mentioned results by proving a general version of the Morse-Sard Theorem involving Hausdorff measures. Let $k \ge 1$ be an integer and $\alpha \in [0,1]$. We say that a function $F\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ is of class $C^{k+(\alpha)}$ at a subset A of U if F is C^k in U and for each $x \in A$ there are $\varepsilon_x > 0$, $K_x > 0$ such that $|y - x| < \varepsilon_x \Rightarrow |D^kF(y) - D^kF(x)| \le K_x|y-x|^\alpha$ (this is less restrictive than supposing $F \in C^{k+\alpha}$). Our main result is the following

Theorem. *Let* $F\colon U \subset \mathbb{R}^n \xrightarrow{C^k} \mathbb{R}^m$ *and let* $p < m$ *be an integer. If* $C_p(F) := \{x \in U \mid \operatorname{rank}(DF(x)) \le p\}$ *and if* F *is of class* $C^{k+(\alpha)}$ *at* $C_p(F)$ *then the Hausdorff* $(p + \frac{n-p}{k+\alpha})$*-measure of* $F(C_p(F))$ *is zero.*

In particular, if $k + \alpha = \frac{n-p}{m-p}$, we recover the result of [**B2**], with a weaker hypothesis. We remark that if $p + \frac{n-p}{k+\alpha} < m$, the Hausdorff $(p + \frac{n-p}{k+\alpha})$-measure is not the Lebesgue measure or a product measure in $\mathbb{R}^m$, and so we can not use Fubini's Theorem. This difficulty is solved in the present paper by replacing the use of Fubini's theorem by a careful decomposition of the critical set, combined with a parametrized strong version of the main lemma of Morse's paper ([**M**, Theorem 2.1]).

We shall also give examples that show that our result is quite sharp, by giving counterexamples to slight changes of the hypothesis or of the conclusion.

2. Functions whose zeros include a given set

We shall prove here a version of Theorem 3.6 of [**M**] and Lemma 3.4.2 of [**F**], which will be fundamental for the later results.

Theorem 2.1. *Let $k \geq 1$, $\alpha \in [0,1]$, $n > p$ and $A \subset U \subset \mathbb{R}^n$, where U is an open set. Then there are sets $A_1, A_2 \ldots \subset A$ such that $A = \bigcup_{i=1}^{\infty} A_i$, where for each $i = 1, 2, \ldots$ there is a function $\psi_i \colon B_i \times V_i \xrightarrow{C^1} U$ where B_i is a ball in some $\mathbb{R}^{r_i}$, $r_i \geq 0$ and V_i is a ball in $\mathbb{R}^p$ such that $\psi_i(x,y) = (\widetilde{\psi}_i(x,y), y)$, and $|\psi_i(x_1,y_1) - \psi_i(x_2,y_2)| \geq |(x_1,y_1) - (x_2,y_2)|$, $\forall\,(x_1,y_1),(x_2,y_2) \in B_i \times V_i$ and $A_i \subset \psi_i(B_i \times V_i)$, with the following property: We can write $A_i = A_i' \cup A_i''$ so that $\psi_i^{-1}(A_i'')$ has measure zero in $B_i \times V_i$, and if $f \colon U \to \mathbb{R}$ vanishes in A and f is $C^{k+(\alpha)}$ at A we have:*

- $\displaystyle\limsup_{(x,y_0)\to(x_0,y_0)} \frac{f(\psi_i(x,y_0))}{|x-x_0|^{k+\alpha}} < +\infty$, *$\forall\,(x_0,y_0) \in B_i \times V_i$ such that $\psi_i(x_0,y_0) \in A_i$,*
- $\displaystyle\lim_{(x,y_0)\to(x_0,y_0)} \frac{f(\psi_i(x,y_0))}{|x-x_0|^{k+\alpha}} = 0$, *$\forall\,(x_0,y_0) \in B_i \times V_i$ such that $\psi_i(x_0,y_0)) \in A_i'$.*

Proof: Let us consider first the case $k = 1$ and $df(x) \cdot v = 0\ \forall\, x \in A$, $v \in \mathbb{R}^{n-p} \times \{0\}$. In this case we take $A = (A' \cap A) \cup A''$ where A' is the set of density points of $\overline{A}$ in the direction of $\mathbb{R}^{n-p} \times \{0\}$ ($(x,y) \in A' \Rightarrow \displaystyle\lim_{\varepsilon\to 0} \frac{m((B_\varepsilon(x) \times \{y\}) \cap \overline{A})}{m(B_\varepsilon(x))} = 1$, where m is the $(n-p)$-dimensional measure). The measure of $A'' = A - A'$ is zero, since it is zero in each plane $\mathbb{R}^{n-p} \times \{y\}$.

For $(x_0,y_0) \in A$ take $B((x_0,y_0),\varepsilon(x_0,y_0))$ a ball contained in U and $\psi = \mathrm{Id}|_{B((x_0,y_0),\varepsilon(x_0,y_0))}$. We have $\displaystyle\limsup_{(x,y_0)\to(x_0,y_0)} \frac{f(x,y_0)}{|x-x_0|^{1+\alpha}} < +\infty$, since $f(x,y_0) = f(x,y_0) - f(x_0,y_0) = df(tx_0 + (1-t)x)(x-x_0)$, $t \in (0,1) \Rightarrow |f(x,y_0)| \leq K_{x_0}|x-x_0|^{1+\alpha}$. For $(x_0,y_0) \in A'$,

$$\lim_{\delta\to 0} \frac{1}{\mathrm{vol}(S^{n-p-1})} \int_{S^{n-p-1}} \left(\frac{1}{\delta} \int_0^\delta \chi_{\overline{A}}(x_0 + tv, y_0)\,dt \right) dv = 1,$$

so $\forall\,\varepsilon > 0\ \exists\,\delta_0 > 0$ s.t. $|x - x_0| < \delta_0 \Rightarrow \exists\, v \in S^{n-p-1}$ with

