Publicacions Matemàtiques, Vol. **44** (2000), 641–647

A SHORT PROOF OF A THEOREM OF BRODSKIĬ

James Howie

Abstract

A short proof, using graphs and groupoids, is given of Brodskiĭ's theorem that torsion-free one-relator groups are locally indicable.

1. Introduction

In 1980, Sergei Brodskiĭ announced [**2**] the result, previously conjectured by Gilbert Baumslag [**1**], that every torsion-free one-relator group is *locally indicable*, that is, every nontrivial, finitely generated subgroup has an infinite cyclic homomorphic image. His algebraic proof was published in full in 1984 [**3**]. Around the same time, I independently obtained Brodskiĭ's theorem, and published a slightly more general version in [**7**], with a topological proof: a one-relator quotient of a free product of locally indicable groups is locally indicable, provided the relator is neither a proper power nor conjugate to an element of one of the free factors. A further version of the theorem was later proved by John Hempel [**5**]: the quotient of a surface group by a single relator that is not a proper power is locally indicable.

This paper arose as a response to requests from colleagues —notably Warren Dicks— for a proof of Brodskiĭ's theorem more accessible than those in [**3**], [**7**]. In particular the topology used in [**7**] seemed to cause some difficulty. Here I present a straightforward proof of the theorem, using groupoids. It is essentially my proof from [**7**], restricted to the original case of a torsion-free one-relator group, with as much of the topology as possible translated into algebra. The only remaining topology is the notion of an infinite cyclic cover of a graph or groupoid. For more detailed background material on graphs and groupoids, the best reference is [**6**], but for completeness I have included some elementary definitions in §2 below, and a description of the construction of infinite cyclic covers in §3.

Acknowledgement. I first presented a version of this proof in a seminar to mark the retirement of my PhD supervisor, Philip Higgins. I am grateful to him for his support. I am also grateful to Warren Dicks for encouraging me to publish this material, and for pointing out Corollary 3.2.

2. Preliminaries

A *graph* Γ consists of a set $V = V(\Gamma)$ of *vertices* and a set $E = E(\Gamma)$ of *edges*, together with a map $i\colon E \to V$ (the *initial vertex map*), and a fixed-point-free involution $e \mapsto e^{-1}\colon E \to E$. The *terminal vertex map* $t\colon E \to V$ is defined by $t(e) = i(e^{-1})$. A *path* in Γ from the vertex u to the vertex v is a sequence $e_1, \dots, e_n$ of edges, with $i(e_1) = u$, $i(e_j) = t(e_{j-1})$ for $2 \le j \le n$, and $t(e_n) = v$. (We also call u the *initial vertex*, and v the *terminal vertex* of P.) The path P is *reduced* if $e_j \ne e_{j-1}^{-1}$ for all $2 \le j \le n$, and *closed* if $u = v$. A reduced closed path is *cyclically reduced* if, in addition, $e_n \ne e_1^{-1}$. A closed path is a *proper power* if it is obtained by repeating a closed path two or more times. A graph is *connected* if any two vertices are joined by a path. The set of all reduced paths forms a groupoid $F(\Gamma)$ under juxtaposition (followed by cancellation of any resulting inverse pairs of consecutive edges), called the *free groupoid* on Γ (see [**6**] for details). The set of all reduced closed paths at a vertex v forms a group $\pi(\Gamma, v)$, called the *path group*, or *fundamental group*, of Γ (based at v). It is equal to the vertex group at v of the groupoid $F(\Gamma)$. It is a free group, and every free group arises in this way.

A presentation $\langle \Gamma \mid R \rangle$ of a groupoid G consists of:

1) a graph Γ; and

2) a set R of cyclically reduced closed paths in Γ,

such that $G = F(\Gamma)/N(R)$, where $N(R)$ denotes the smallest normal subgroupoid containing R. The presentation is *staggered* if there are linear orderings on the sets R and $E = E(\Gamma)$ which are *compatible* in the sense that, if $\alpha, \beta \in R$ with $\alpha < \beta$, then $\max(\alpha) < \max(\beta)$ and $\min(\alpha) < \min(\beta)$, where max and min denote the greatest and least edges occurring in a path (under the given linear ordering on E).

A group G is *indicable* if it admits an infinite cyclic homomorphic image. It is *locally indicable* if every non-trivial, finitely generated subgroup is indicable.

