Publicacions Matemàtiques, Vol. **44** (2000), 135–155

HOMOGENOUS BANACH SPACES ON THE UNIT CIRCLE

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Abstract

We prove that a homogeneous Banach space $\mathcal{B}$ on the unit circle $\mathbb{T}$ can be embedded as a closed subspace of a dual space $\Xi^*_{\mathcal{B}}$ contained in the space of bounded Borel measures on $\mathbb{T}$ in such a way that the map $\mathcal{B} \mapsto \Xi^*_{\mathcal{B}}$ defines a bijective correspondence between the class of homogeneous Banach spaces on $\mathbb{T}$ and the class of prehomogeneous Banach spaces on $\mathbb{T}$.

We apply our results to show that the algebra of all continuous functions on $\mathbb{T}$ is the only homogeneous Banach algebra on $\mathbb{T}$ in which every closed ideal has a bounded approximate identity with a common bound, and that the space of multipliers between two homogeneous Banach spaces is a dual space. Finally, we describe the space $\Xi^*_{\mathcal{B}}$ for some examples of homogeneous Banach spaces $\mathcal{B}$ on $\mathbb{T}$.

1. Introduction

Every Banach space $\mathcal{Z}$ is a closed subspace of its second dual space $\mathcal{Z}^{**}$, but the restriction of the weak star (wk*) topology on $\mathcal{Z}^{**}$ to $\mathcal{Z}$ (which is of course just the weak topology on $\mathcal{Z}$) need not be particularly useful. Loosely speaking, this is because the space $\mathcal{Z}^{**}$ is too large and we are therefore interested in embedding $\mathcal{Z}$ in a smaller dual space. In this paper, we show that a Banach space $\mathcal{B}$ of functions on the unit circle $\mathbb{T}$ satisfying certain conditions can be embedded in a dual space contained in the space $\mathcal{M}$ of bounded Borel measures on $\mathbb{T}$ in such a way, that convergence of a sequence in $\mathcal{B}$ in the wk* topology depends only on certain simple properties of the sequence. For some Banach spaces $\mathcal{B}$ of functions on $\mathbb{T}$, there is a "natural" dual space in which $\mathcal{B}$ is embedded. Examples of such embeddings are that of L^1 in $\mathcal{M}$, of $\mathcal{C}$ (the algebra

1991 *Mathematics Subject Classification.* 46E30, 46E15, 46J10.

*Supported by the Danish Natural Science Research Council and by a TMR Marie Curie postdoctoral grant from the European Commission.

of all continuous functions on $\mathbb{T}$) in L^∞ and of the little Lipschitz algebra λ_γ in the big Lipschitz algebra Λ_γ. We shall see that our general approach reproduces these examples. We wish to thank the referee for pointing out a mistake in our original formulation of Theorem 4.3.

Following Shilov ([**20**], see also [**13**]), we say that a Banach space $\mathcal{B}$ continuously embedded in L^1 is a homogeneous Banach space on $\mathbb{T}$ if it satisfies the following condition. For $f \in \mathcal{B}$ and $s \in \mathbb{T}$, the translate

$$f_s(t) = f(t-s) \quad (t \in \mathbb{T})$$

belongs to $\mathcal{B}$ with $\|f_s\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ and $f_s \to f$ in $\mathcal{B}$ as $s \to 0$. For $s \in \mathbb{T}$, the translation operator $f \mapsto f_s$ $(f \in \mathcal{B})$ is denoted by R_s.

For $f \in L^1$, let $\widehat{f}(n)$ $(n \in \mathbb{Z})$ be the Fourier coefficients of f and, for $N \in \mathbb{N}$, let

$$\sigma_N(f) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n)\alpha^n$$

be the N'th Fejér sum of f (where $\alpha^n(t) = e^{int}$ for $t \in \mathbb{T}$). The following result ([**20**, pp. 12–13] or [**13**, Theorem I.2.11]) is essential for our approach.

Theorem 1.1 (Shilov). *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and let $f \in \mathcal{B}$. If $\widehat{f}(n) \neq 0$ for some $n \in \mathbb{Z}$, then $\alpha^n \in \mathcal{B}$. Moreover,*

$$\sigma_N(f) \to f$$

in $\mathcal{B}$ as $N \to \infty$ and $\|\sigma_N(f)\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}}$ for $N \in \mathbb{N}$. In particular, the trigonometric polynomials in $\mathcal{B}$ are dense in $\mathcal{B}$.