$$\left| v - \frac{x-x_0}{|x-x_0|} \right| < \varepsilon$$

and

$$\left| \frac{1}{|x-x_0|} \int_0^{|x-x_0|} \chi_{\overline{A}}(x_0 + tv, y_0)\,dt - 1 \right| < \varepsilon,$$

so, if $\tilde{x} = x_0 + |x - x_0|v$,

$$|f(x, y_0) - f(x_0, y_0)| \le |f(x, y_0) - f(\tilde{x}, y_0)| + |f(\tilde{x}, y_0) - f(x_0, y_0)|,$$

but

$$\begin{aligned} &f(x, y_0) - f(\tilde{x}, y_0) = df(\theta t + (1-\theta)\tilde{x}, y_0) \cdot (x - \tilde{x}), \quad \theta \in (0,1) \\ &\Rightarrow |f(x, y_0) - f(\tilde{x}, y_0)| \le K_{x_0}|x - x_0|^\alpha \cdot \varepsilon |x - x_0| = \varepsilon K_{x_0}|x - x_0|^{1+\alpha} \end{aligned}$$

and

$$\begin{aligned} f(\tilde{x}, y_0) &- f(x_0, y_0) \\ &= \int_0^{|\tilde{x} - x_0|} df(x_0 + tv, y_0) \cdot v\, dt \\ &\le K_{x_0}|x - x_0|^\alpha \cdot m \left\{ t \in [0, |\tilde{x} - x_0|] \mid \frac{\partial f}{\partial x}(x_0 + tv, y_0) \ne 0 \right\} \\ &\le K_{x_0}|x - x_0|^\alpha \cdot \varepsilon |x - x_0| = \varepsilon K_{x_0}|x - x_0|^{1+\alpha}. \end{aligned}$$

So

$$|f(x, y_0)| = |f(x, y_0) - f(x_0, y_0)| \le 2\varepsilon K_{x_0}|x - x_0|^{1+\alpha} \\ \Rightarrow \lim_{(x, y_0) \to (x_0, y_0)} \frac{f(x, y_0)}{|x - x_0|^{1+\alpha}} = 0.$$

We can take a countable subcovering of A by the $B((x_0, y_0), \varepsilon(x_0, y_0))$ to finish the proof in this case.

Consider now the case $k \ge 1$, n arbitrary. We have $A = A^* \cup A^{**}$ where $A^* = \{x \in A \mid \exists\, g \colon U \xrightarrow{C^k} \mathbb{R},\ g|_A \equiv 0,\ \exists\, v \in \mathbb{R}^{n-p} \times \{0\},\ dg(x) \cdot v \ne 0\}$. $A^{**} = A \backslash A^*$. If $(x_0, y_0) \in A^*$ there is g as above, so there is $\varepsilon > 0$ such that $g^{-1}(0) \cap B_\varepsilon(x_0, y_0)$ is contained in the image of $\psi \colon B \times V \xrightarrow{C^k} U$ where B is a ball in $\mathbb{R}^{n-p-1}$, as in the statement, and $A \subset g^{-1}(0)$. Taking a countable subcovering of A^* by these balls we reduce the proof in this case to a case with smaller n. If $k = 1$, the result was yet proved for A^{**}. If $k > 1$, and assuming by induction the result for $k - 1$, we have

$$A^{**} = \bigcup_{i=1}^\infty A_i^{**},\ A_i^{**} = (A_i^{**})' \cup (A_i^{**})'',\ A_i^{**} \subset \psi_i(B_i \times V_i), \quad \psi_i \in C^1,$$

$$\psi_i(x_0, y_0) \in A_i^{**} \Rightarrow \limsup_{x \to x_0} \frac{||df(\psi_i(x, y_0))|_{\mathbb{R}^{n-p} \times \{0\}}||}{|x - x_0|^{k-1+\alpha}} < +\infty$$

$$\Rightarrow \limsup_{x \to x_0} \frac{|f(\psi_i(x, y_0))|}{|x - x_0|^{k+\alpha}} < +\infty$$

and

$$\psi_i(x_0, y_0) \in (A_i^{**}) \Rightarrow \lim_{x \to x_0} \frac{||df(\psi_i(x, y_0))|_{\mathbb{R}^{n-p} \times \{0\}}||}{|x - x_0|^{k-1+\alpha}} = 0$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(\psi_i(x, y_0))}{|x - x_0|^{k+\alpha}} = 0,$$

both by the mean value theorem, and the proof is finished by induction. □

Corollary 2.2. *Let $k \geq 1$, $\alpha \in [0, 1]$, $n > p$ and $A \subset U \subset \mathbb{R}^n$, where U is an open set. Then there are sets $A_1, A_2 \ldots \subset A$ such that $A = \bigcup_{i=1}^{\infty} A_i$, where for each $i = 1, 2, \ldots$ there is a function $\psi_i \colon B_i \times V_i \xrightarrow{C^1} U$ where B_i is a ball in some $\mathbb{R}^{r_i}$, $r_i \geq 0$ and V_i is a ball in $\mathbb{R}^p$ such that $\psi_i(x, y) = (\widetilde{\psi}_i(x, y), y)$, and $|\psi_i(x_1, y_1) - \psi_i(x_2, y_2)| \geq |(x_1, y_1) - (x_2, y_2)|$, $\forall\, (x_1, y_1), (x_2, y_2) \in B_i \times V_i$ and $A_i \subset \psi_i(B_i \times V_i)$, with the following property: We can write $A_i = A_i' \cup A_i''$ so that $\psi_i^{-1}(A_i'')$ has measure zero in $B_i \times V_i$, and if $f \colon U \to \mathbb{R}$ is $C^{k+(\alpha)}$ at A and $D_x f \equiv 0$ in A we have:*