3. The main result

Theorem 3.1. *Let G be a groupoid given by a staggered presentation $\langle \Gamma \mid R \rangle$ in which no element of R is a proper power. Then every vertex group of G is locally indicable.*

Brodskiĭ's theorem is the special case of Theorem 3.1 in which $V(\Gamma)$ and R are singleton sets (and $E(\Gamma)$ has an arbitrary ordering).

Corollary 3.2. *Any torsion-free subgroup of a one-relator group is locally indicable.*

Proof: Let $G = \langle X \mid r^m \rangle$ be a one-relator group, where $m \geq 2$ and r is not a proper power. Let $\bar{G} = \langle X \mid r \rangle$ be the corresponding torsion-free one-relator group. Then there is a short exact sequence

$$1 \to F \to G \to \bar{G} \to 1$$

in which F is a free product of cyclic groups [**4**]. If H is a torsion-free subgroup of G, then $H \cap F$ is free, so locally indicable. Hence H is an extension of a locally indicable group by a locally indicable group, so is locally indicable. □

Proof of Theorem 3.1: Suppose the theorem were false. Then for some $G = \langle \Gamma \mid R \rangle$ as in the theorem, and some vertex $v \in V(\Gamma)$, there would be a finitely generated, non-indicable subgroup $H \neq \{1\}$ of the vertex group G_v of G at v. Suppose H is generated by reduced closed paths $\gamma_1, \ldots, \gamma_n$ at v. Since H is non-indicable, it has finite abelianisation, and so there are n words $W_1, \ldots, W_n$ in the free group on n generators $x_1, \ldots, x_n$, such that:

1) the abstract group $\langle x_1, \ldots, x_n \mid W_1, \ldots, W_n \rangle$ has finite abelianisation; and
2) each path $W_j(\gamma_1, \ldots, \gamma_n)$ $(1 \leq j \leq n)$ belongs to $N(R)$.

Because of 2) there is an identity:

$$(1) \qquad W_j(\gamma_1, \ldots, \gamma_n) = (\delta_{j,1}\alpha_{j,1}\delta_{j,1}^{-1}) \cdots (\delta_{j,m(j)}\alpha_{j,m(j)}\delta_{j,m(j)}^{-1})$$

for each j, where each $\alpha_{j,k}$ is an element of R or its inverse, and each $\delta_{j,k}$ is a path in Γ from v to the initial (and terminal) vertex of $\alpha_{j,k}$. We will refer to the collection of paths γ_j, words W_j and identities (1) as a *datum*, Δ say. There is nothing in the definition of a datum which enforces the nontriviality of the subgroup H generated by the γ_j, so data exist for the trivial subgroup also. In fact, we will prove the theorem by showing that, for any datum as above, the corresponding subgroup H vanishes. We will do this by induction on $L(\Delta) - M(\Delta)$, where $L(\Delta)$ is the sum of the lengths of all the paths $\delta_{j,k}$ and $\alpha_{j,k}$, and $M(\Delta)$ is the number of *distinct* vertices visited by these paths. Clearly $M(\Delta) \leq L(\Delta)$, so induction on $L(\Delta) - M(\Delta)$ makes sense.

The first step is to replace Γ by the smallest subgraph Γ_0 containing all the paths $\delta_{j,k}$ and $\alpha_{j,k}$ (and hence all the γ_j), and R by the subset $R_0 = \{\alpha_{j,k} \mid 1 \leq j \leq n, 1 \leq k \leq m(j)\}$. Note that Γ_0 is a finite graph, and R_0 is a finite set. This gives a new (finite) presentation $\langle \Gamma_0 \mid R_0 \rangle$ of a groupoid G_0; the paths γ_j generate a subgroup H_0 of G_0; the inclusion of Γ_0 in Γ induces a natural homomorphism $G_0 \to G$ which maps H_0 onto H; and the presentation of G_0 is staggered under the restriction of the orders on $E(\Gamma)$ and R to $E(\Gamma_0)$ and R_0 respectively. In particular, if we prove that $H_0 = \{1\}$, then it follows that $H = \{1\}$, as desired.

From now on, we assume that $G = G_0$, etc.

Case 1: Assume that the vertex group G_v of G at v is indicable.