For $n \in \mathbb{Z}$, define $\xi_n \in \mathcal{B}^*$ by

$$\langle f, \xi_n \rangle = \widehat{f}(n) \quad (f \in \mathcal{B}).$$

In passing, we remark that the definition of a homogeneous Banach space $\mathcal{B}$ on $\mathbb{T}$ does not exclude the case where $\widehat{f}(n) = 0$ for every $f \in \mathcal{B}$ (that is $\xi_n = 0$) for some values of n. In particular, certain spaces of analytic functions are homogeneous Banach spaces on $\mathbb{T}$. (See Section 5 for further details.) For $\varphi \in \mathcal{B}^*$ and $n \in \mathbb{Z}$, let

$$\widehat{\varphi}(n) = \langle \alpha^n, \varphi \rangle.$$

Since the trigonometric polynomials in $\mathcal{B}$ are dense in $\mathcal{B}$, the map $\varphi \mapsto (\widehat{\varphi}(n))$ is 1-1 on $\mathcal{B}^*$.

In this paper, the subspace

$$\Xi_{\mathcal{B}} = \overline{\operatorname{span}\{\xi_n : n \in \mathbb{Z}\}}^{\mathcal{B}^*}$$

of $\mathcal{B}^*$ plays an important part. For $\varphi \in \mathcal{B}^*$, define the Fejér sums of φ by

$$\sigma_N(\varphi) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{\varphi}(n)\xi_n \quad (N \in \mathbb{N}),$$

and, for $s \in \mathbb{T}$, define the translate $\varphi_s \in \mathcal{B}^*$ by $\langle f, \varphi_s \rangle = \langle f_s, \varphi \rangle$ $(f \in \mathcal{B})$. Since $(\xi_n)_s = e^{-ins}\xi_n$ $(s \in \mathbb{T}, n \in \mathbb{Z})$, it follows that $\Xi_{\mathcal{B}}$ is translation invariant. The previous theorem leads to the following result in the dual space.

Proposition 1.2. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and let $\varphi \in \mathcal{B}^*$. Then $\sigma_N(\varphi) \to \varphi$ wk^* in $\mathcal{B}^*$ as $N \to \infty$ and $\|\sigma_N(\varphi)\|_{\mathcal{B}^*} \leq \|\varphi\|_{\mathcal{B}^*}$ for $N \in \mathbb{N}$. In particular, the unit ball of $\Xi_{\mathcal{B}}$ is wk^* sequentially dense in the unit ball of $\mathcal{B}^*$. Moreover, $\varphi_s \to \varphi$ wk^* in $\mathcal{B}^*$ as $s \to 0$. For $\varphi \in \Xi_{\mathcal{B}}$, we have $\sigma_N(\varphi) \to \varphi$ in $\Xi_{\mathcal{B}}$ as $N \to \infty$ and $\varphi_s \to \varphi$ in $\Xi_{\mathcal{B}}$ as $s \to 0$.*

Proof: For $f \in \mathcal{B}$ and $N \in \mathbb{N}$, we have

$$\langle f, \sigma_N(\varphi) \rangle = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n)\widehat{\varphi}(n) = \langle \sigma_N(f), \varphi \rangle,$$

so the first two conclusions follows from the previous theorem. Also, the maps $\varphi \mapsto \sigma_N(\varphi)$ define a bounded sequence of operators on $\Xi_{\mathcal{B}}$, and since $\sigma_N(\xi_m) \to \xi_m$ in $\Xi_{\mathcal{B}}$ as $N \to \infty$ for $m \in \mathbb{Z}$, it follows that $\sigma_N(\varphi) \to \varphi$ in $\Xi_{\mathcal{B}}$ for $\varphi \in \Xi_{\mathcal{B}}$. The results about φ_s follow in the same way. □

2. Embedding a homogeneous Banach space in a dual space

We shall now focus on the space $\Xi_{\mathcal{B}}^*$. For $\mu \in \Xi_{\mathcal{B}}^*$ and $n \in \mathbb{Z}$, let

$$\widehat{\mu}(n) = \langle \xi_n, \mu \rangle.$$

By definition of $\Xi_{\mathcal{B}}$, it follows that the map $\mu \mapsto (\widehat{\mu}(n))$ is 1-1 on $\Xi_{\mathcal{B}}^*$. The following is the key result of this paper.