- $\displaystyle\limsup_{(x,y_0) \to (x_0,y_0)} \frac{|f(\psi_i(x, y_0)) - f(\psi_i(x_0, y_0))|}{|x - x_0|^{k+\alpha}} < +\infty$, *$\forall\, (x_0, y_0) \in B_i \times V_i$ such that $\psi_i(x_0, y_0) \in A_i$,*
- $\displaystyle\lim_{(x,y_0) \to (x_0,y_0)} \frac{|f(\psi_i(x, y_0)) - f(\psi_i(x_0, y_0))|}{|x - x_0|^{k+\alpha}} = 0$, *$\forall\, (x_0, y_0) \in B_i \times V_i$ such that $\psi_i(x_0, y_0)) \in A_i'$.*

Proof: If $k \geq 2$ this is an immediate consequence of Theorem 2.1 applied to $D_x f$ and of the mean value theorem. If $k = 1$ this can be proved exactly as the case $k = 1$ of the Theorem 2.1. □

Corollary 2.3. *In the statements of Theorem 2.1 and Corollary 2.2, for any $x \in B_i$ s.t. $\psi_i(x) \in A_i$ there are $\varepsilon_x > 0$, $K_x > 0$ such that $|y - x| < \varepsilon_x \Rightarrow |f(\psi_i(y)) - f(\psi_i(x))| \leq K_x |y - x|^{k+\alpha}$, and for any $\varepsilon > 0$ there is a $\delta > 0$ so that $\frac{\lambda(\psi_i^{-1}(A_i) \cap B_r(x))}{\lambda(B_r(x))} > 1 - \delta \Rightarrow |f(\psi_i(y)) - f(\psi_i(x))| \leq \varepsilon K_x r^{k+\alpha}$, if $r \leq \varepsilon_x$ and $|y - x| \leq r$ (δ depends only on ε and n, but not on f or on x).*

Proof: This is only a more precise formulation of the results proved in the demonstration of the theorem. □

Remark 2.1. For $k = 0$ we have the same results, except the statement $\lim_{y\to x} \frac{f(\psi_i(y))}{|y-x|^{k+\alpha}} = 0$, for each $x \in B_i$ such that $\psi_i(x) \in A_i'$.

3. The main results

Lemma 3.1. *Let $A \subset \mathbb{R}^m$ with $\lambda(A) < \infty$ and let $\mathcal{U}$ be a family of balls $B_r(x)$, $x \in A$ such that for each $x \in A$ there is an $\varepsilon_x > 0$ such that $r \le \varepsilon_x \Rightarrow B_r(x) \in \mathcal{U}$. Then for each $\varepsilon > 0$ there are $x_n \in A$, $r_n > 0$ with $B_{r_n}(x_n) \in \mathcal{U}$ and $A \subset \bigcup_{n=1}^{\infty} B_{r_n}(x_n)$ such that $\sum_{n=1}^{\infty} \lambda(B_{r_n}(x_n)) < \lambda(A) + \varepsilon$.*

Proof: This lemma is essentially the Vitali covering theorem from measure theory. Take $U \supset A$ an open set with $\lambda(U) < \lambda(A) + \frac{\varepsilon}{2}$. If we choosed $B_{\tilde r_1}(x_1), \dots, B_{\tilde r_n}(x_n)$, define $s_n = \sup\{r > 0 \mid \exists\, x \in A \text{ s.t. } r < \frac{\varepsilon_x}{5},\ B_r(x) \subset U \text{ and } B_r(x) \cap (B_{\tilde r_1}(x_1) \cup \cdots \cup B_{\tilde r_n}(x_n)) = \emptyset\}$. Choose $B_{\tilde r_{n+1}}(x_{n+1})$ such that $\tilde r_{n+1} > \frac{s_n}{2}$, $\tilde r_{n+1} < \frac{\varepsilon_{x_{n+1}}}{5}$, $B_{\tilde r_{n+1}}(x_{n+1}) \subset U$ and $B_{\tilde r_{n+1}}(x_{n+1}) \cap (B_{\tilde r_1}(x_1) \cup \cdots \cup B_{\tilde r_n}(x_n)) = \emptyset$. Since the $B_{\tilde r_i}(x_i)$ are disjoint and contained in U we have $\sum_{i=1}^{\infty} \lambda(B_{\tilde r_i}(x_i)) < \lambda(A) + \frac{\varepsilon}{2}$, and so there is a $n_0 \in \mathbb{N}$ such that $\sum_{i=n_0}^{\infty} \lambda(B_{5\tilde r_i}(x_i)) < \frac{\varepsilon}{2}$. We take $B_{r_i}(x_i) = B_{\tilde r_i}(x_i)$, $i < n_0$ and $B_{r_i}(x_i) = B_{5\tilde r_i}(x_i)$, $i \ge n_0$.

Clearly we have $\sum_{i=1}^{\infty} \lambda(B_{r_i}(x_i)) < \lambda(A) + \varepsilon$. To prove that $A \subset \bigcup_{n=1}^{\infty} \overline{B_{r_n}(x_n)}$, take $x \in A$ and $r = \min\{\tilde r_{n_0}, \varepsilon_x/5,\ d(x, U^c \cup \bigcup_{i<n_0} B_{r_i}(x_i))\}$. If $r > 0$, take $n \ge n_0$ such that $s_n < r \le s_{n-1}$ (we have $r \le \tilde r_{n_0} \le s_{n_0-1}$), and note that $s_n < r \Rightarrow B_r(x) \cap (B_{\tilde r_1}(x_1) \cup \cdots \cup B_{\tilde r_n}(x_n)) \ne \emptyset \Rightarrow \exists\, i \le n$ such that $B_r(x) \cap B_{\tilde r_i}(x_i) \ne \emptyset$. We have $n \ge n_0$ since $r \le d(x, B_{\tilde r_i}(x_i))$, and $\tilde r_i > \frac{s_{n-1}}{2} \ge \frac{r}{2}$, since $i \le n$. Therefore, we have $x \in B_{5\tilde r_i}(x_i)$. If $r = 0$ then $x \in \overline{B_{r_i}(x_i)}$ for some $i < n_0$. This proves that $A \subset \bigcup_{n=1}^{\infty} \overline{B_{r_n}(x_n)}$. Taking $\tilde r_n = \left(\frac{\lambda(A)+\varepsilon}{\sum_{i=1}^{\infty} \lambda(B_{r_i}(x_i))}\right)^{1/2m} \cdot r_n$, we have $A \subset \bigcup_{n=1}^{\infty} B_{\tilde r_n}(x_n)$, with $\sum_{n=1}^{\infty} \lambda(B_{\tilde r_n}(x_n)) = (\lambda(A) + \varepsilon)^{1/2} (\sum_{i=1}^{\infty} \lambda(B_{r_i}(x_i)))^{1/2} < \lambda(A) + \varepsilon$. □