Choose an epimorphism $G_v \to \mathbb{Z}$ of groups and extend it to an epimorphism $\theta\colon G \to \mathbb{Z}$ of groupoids. Corresponding to θ we construct *infinite cyclic coverings* Γ' of Γ and G' of G as follows. Firstly we define $V(\Gamma') := V(\Gamma) \times \mathbb{Z}$; $E(\Gamma') := E(\Gamma) \times \mathbb{Z}$; $i(e,n) := (i(e),n)$ and $(e,n)^{-1} := (e^{-1}, n + \theta(e))$ to get a graph Γ'. The projections onto the first coordinates determine a graph homomorphism $\pi\colon \Gamma' \to \Gamma$, called a *covering projection*. This satisfies the (easily verified) *path lifting property*: given any path P in Γ, beginning at a vertex v, say, and any integer n, there is a unique path P_n in Γ', beginning at (v,n), with $\pi(P_n) = P$. (We call P_n the *lift* of P beginning at (v,n).) If P ends at a vertex u, then P_n ends at $(u, n + \theta(P))$. In particular, for each $r \in R$, each r_n is a closed path, since r is closed and $\theta(r) = 0$. Let R' be the set $\{r_n \mid r \in R, n \in \mathbb{Z}\}$ of closed paths in Γ', and define G' to be the groupoid $\langle \Gamma' \mid R' \rangle$.

Since H is non-indicable, we must have $\theta(H) = 0$. Hence each path γ_j lifts to a closed path γ'_j at $v' := (v,0)$. If $\delta'_{j,k}$ is the lift of $\delta_{j,k}$ that begins at v', and $\alpha'_{j,k}$ is the lift of $\alpha_{j,k}$ that begins at the terminal vertex of $\delta_{j,k}$, then we have identities

$$(2) \quad W_j(\gamma'_1, \ldots, \gamma'_n) = (\delta'_{j,1}\alpha'_{j,1}(\delta'_{j,1})^{-1}) \cdots (\delta'_{j,m(j)}\alpha'_{j,m(j)}(\delta'_{j,m(j)})^{-1})$$

for each $1 \leq j \leq n$.

We introduce linear orderings on $E(\Gamma')$ and R' by:

$$(e,n) < (f,m) \quad \text{if } e < f \quad \text{or if } e = f \text{ and } n < m;$$
$$r_n < s_m \quad \text{if } r < s \quad \text{or if } r = s \text{ and } n < m.$$

It is clear that these are compatible, and hence that $\langle \Gamma' \mid R' \rangle$ is a staggered presentation.

Let H' be the subgroup of G' generated by the paths γ'_j $(1 \le j \le n)$, and let Δ' be the datum consisting of the γ'_j, W_j and identities (2). Then H' is non-indicable, by the identities (2), and H' is mapped onto H by π. We show that the inductive hypothesis applies to H'. It follows that H', and hence H, vanishes.

Clearly $L(\Delta') = L(\Delta)$. Let Γ_1 be the smallest subgraph of Γ' containing all the paths $\delta'_{j,k}$ and $\alpha'_{j,k}$. Then $M(\Delta') = |V(\Gamma_1)|$ and $M(\Delta) = |V(\Gamma)|$. Moreover, Γ_1 is mapped surjectively onto Γ by π, and it suffices to show that this surjection is proper on vertices. If not, then by construction the restriction of π to Γ_1 must be bijective both on edges and on vertices, so a graph isomorphism $\Gamma_1 \to \Gamma$. But there is at least one closed path β in Γ at v with $\theta(\beta) = 1$. Under the graph isomorphism $\pi^{-1}\colon \Gamma \to \Gamma_1$, β is mapped onto the unique path β' beginning at $v' = (v, 0)$ such that $\pi(\beta') = \beta$. But by definition β' ends at $(v, \theta(\beta)) = (v, 1) \neq (v, 0)$. Hence $(v, 0), (v, 1) \in V(\Gamma_1)$ with $\pi(v, 0) = \pi(v, 1) = v$, contradicting the assumption that $\pi\colon V(\Gamma_1) \to V(\Gamma)$ is injective.

This contradiction completes the proof in Case 1.

Case 2: Now assume that G_v is not indicable.

Note that the argument in Case 1 shows that this must include the initial case of the induction.