Theorem 2.1. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and define*

$$\Phi : \mathcal{B} \to \Xi_{\mathcal{B}}^*$$

by

$$\langle \varphi, \Phi(f) \rangle = \langle f, \varphi \rangle$$

for $f \in \mathcal{B}$ and $\varphi \in \Xi_{\mathcal{B}}$. (That is, $\Phi(f) = \iota(f)|_{\Xi_{\mathcal{B}}}$ $(f \in \mathcal{B})$, where $\iota : \mathcal{B} \to \mathcal{B}^{**}$ is the canonical embedding.) Then Φ is an isometry and

$$\widehat{\Phi(f)}(n) = \widehat{f}(n)$$

for $f \in \mathcal{B}$ and $n \in \mathbb{Z}$. Moreover, a sequence (μ_m) in $\Xi_{\mathcal{B}}^*$ converges wk* *to* 0 *in $\Xi_{\mathcal{B}}^*$ as $m \to \infty$ if and only if the sequence is bounded in $\Xi_{\mathcal{B}}^*$ and $\widehat{\mu_m}(n) \to 0$ as $m \to \infty$ for every $n \in \mathbb{Z}$.*

Proof: Let $f \in \mathcal{B}$. It follows from Proposition 1.2 that

$$\|f\|_{\mathcal{B}} = \sup\{|\langle f, \varphi\rangle| : \varphi \in \Xi_{\mathcal{B}},\ \|\varphi\|_{\mathcal{B}^*} \le 1\} = \|\Phi(f)\|_{\Xi_{\mathcal{B}}^*}.$$

Also,

$$\widehat{\Phi(f)}(n) = \langle \xi_n, \Phi(f)\rangle = \langle f, \xi_n\rangle = \widehat{f}(n) \quad (n \in \mathbb{Z}),$$

so $\widehat{\Phi(f)} = \widehat{f}$. The last statement follows immediately, since $\operatorname{span}\{\xi_n : n \in \mathbb{Z}\}$ is dense in $\Xi_{\mathcal{B}}$. □

Because of the theorem, we shall from now on consider $\mathcal{B}$ as a closed subspace of $\Xi_{\mathcal{B}}^*$ and identify $\Phi(f)$ and f for $f \in \mathcal{B}$. We shall use the following simple result several times.

Lemma 2.2. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and suppose that $\widehat{\mu}(n) \ne 0$ for some $\mu \in \Xi_{\mathcal{B}}^*$ and $n \in \mathbb{Z}$. Then $\alpha^n \in \mathcal{B}$. In particular, if $p \in \Xi_{\mathcal{B}}^*$ is a trigonometric polynomial, then $p \in \mathcal{B}$.*

Proof: Since $\langle \xi_n, \mu\rangle = \widehat{\mu}(n) \ne 0$, we have $\xi_n \ne 0$ in $\Xi_{\mathcal{B}} \subseteq \mathcal{B}^*$ and thus $\widehat{f}(n) \ne 0$ for some $f \in \mathcal{B}$. Hence $\alpha^n \in \mathcal{B}$ by Theorem 1.1. The last statement is now obvious. □

For $\mu \in \Xi_{\mathcal{B}}^*$ and $N \in \mathbb{N}$, let

$$\sigma_N(\mu) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{\mu}(n)\alpha^n$$

be the Fejér sums of μ. Then $\sigma_N(\mu) \in \mathcal{B}$ by the previous lemma.

Corollary 2.3. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$. For $\mu \in \Xi_{\mathcal{B}}^*$, we have*

$$\sigma_N(\mu) \to \mu \quad \text{wk}^*$$

in $\Xi_{\mathcal{B}}^*$ as $N \to \infty$ and $\|\sigma_N(\mu)\|_{\mathcal{B}} \le \|\mu\|_{\Xi_{\mathcal{B}}^*}$ for $N \in \mathbb{N}$.