Remark 3.1. In the Lemma 3.1 we can replace a family of balls $B_r(x)$ by a family of cubes $C_r(x) = \prod_{i=1}^{m} [x_i - r, x_i + r]$, where $x = (x_1, \dots, x_m)$, using the same proof.

Lemma 3.2. *Let $F\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function, $A \subset U$ and $d > 0$ such that for any $x \in A$ there are $\varepsilon_x > 0$, $K_x > 0$ such that $m_d(F(B_\varepsilon(x) \cap A)) \le K_x.\lambda(B_\varepsilon(x))$, $\forall\, \varepsilon < \varepsilon_x$, where m_d is the Hausdorff measure of dimension d, and there is $A' \subset A$ such that $\lambda(A \backslash A') = 0$ and $\lim_{\varepsilon\to 0} \frac{m_d(F(B_\varepsilon(x)\cap A))}{\lambda(B_\varepsilon(x))} = 0$, $\forall\, x \in A'$. Then $m_d(F(A)) = 0$.*

Remark 3.2. The same result is true if we replace $B_\varepsilon(x)$ by $C_\varepsilon(x)$.

Remark 3.3. We can replace the condition

$$\text{“}m_d(F(B_\varepsilon(x) \cap A)) \leq K_x\, \lambda(B_\varepsilon(x)), \quad \forall\, \varepsilon < \varepsilon_x\text{”}$$

by

$$\text{“}F(B_\varepsilon(x) \cap A) \text{ can be covered by balls } B_{\delta_i}(y_i), \quad i \in \mathbb{N},$$
$$\text{with } \sum_{i=1}^{\infty} \delta_i^d \leq K_x\, \lambda(B_\varepsilon(x)), \quad \forall\, \varepsilon < \varepsilon_x\text{”},$$

and the condition

$$\text{“}\lim_{\varepsilon \to 0} \frac{m_d(F(B_\varepsilon(x) \cap A)}{\lambda(B_\varepsilon(x))} = 0, \quad \forall\, x \in A'\text{”}$$

by

$$\text{“}F(B_\varepsilon(x) \cap A) \text{ can be covered by balls } B_{\delta_i^{(\varepsilon)}}(y_i), \quad i \in \mathbb{N}$$
$$\text{with } \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{\infty} (\delta_i^{(\varepsilon)})^d}{\lambda(B_\varepsilon(x))} = 0, \quad \forall\, x \in A'\text{”}.$$

The proof remains essentially the same, and Remark 3.2 is still valid.

Remark 3.4. If we replace the conditions of this lemma by “$F(B_\varepsilon(x) \cap A)$ can be covered by balls $B_{\delta_i}(y_i)$, $i \in \mathbb{N}$, with $\sum_{i=1}^{\infty} \delta_i^d \leq k\, \lambda(B_\varepsilon(x))$, $\forall\, \varepsilon < \varepsilon_x$ (note that here k does not depend on x), and $\lambda(A) < \infty$”, then we can conclude, using the same proof, that $m_d(F(A)) \leq k\, \lambda(A)$.

Proof: We may suppose that A has finite Lebesgue measure, since A is a countable union of sets with finite measure, and a countable union of sets with Hausdorff d-measure zero has Hausdorff d-measure zero. Moreover, since $A = \bigcup_{k=1}^{\infty} A_k$, where $A_k = \{x \in A \mid K_x \leq k\}$, we may suppose $K_x \leq K$, $\forall\, x \in A$. Let C be the Lebesgue measure of A.

Let $\varepsilon > 0$. For each $x \in A'$ take $\delta_x > 0$ such that $B_{\delta_x}(x) \subset U$ and $r \leq \delta_x \Rightarrow \frac{m_d(F(B_r(x) \cap A))}{\lambda(B_r(x))} \leq \frac{\varepsilon}{2(C+1)}$. By the Lemma 3.1 we can cover A' by $\bigcup_{n=1}^{\infty} B_{r_n}(x_n)$ with

$$\sum_{n=1}^{\infty} \lambda(B_{r_n}(x_n)) < C + 1$$

and

$$r_n \le \delta_{x_n} \Rightarrow \sum_{n=1}^{\infty} m_d(F(B_{r_n}(x_n) \cap A)) \le \frac{\varepsilon}{2(C+1)} \cdot (C+1) = \frac{\varepsilon}{2} \Rightarrow m_d(F(A')) \le \frac{\varepsilon}{2}.$$

By Lemma 3.1 we can cover $A \backslash A'$ by $\bigcup_{n=1}^{\infty} B_{\tilde{r}_n}(\widetilde{x}_n)$ such that $B_{\tilde{r}_n}(\widetilde{x}_n) \subset U$ and $\tilde{r}_n < \varepsilon_{\widetilde{x}_n}$, $\forall\, n \in \mathbb{N}$, with

$$\sum_{n=1}^{\infty} \lambda(B_{\tilde{r}_n}(\widetilde{x}_n)) < \frac{\varepsilon}{2K} \Rightarrow \sum_{n=1}^{\infty} m_d(F(B_{\tilde{r}_n}(\widetilde{x}_n))) \le \frac{\varepsilon}{2K} \cdot K = \frac{\varepsilon}{2} \Rightarrow m_d(F(A \backslash A')) \le \frac{\varepsilon}{2} \Rightarrow m_d(F(A)) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have $m_d(F(A)) = 0$. □

We first use Lemma 3.2 to prove the following strong version of Constantin's result ([**Co**]), that does not suppose continuity of the derivatives. Here we do not suppose differentiability in every point, but only in the set of critical points under consideration.