The proof in this case is a second induction, this time on the number of elements in the relation set R. If $R = \emptyset$, then $G = F(\Gamma)$ is a free groupoid, so G_v is a free group. But G_v is also non-indicable, so $G_v = \{1\}$ and hence $H = \{1\}$.

If $R = \{r\}$ is a singleton set, then G_v is a one-relator group. Since G_v is non-indicable, it must be finite cyclic, and so $\pi(\Gamma, v)$ is cyclic. In other words, Γ has first Betti number 1, and so contains a single nontrivial cycle. Since r is cyclically reduced and not a proper power, r is this cycle (traversed in one of the two possible directions), so again $H = G_v = \{1\}$. Moreover, note that each edge in r occurs precisely once in r.

For the general case, we take the slightly stronger property noted above to be the inductive hypothesis: namely that $G_v = \{1\}$ and every edge occurring in any relation $r \in R$ occurs precisely once in r.

Now suppose that $r_{\max}$ is the greatest relation in R (with respect to the given linear ordering). Let $e = \max(r_{\max})$. Suppose first that $\Gamma'' = \Gamma \setminus \{e\}$ is connected. Then $G'' = \langle \Gamma'' \mid R \setminus \{r_{\max}\} \rangle$ cannot have indicable vertex groups, for then so would G. By induction each vertex group of G'' is trivial, and each edge occurring in each relator occurs precisely once in that relator. In particular $f = \min(r_{\min})$ occurs precisely once in $r_{\min}$, where $r_{\min}$ is the least relator in $R \setminus \{r_{\max}\}$ (and hence in R). Thus $G_2 = \langle \Gamma \setminus \{f\} \mid R \setminus \{r_{\min}\} \rangle$ is isomorphic to G, so has nonindicable vertex groups. By inductive hypothesis the vertex groups of G_2 are trivial, and each edge occurring in any of its relators occurs precisely once in that relator. Hence the same is true for G, and we are done.

A similar argument works if we suppose that G'' has two components Γ_3 and Γ_4, say. For each relator other than $r_{\max}$ must be a path in one of Γ_3, Γ_4, so $R \setminus \{r_{\max}\}$ splits as a disjoint union $R_3 \cup R_4$, and we have two groupoids $G_3 = \langle \Gamma_3 \mid R_3 \rangle$ and $G_4 = \langle \Gamma_4 \mid R_4 \rangle$. Since G has nonindicable vertex groups, so does at least one of G_3, G_4 (say G_3). Now R_3 cannot be empty, for then Γ_3 would be a tree, and no cyclically reduced closed path could contain $e = \max(r_{\max})$, a contradiction. Now apply the same argument as above, taking $r_{\min}$ to be the least relator in R_3.

This completes the proof. □

References

[1] G. Baumslag, Some problems on one-relator groups, in: *"Proceedings of the Second International Conference on the Theory of Groups"* (Australian Nat. Univ., Canberra, 1973), Lecture Notes in Math. **372**, Springer, Berlin, 1974, pp. 75–81.

[2] S. D. Brodskiĭ, Equations over groups and groups with one defining relation, *Uspekhi Mat. Nauk* **35(4)** (1980), 183; *Russian Math. Surveys* **35(4)** (1980), 165.

[3] S. D. Brodskiĭ, Equations over groups and groups with one defining relation, *Sibirsk. Mat. Zh.* **25(2)** (1984), 84–103; *Siberian Math. J.* **25(2)** (1984), 235–251.

[4] J. Fischer, A. Karrass and D. Solitar, On one-relator groups having elements of finite order, *Proc. Amer. Math. Soc.* **33** (1972), 297–301.

[5] J. Hempel, One-relator surface groups, *Math. Proc. Cambridge Philos. Soc.* **108(3)** (1990), 467–474.

[6] P. J. Higgins, *"Notes on categories and groupoids"*, Van Nostrand Reinhold Mathematical Studies **32**, Van Nostrand Reinhold Co., London-New York-Melbourne, 1971.

[7] J. Howie, On locally indicable groups, *Math. Z.* **180(4)** (1982), 445–461.

Department of Mathematics
Heriot-Watt University
Riccarton
Edinburgh EH14 4AS
United Kingdom
E-mail address: J.Howie@hw.ac.uk

Primera versió rebuda el 27 de juny de 2000,
darrera versió rebuda el 2 de setembre de 2000.