Proof: For $\varphi \in \Xi_{\mathcal{B}}$ and $N \in \mathbb{N}$, we have

$$\langle \varphi, \sigma_N(\mu) \rangle = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{\varphi}(n)\widehat{\mu}(n) = \langle \sigma_N(\varphi), \mu \rangle.$$

Since $\|\sigma_N(\varphi)\|_{\Xi_{\mathcal{B}}} \leq \|\varphi\|_{\Xi_{\mathcal{B}}}$ by Proposition 1.2, it follows that $\|\sigma_N(\mu)\|_{\mathcal{B}} \leq \|\mu\|_{\Xi^*_{\mathcal{B}}}$. Also, $(\sigma_N(\mu))\widehat{\ }(m) \to \widehat{\mu}(m)$ as $N \to \infty$ for $m \in \mathbb{Z}$, so $\sigma_N(\mu) \to \mu$ wk* in $\Xi^*_{\mathcal{B}}$ as $N \to \infty$ by Theorem 2.1. □

When $(L^1)^*$ and L^∞ are identified via the duality

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(t)g(-t)\,dt \quad (f \in L^1,\ g \in L^\infty),$$

the functional ξ_n is identified with the function α^n. Hence

$$\Xi_{L^1} = \overline{\operatorname{span}\{\alpha^n : n \in \mathbb{Z}\}}^{L^\infty} = \mathcal{C}$$

and thus

$$\Xi^*_{L^1} = \mathcal{M}.$$

In the general case, we shall now see that $\Xi^*_{\mathcal{B}}$ can be regarded as a subspace of $\mathcal{M}$. We first prove a more general version of the result.

Proposition 2.4. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be homogeneous Banach spaces on $\mathbb{T}$ and suppose that $\mathcal{B}_1$ is continuously embedded in $\mathcal{B}_2$. Let $\iota : \mathcal{B}_1 \to \mathcal{B}_2$ be the inclusion map. Then $\rho = \iota^*|_{\Xi_{\mathcal{B}_2}}$ maps $\Xi_{\mathcal{B}_2}$ into a dense subset of $\Xi_{\mathcal{B}_1}$ and*

$$\rho^* : \Xi^*_{\mathcal{B}_1} \to \Xi^*_{\mathcal{B}_2}$$

is 1-1. Moreover, if $\mathcal{B}_1$ is dense in $\mathcal{B}_2$, then ρ is 1-1 and the range of ρ^* is wk* *dense in $\Xi^*_{\mathcal{B}_2}$.*

Proof: We have $\rho(\xi_n) = \xi_n$ for $n \in \mathbb{Z}$, so ρ maps $\Xi_{\mathcal{B}_2}$ into $\Xi_{\mathcal{B}_1}$ and has dense range. Hence ρ^* is 1-1. If $\mathcal{B}_1$ is dense in $\mathcal{B}_2$, then ρ is clearly 1-1 and the range of ρ^* is thus wk* dense in $\Xi^*_{\mathcal{B}_2}$. □

With $\mathcal{B}_2 = L^1$, we obtain the following.

Corollary 2.5. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and let $\iota : \mathcal{B} \to L^1$ be the inclusion map. Then $\rho = \iota^*|_{\mathcal{C}}$ maps $\mathcal{C}$ into a dense subset of $\Xi_{\mathcal{B}}$ and*

$$\rho^* : \Xi^*_{\mathcal{B}} \to \mathcal{M}$$

is 1-1. Moreover, if $\mathcal{B}$ is dense in L^1, then ρ is 1-1 and the range of ρ^* is wk* *dense in $\mathcal{M}$.*

From now on, we consider $\Xi_{\mathcal{B}}^*$ as a subspace of $\mathcal{M}$. For $\mu \in \Xi_{\mathcal{B}}^*$ and $s \in \mathbb{T}$, the translate μ_s is defined as a measure by $\mu_s(E) = \mu(E - s)$ for measurable sets $E \subseteq \mathbb{T}$. Since $\sigma_N(\mu_s) = \sigma_N(\mu)_s$ is a bounded sequence in $\mathcal{B}$ and since $\widehat{\sigma_N(\mu_s)}(m) \to \widehat{\mu_s}(m)$ as $N \to \infty$ for $m \in \mathbb{Z}$, we deduce that $\sigma_N(\mu_s) \to \mu_s$ wk* in $\Xi_{\mathcal{B}}^*$ as $N \to \infty$, so $\mu_s \in \Xi_{\mathcal{B}}^*$ with $\|\mu_s\|_{\Xi_{\mathcal{B}}^*} = \|\mu\|_{\Xi_{\mathcal{B}}^*}$. (Alternatively, $\mu_s \in \Xi_{\mathcal{B}}^*$ could be defined by $\langle \varphi, \mu_s \rangle = \langle \varphi_s, \mu \rangle$ $(\varphi \in \Xi_{\mathcal{B}})$.)