Theorem 3.3. *Let $F: X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function, and let $A = \{x \in X \mid DF(x)$ exists and is not surjective$\}$. Then $\lambda(F(A)) = 0$.*

Proof: It is a simple consequence of Lemma 3.2, since if $x \in A$ then $\lim_{r \to 0} \frac{\lambda(F(B_r(x)))}{\lambda(B_r(x))} = 0$. Indeed, $x \in A \Rightarrow F(x+h) = F(x) + DF(x).h + r(h)$, where $\lim_{h \to 0} \frac{r(h)}{|h|} = 0$. Let $K = ||DF(x)||$, and let $\varepsilon \in (0,1)$. Let $\delta > 0$ such that $|h| \le \delta \Rightarrow \frac{|r(h)|}{|h|} < \frac{\varepsilon}{2(K+1)^{n-1}}$. Then, if $|h| \le \delta$, $F(x+h) - F(x)$ belongs to an $\frac{\varepsilon.|h|}{2(K+1)^{n-1}}$ neighbourhood of a ball of radius $K|h|$ in a subspace of $\mathbb{R}^n$ of dimension $n-1$ (a fixed subspace of $\mathbb{R}^n$ of dimension $n-1$ which contains the image of DF), and thus belongs to the orthogonal product of a ball of radius $(K+1)|h|$ in this subspace by an interval of radius $\frac{\varepsilon|h|}{2(K+1)^{n-1}}$. Therefore, $\lambda(F(B_r(x)) \le \frac{\varepsilon.r.r^{n-1}(K+1)^{n-1}}{(K+1)^{n-1}} v_{n-1} = \varepsilon r^n v_{n-1}$, where v_{n-1} is the volume of the unitary ball in $\mathbb{R}^{n-1}$, and, since $\varepsilon > 0$ is arbitrary, $\lim_{r \to 0} \frac{\lambda(F(B_r(x)))}{\lambda(B_r(x))} = 0$. □

Theorem 3.4. *Let $F: U \subset \mathbb{R}^n \xrightarrow{C^k} \mathbb{R}^m$ be a function of class $C^{k+(\alpha)}(\alpha \in (0,1])$ at $C_p(F) := \{x \in U \mid \operatorname{rank}(DF(x)) \le p\}$. Then the Hausdorff measure of dimension $d = p + \frac{n-p}{k+\alpha}$ of $F(C_p(F))$ is zero, $\forall\, p < \min\{m,n\}$.*

Proof: Since $C_p(F) = \bigcup_{r=0}^{p}\{x \in U \mid \operatorname{rank}(DF(x)) = r\}$, and $r + \frac{n-r}{k+\alpha} \leq p + \frac{n-p}{k+\alpha}$ for $0 \leq r \leq p$, we may restrict our attention to $\widetilde{C}_p(F) = \{x \in U \mid \operatorname{rank}(DF(x)) = p\}$. If $x_0 \in C_p(F)$, we have, after a change of coordinates of class C^k, $F(z,y) = (z, G(z,y))$, with $(z,y) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ and $G(z,y) \in \mathbb{R}^{m-p}$, in a neighbourhood V of $x_0 = (z_0, y_0)$. We shall restrict our attention to this neighbourhood. We have $x = (z,y) \in C_p(F) \Leftrightarrow D_y\, G(z,y) = 0$. We can apply the results of the Section 2 (Theorem 2.1, Corollary 2.3 and Remark 2.1) to the function $D_y\, G$, and obtain the decomposition $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset \psi_i(V_i \times B_i)$, where $A = \{(z,y) \in V \mid D_y G(z,y) = 0\}$. Let us fix such an A_i.

Since $\psi_i^{-1}(A_i) = \bigcup_{m\in\mathbb{N}}\{x \in \psi_i^{-1}(A_i) \mid \varepsilon_x \geq \frac{1}{m},\, K_x \leq m\}$, we may suppose $\varepsilon_x \geq \frac{1}{M}$, $K_x \leq M$, $\forall\, x \in \psi_i^{-1}(A_i)$, for some fixed M and also that V has finite Lebesgue measure $\lambda(V)$.

With these assumptions, we shall prove that there is a constant K_0 such that for any $X \subset V$, $\nu > 0$, we can cover $F(A_i \cap X)$ by balls $B_{\delta_i}(p_i)$ so that $\sum_{i=1}^{\infty} \delta_i^d \leq K_0(\lambda(X) + \nu)$. For this, given a point $x \in A_i \cap X$ and an $\varepsilon < \frac{1}{2\sqrt{n}M}$, we can divide the cube $C_\varepsilon(x) = C_\varepsilon(z) \times C_\varepsilon(y)$ into $([\varepsilon^{1-(k+\alpha)}] + 1)^p$ boxes $C_\delta(\widetilde{z}_i) \times C_\varepsilon(y)$, $\delta < \varepsilon^{k+\alpha}$. If there is some point (z_i, y_i) in $(C_\delta(\widetilde{z}_i) \times C_\varepsilon(y)) \cap (A_i \cap X)$, then for any point (z_i', y_i') in $(C_\delta(\widetilde{z}_1) \times C_\varepsilon(y)) \cap (A_i \cap X)$, we have $|F(z_i', y') - F(z_i, y_i)| \leq |F(z_i', y') - F(z_i, y')| + |F(z_i, y') - F(z_i, y_i)| \leq K'\delta + |F(z_i, y_i') - F(z_i, y_i)|$ (where K' is $\sqrt{p}$ times a Lipschitz constant of $F|_V$ which we may suppose to exist) $\leq K'\delta + |G(z_i, y_i') - G(z_i, y_i)|$.

Observe now that $(z_i, y_i) = (z_i, \widetilde{\psi}_i(p_1))$ and $(z_i', y_i') = (z_i, \widetilde{\psi}_i(p_2))$, for some p_1, p_2 in $\{z_i\} \times B_i$ with $|p_1 - p_2| \leq |y_i - y_i'| \leq 2\varepsilon\sqrt{n}$. Let $\gamma\colon [0,1] \to V_i \times B_i$ be a straight path joining p_1 and p_2. Then $G(z_i, y_i') - G(z_i, y_i) = \int_0^1 \frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) \cdot \widetilde{\gamma}'(t)\, dt$, where $\widetilde{\gamma} := \psi_i \circ \gamma$. We have $\frac{\partial G}{\partial y}(\widetilde{\gamma}(0)) = 0$, so

$$\begin{aligned}\left\|\frac{\partial G}{\partial y}(\widetilde{\gamma}(t))\right\| &= \left\|\frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) - \frac{\partial G}{\partial y}(\widetilde{\gamma}(0))\right\| \\ &\leq M|p_1 - p_2|^{k+\alpha-1} \\ &\leq M(2\varepsilon\sqrt{n})^{k+\alpha-1} \Rightarrow \left\|\frac{\partial G}{\partial y}(\widetilde{\gamma}(t))\right\| |\gamma'(t)| \\ &\leq K''\varepsilon^{k+\alpha},\end{aligned}$$

for some constant K''. Indeed, $|\widetilde{\gamma}'(t)|$ is limited by a constant multiple of $|p_1 - p_2| \leq 2\sqrt{n}\varepsilon$. So

$$|G(z_1, y_i') - G(z_i, y_i)| \leq \int_0^1 \left|\frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) \circ \widetilde{\gamma}(t)\right| dt \leq K''\varepsilon^{k+\alpha}$$

and

$$|F(z'_i, y') - F(z_i, y_i)| \le K'\delta + K''\varepsilon^{k+\alpha} \le K'_0 \cdot \varepsilon^{k+\alpha},$$

where $K'_0 = K' + K''$.

Therefore, $F(C_\delta(\widetilde{z}_i) \times C_\varepsilon(y))$ is contained in some ball $B_{\delta_i}(q_i)$, with $\delta_i \le K'_0 \cdot \varepsilon^{k+\alpha}$, so

$$\begin{aligned}\sum_i \delta_i^d &\le ([\varepsilon^{1-(k+\alpha)}] + 1)^p (K'_0\, \varepsilon^{k+\alpha})^d \\ &= (K'_0)^d ([\varepsilon^{1-(k+\alpha)}] + 1)^p (\varepsilon^{k+\alpha})^{p + \frac{n-p}{k+\alpha}} \\ &\le \widetilde{K}_0\, \varepsilon^n\end{aligned}$$

for some constant $\widetilde{K}_0$. So, $F(C_\varepsilon(x))$ can be covered by balls $B_{\delta_i}(q_i)$ with $\sum_i \delta_i^d \le K_0.\lambda(C_\varepsilon(x))$, and by the Lemma 3.2, Remarks 3.2 and 3.4, we can conclude that $m_d(F(A_i \cap X)) \le K_0 \lambda(X)$ where $K_0 = 2^n \widetilde{K}_0$, and so we can cover $F(A_i \cap X)$ by balls $B_{\delta_i}(p_i)$ so that $\sum_{i=1}^\infty \delta_i^d \le K_0(\lambda(X) + \nu)$, as we stated.

We shall prove now that there is an $A'_i \subset A_i \subset V$ with $\lambda(A_i \backslash A'_i) = 0$ such that $F(C_\varepsilon(x) \cap A_i)$ can be covered by balls $B_{\delta_i^{(\varepsilon)}}(W_i)$, $i \in \mathbb{N}$ with $\lim\limits_{\varepsilon \to 0} \frac{\sum_i (\delta_i^{(\varepsilon)})^d}{\lambda(C_\varepsilon(x))} = 0$, $\forall x \in A'_i$. This will imply our theorem, by the Lemma 3.2, Remarks 3.2 and 3.3, since we have proved above that $m_d(F(C_\varepsilon(x) \cap A_i) \le K_0 \lambda(C_\varepsilon(x))$, $\forall \varepsilon < \frac{1}{2\sqrt{n}M}$. For this, since $A_i \subset \psi_i(V_i \times B_i)$, $B_i \subset \mathbb{R}^{r_i}$, $r_i \le n-p$, we may suppose $r_i = n-p$ and $\psi_i =$ identity, because $r_i < n-p \Rightarrow \lambda(A_i) = 0$ and we can take $A'_i = A_i$. Let us take A'_i equal to the set of the density points of A_i. Given a point $x \in A'_i$, and an $\eta' > 0$, we want to find an $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0 \Rightarrow F(C_\varepsilon(x) \cap A_i)$ can be covered by balls $B_{\delta_i^{(\varepsilon)}}(W_i)$, $i \in \mathbb{N}$ such that $\sum_i (\delta_i^{(\varepsilon)})^d \le \eta' \lambda(C_\varepsilon(x))$. Let $\eta, \tilde{\eta} > 0$, $\tilde{\varepsilon} < \frac{1}{2\sqrt{n}M}$ such that $\frac{\lambda(C_\varepsilon(x) \cap A_i)}{\lambda(C_\varepsilon(x))} > 1 - \tilde{\eta}^2$, $\forall \varepsilon \le \tilde{\varepsilon}$. Divide the cube $C_\varepsilon(x) = C_\varepsilon(\widetilde{z}) \times C_\varepsilon(\widetilde{y})$, $\varepsilon < \tilde{\varepsilon}$ into $N = ([\varepsilon^{1-(K+\alpha)}\eta^{-1}] + 1)^p$ boxes $C_\delta(\widetilde{z}_i) \times C_\varepsilon(\widetilde{y})$, $\delta < \eta\, \varepsilon^{k+\alpha}$, $1 \le i \le N$. Then for at least $(1-\tilde{\eta})N$ values of i, there is a $z_i \in C_\delta(\widetilde{z}_i)$ such that $\lambda(\{y \in C_\varepsilon(\widetilde{y}) \mid (z_i, y) \in A_i\} / \lambda(C_\varepsilon(\widetilde{y})) > 1 - \tilde{\eta}$ (here λ is the Lebesgue measure in $\mathbb{R}^{n-p}$), because $\lambda(C_\varepsilon(x) \cap A_i) > (1 - \tilde{\eta}^2)\lambda(C_\varepsilon(x))$.