As a special case of strong Bochner integrals ([**10**, Chapter 3]), recall the following definition. Let $\mathcal{Z}$ be a Banach space and let $h : \mathbb{T} \to \mathcal{Z}^*$ be a wk* measurable map (that is, the map $t \mapsto \langle z, h(t) \rangle$ $(\mathbb{T} \to \mathbb{C})$ is measurable for every $z \in \mathcal{Z}$) with $\int_{\mathbb{T}} \|h(t)\|\,dt < \infty$. Define the wk* Bochner integral $\int_{\mathbb{T}} h(t)\,dt \in \mathcal{Z}^*$ by

$$\left\langle z, \int_{\mathbb{T}} h(t)\,dt \right\rangle = \int_{\mathbb{T}} \langle z, h(t) \rangle\,dt$$

for $z \in \mathcal{Z}$. Also, for $f \in L^1$ and $\mu \in \mathcal{M}$, let

$$(f * \mu)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t-s)\,d\mu(s) \quad (t \in \mathbb{T})$$

be the convolution of f and μ.

Theorem 2.6. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and let $\mu \in \Xi_{\mathcal{B}}^*$. Then the map $s \mapsto \mu_s$ $(\mathbb{T} \to \Xi_{\mathcal{B}}^*)$ is* wk* *continuous, and, for $f \in L^1$, the integral*

$$\frac{1}{2\pi} \int_{\mathbb{T}} f(s) \mu_s\,ds$$

exists as a wk* *Bochner integral in $\Xi_{\mathcal{B}}^*$ and equals $f * \mu$. Also, $\mu_s \to \mu$ in $\Xi_{\mathcal{B}}^*$ as $s \to 0$ if and only if $\mu \in \mathcal{B}$.*

Proof: We have $\|\mu_s\|_{\Xi_{\mathcal{B}}^*} = \|\mu\|_{\Xi_{\mathcal{B}}^*}$ for $s \in \mathbb{T}$ and $\widehat{\mu_s}(n) = e^{-ins}\widehat{\mu}(n) \to \widehat{\mu}(n)$ as $s \to 0$ for $n \in \mathbb{Z}$, so $\mu_s \to \mu$ wk* in $\Xi_{\mathcal{B}}^*$ as $s \to 0$ by Theorem 2.1. For $f \in L^1$, it thus follows that $\int_{\mathbb{T}} f(s)\mu_s\,ds$ exists as a wk* Bochner integral in $\Xi_{\mathcal{B}}^*$. Furthermore, Bochner integrals commute with continuous linear functionals, so

$$\begin{aligned}\left(\frac{1}{2\pi} \int_{\mathbb{T}} f(s)\mu_s\,ds\right)^{\wedge}(n) &= \frac{1}{2\pi} \int_{\mathbb{T}} \langle \xi_n, f(s)\mu_s \rangle\,ds = \frac{1}{2\pi} \int_{\mathbb{T}} f(s)\widehat{\mu_s}(n)\,ds \\ &= \widehat{\mu}(n) \frac{1}{2\pi} \int_{\mathbb{T}} f(s) e^{-ins}\,ds = \widehat{f}(n)\widehat{\mu}(n)\end{aligned}$$

for $n \in \mathbb{Z}$. Consequently $1/(2\pi) \int_{\mathbb{T}} f(s)\mu_s\,ds = f * \mu$.

Let $\mathcal{Z}$ be the space of those $\mu \in \Xi_{\mathcal{B}}^*$ for which $\mu_s \to \mu$ in $\Xi_{\mathcal{B}}^*$ as $s \to 0$. Then $\mathcal{B} \subseteq \mathcal{Z}$. On the other hand, for $\mu \in \mathcal{Z}$, we have $\mu_s \to \mu$ in $\mathcal{M}$ as $s \to 0$, so it follows from [**9**, V.19.27] that $\mu \in L^1$. Also, $\mathcal{Z}$ is closed in $\Xi^*_{\mathcal{B}}$, so we deduce that $\mathcal{Z}$ is a homogeneous Banach space on $\mathbb{T}$. Hence the trigonometric polynomials in $\mathcal{Z}$ are dense in $\mathcal{Z}$, so $\mathcal{Z} \subseteq \mathcal{B}$ by Lemma 2.2. □

Our aim now is to describe the spaces $\Xi^*_{\mathcal{B}}$ as so-called prehomogeneous Banach spaces on $\mathbb{T}$. To suit our purpose, we have weakened condition (i) in the following definition compared to the definition in [**12**, p. 64] (where it is required that $\mathcal{Y}$ contains every trigonometric polynomial).