For such an i, take an y_i such that $(z_i, y_i) \in A_i$. Then, applying Theorem 2.1, Corollaries 2.2 and 2.3, given η we can choose $\tilde{\eta}$ so that $|F(z_i, y) - F(z_i, y_i)| < \eta \cdot \varepsilon^{k+\alpha} \Rightarrow |F(z, y) - F(z_i, y_i)| \leq 2K'\sqrt{n}.\eta\varepsilon^{K+\alpha} + \eta\varepsilon^{K+\alpha} \leq K''\eta\varepsilon^{k+\alpha}$, for some constant K'' and for any $z \in C_\varepsilon(\widetilde{z}_i)$, where K' is a Lipschitz constant for $F \Rightarrow F(C_\delta(\widetilde{z}_i) \times C_\varepsilon(\widetilde{y}))$ is contained in a ball $B_{\delta_i}(Q_i)$, with $\sum_i \delta_i^d$ over these values of i less than

$$
\begin{aligned}
&([\varepsilon^{1-(K+\alpha)}\eta^{-1}] + 1)^p (K''\eta\varepsilon^{k+\alpha})^d \\
&\qquad = (K'')^d([\varepsilon^{1-(k+\alpha)}.\eta^{-1}] + 1)^p(\eta\varepsilon^{k+\alpha})^{p+\frac{n-p}{k+\alpha}} \leq \widetilde{K}_0\eta^{\frac{n-p}{k+\alpha}}.\varepsilon^n
\end{aligned}
$$

for some constant $\widetilde{K}_0$. The union of the remaining (at most $\tilde{\eta}N$) boxes has volume at most $\tilde{\eta}\varepsilon^n \Rightarrow$ the union of the image of the intersection of A_i with the union of these boxes by F is contained in a union of balls $B_{\delta_i}(Q_i)$ with $\sum_i \delta_i^d \leq 2K_0\tilde{\eta}\varepsilon^n$, by the statement proved before, and so $F(C_\varepsilon(x))$ can be covered by balls $B_{\delta_i}(\widetilde{Q}_i)$ with $\Sigma\delta_i^d \leq (\widetilde{K}_0\eta^{\frac{n-p}{k+\alpha}} + 2K_0\tilde{\eta})\varepsilon^n$. Choosing η, $\tilde{\eta}$ so small that $\widetilde{K}_0\eta^{\frac{n-p}{k+\alpha}} + 2K_0\tilde{\eta} \leq \eta'$, we obtain the desired result with $\varepsilon_0 = \tilde{\varepsilon}$. □

Remark 3.5. In the cases of functions of class C^k ($C^{k+\alpha}$ with $\alpha = 0$) we have the same result. If $k \geq 2$, it follows from the case of class $C^{k-1+(1)}$ of the theorem. If $k = 1$, $p + \frac{n-p}{k+\alpha} = n$, and the proof of the Theorem 3.3 shows that if $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is a function and $C(F) = \{x \in U \mid DF(x) \text{ exists and } \operatorname{rank} DF(x) < n\}$ then $m_n(F(C(F))) = 0$, where m_n is the Hausdorff measure of dimension n.

4. Examples

In this section we give some examples which show that the previous results are quite sharp. In all these examples we shall use a certain kind of functions of the real line that we shall describe below.

Definition 4.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $0 < \lambda_i < \frac{1}{2}$, $\forall\, i \in \mathbb{N}$. The central Cantor set $K_{\underline{\lambda}}$ is the Cantor set constructed as follows: We remove from the interval $[0, 1]$ the central open interval $U_{1,1}$ of proportion $1 - 2\lambda_1$, then we remove from the two remaining intervals the central open intervals $U_{2,1}$ and $U_{2,2}$ of proportion $1 - 2\lambda_2$, and so on. After the r-th step of the construction there will remain 2^r intervals of length $\lambda_1\lambda_2 \ldots \lambda_r$. The intersection of all these sets is the central Cantor set $K_{\underline{\lambda}}$. The open intervals removed in the r-th step of the construction have length $\lambda_1\lambda_2 \ldots \lambda_{r-1}(1 - 2\lambda_r)$.

Let $\psi\colon \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$ be a fixed function such that $\psi(\mathbb{R}) \subset [0,1]$, $\psi(x) = 0$, $\forall\, x \le 0$, $\psi(x) = 1$, $\forall\, x \ge 1$. Given two central Cantor sets $K_{\underline{\lambda}}$ and $K_{\underline{\mu}}$, we construct the function $f_{\underline{\lambda},\underline{\mu}}\colon \mathbb{R} \to \mathbb{R}$ as follows: $f_{\underline{\lambda},\underline{\mu}}(x) = 0$, $\forall\, x \le 0$, $f_{\underline{\lambda},\underline{\mu}}(x) = 1$, $\forall\, x \ge 1$, and if $U_{i,j} = (a,b)$ and $v_{i,j} = (c,d)$ are corresponding removed intervals in the constructions of $K_{\underline{\lambda}}$ and $K_{\underline{\mu}}$, respectively, we define $f_{\underline{\lambda},\underline{\mu}}(x) = c + (d-c)\psi(\frac{x-a}{b-a})$, $\forall\, x \in (a,b)$. We extend $f_{\underline{\lambda},\underline{\mu}}$ to $K_{\underline{\lambda}}$ by continuity, obtaining $f_{\underline{\lambda},\underline{\mu}}(K_{\underline{\lambda}}) = K_{\underline{\mu}}$. It is easy to check that if $g_r := \lambda_1\lambda_2\dots\lambda_{r-1}(1-2\lambda_r)$ and $\tilde{g}_r := \mu_1\mu_2\dots\mu_{r-1}(1-2\mu_r)$ satisfy $\lim\limits_{r\to\infty} \frac{\tilde{g}_r}{g_r^k} = 0$ then $f_{\underline{\lambda},\underline{\mu}}$ is C^k (if $k \ge 1$ is an integer). Moreover, if $q > 1$, and $\sup\limits_{r} \to \frac{\tilde{g}_r}{g_r^q} < \infty$ then $f_{\underline{\lambda},\underline{\mu}}$ is $C^{q-1,1}$ if q is integer and is C^q (i.e., it is $C^{[q]+\{q\}}$, where $\{q\} = q - [q] \in (0,1)$) otherwise. See [**BMPV**] for more details.

Example 4.1. Let $\lambda_n = \frac{1}{2} - \frac{1}{2n}$, $\mu_n \equiv a$. Then $\lim\limits_{n\to\infty} \frac{\tilde{g}_n}{g_n^q} = 0$, $\forall\, q < \frac{-\log a}{\log 2}$, and so $f_{\underline{\lambda},\underline{\mu}}$ is C^q, $\forall\, q < \frac{-\log a}{\log 2}$. On the other hand, $m_d(K_{\underline{\mu}}) = 1$ where $d = \frac{-\log 2}{\log a}$ (see [**PT**]). Moreover, since $a \in \left(0, \frac{1}{2}\right)$, $\lim\limits_{n\to\infty} \frac{\tilde{g}_n}{g_n} = 0$, and so $f'_{\underline{\lambda},\underline{\mu}}(x) = 0$, $\forall\, x \in K_{\underline{\lambda}}$. If $F\colon \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ is given by

$$\begin{aligned} &F(x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+p}) \\ &\qquad = (f_{\underline{\lambda},\underline{\mu}}(x_1), f_{\underline{\lambda},\underline{\mu}}(x_2) \dots, f_{\underline{\lambda},\underline{\mu}}(x_n), x_{n+1} \dots, x_{n+p}), \end{aligned}$$

then

$$F(C_p(F)) = F(K_{\underline{\lambda}} \times K_{\underline{\lambda}} \times \dots \times K_{\underline{\lambda}} \times \mathbb{R}^p) = K_{\underline{\mu}} \times K_{\underline{\mu}} \times \dots \times K_{\underline{\mu}} \times \mathbb{R}^p$$

that is a set with positive $(nd+p)$-measure. This shows that given $q > 1$, $p > 0$ and $n > p$ there is a map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $m_d(F(C_p(F))) > 0$, where $d = p + \frac{n-p}{q}$, and F is of class $C^{q'}$ for each $q' < q$.

Remark 4.1. If $a = \frac{1}{2^n}$, $F(x_1, x_2, \dots, x_n) = f_{\underline{\lambda},\underline{\mu}}(x_1) + 2f_{\underline{\lambda},\underline{\mu}}(x_2) + \dots + 2^{n-1}f_{\underline{\lambda},\underline{\mu}}(x_n)$ gives an example of a function $F\colon \mathbb{R}^n \to \mathbb{R}$ which is of class C^q, $\forall\, q < n$ ($q \in \mathbb{R}$) such that $F(\mathbb{R}^n)$ contains an open set, since $K_{\underline{\mu}} + 2K_{\underline{\mu}} + \dots + 2^{n-1}K_{\underline{\mu}} = [0, 2^n - 1]$, which can be proved easily using representation in basis 2^n.

Example 4.2. Let $\lambda_n = \frac{1}{2} - \frac{1}{3n^2}$, $\mu_n = a - \frac{a}{3n}$, $a \in (0, 1/2]$. Then $\lim\limits_{n\to\infty} \frac{\tilde{g}_n}{g_n^q} = 0$, where $q = \frac{-\log a}{\log 2}$, and so $f_{\underline{\lambda},\underline{\mu}}$ is C^q. On the other hand we have $HD(K_{\underline{\mu}}) \ge \frac{-\log 2}{\log a}$. Indeed, if $b < a$ and $\theta_n \equiv b$, $f_{\underline{\mu},\underline{\theta}}$ is clearly C^1, and $f_{\underline{\mu},\underline{\theta}}(K_{\underline{\mu}}) = K_{\underline{\theta}} \Rightarrow HD(K_{\underline{\mu}}) \ge HD(K_{\underline{\theta}}) = \frac{-\log 2}{\log b}$, $\forall\, b < a$. If

$F\colon \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ is given by

$$F(x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+p}) = (f_{\underline{\lambda},\underline{\mu}}(x_1), \dots, f_{\underline{\lambda},\underline{\mu}}(x_n), x_{n+1}, \dots, x_{n+p})$$

then

$$F(C_p(F)) = K_{\underline{\mu}} \times \cdots \times K_{\underline{\mu}} \times \mathbb{R}^p,$$

that is a set with Hausdorff dimension $nd + p$, where $d = \frac{-\log 2}{\log a}$. This shows that given $q \geq 1$, $p > 0$ and $n > p$ there is a map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $HD(F(C_p(F))) = p + \frac{n-p}{q}$, and F is of class C^q.

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