A Banach space $\mathcal{Y}$ continuously embedded in $\mathcal{M}$ is called a prehomogeneous Banach space on $\mathbb{T}$ if the following conditions are satisfied.

(i) If $\mu \in \mathcal{Y}$ and $\widehat{\mu}(n) \neq 0$ for some $n \in \mathbb{Z}$, then $\alpha^n \in \mathcal{Y}$. Moreover, for every trigonometric polynomial p in $\mathcal{Y}$, we have $\|p_s\|_{\mathcal{Y}} = \|p\|_{\mathcal{Y}}$ for $s \in \mathbb{T}$ and $p_s \to p$ in $\mathcal{Y}$ as $s \to 0$.

(ii) For every trigonometric polynomial p and $\mu \in \mathcal{Y}$, we have $\|p * \mu\|_{\mathcal{Y}} \leq \|p\|_{L^1} \|\mu\|_{\mathcal{Y}}$. (Since $p * \mu = \sum \widehat{p}(n)\widehat{\mu}(n)\alpha^n$, we have $p * \mu \in \mathcal{Y}$ by (i).)

(iii) There is a topology, τ, on $\mathcal{Y}$ such that the unit ball of $\mathcal{Y}$ is τ compact and such that the map $\xi_n : \mathcal{Y} \to \mathbb{C}$ is τ continuous for $n \in \mathbb{Z}$.

The following is [**12**, Exercises 2 and 3, p. 64].

Lemma 2.7. *Let $\mathcal{Y}$ be a prehomogeneous Banach space on $\mathbb{T}$.*

(a) *A measure $\mu \in \mathcal{M}$ belongs to $\mathcal{Y}$ if and only if $(\sigma_N(\mu))$ is a bounded sequence in $\mathcal{Y}$.*

(b) *The closure $\mathcal{Y}_h$ in $\mathcal{Y}$ of the trigonometric polynomials in $\mathcal{Y}$ is a homogeneous Banach space on $\mathbb{T}$.*

Proof: (a) Let $\mu \in \mathcal{Y}$. Since $\sigma_N(\mu) = K_N * \mu$, where (K_N) is the Fejér kernel ([**13**, p. 12]), and since $\|K_N\|_{L^1} = 1$ for $N \in \mathbb{N}$, it follows from (ii) that $(\sigma_N(\mu))$ is a bounded sequence in $\mathcal{Y}$. Conversely, if $\mu \in \mathcal{M}$ and $(\sigma_N(\mu))$ is a bounded sequence in $\mathcal{Y}$, then it has a τ clusterpoint $\nu \in \mathcal{Y}$ as $N \to \infty$. Since ξ_n is τ continuous for $n \in \mathbb{Z}$, we deduce that $\mu = \nu \in \mathcal{Y}$.

(b) We have $\mathcal{Y}_h \subseteq L^1$ since L^1 is closed in $\mathcal{M}$. For $f \in \mathcal{Y}_h$, it follows from (i) that $\|f_s\|_{\mathcal{Y}} = \|f\|_{\mathcal{Y}}$ $(s \in \mathbb{T})$ and that $f_s \to f$ in $\mathcal{Y}$ as $s \to 0$. Hence $\mathcal{Y}_h$ is a homogeneous Banach space on $\mathbb{T}$. □

We shall now see that $\Xi^*_{\mathcal{B}}$ is a prehomogeneous Banach space on $\mathbb{T}$.

Theorem 2.8. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$. Equipped with the* wk* *topology, $\Xi^*_{\mathcal{B}}$ is a prehomogeneous Banach space on $\mathbb{T}$ and*

$$(\Xi^*_{\mathcal{B}})_h = \mathcal{B}.$$

Proof: Condition (i) follows from Lemma 2.2. Let p be a trigonometric polynomial and let $\mu \in \Xi^*_{\mathcal{B}}$. Then $p * \mu = 1/(2\pi)\int_{\mathbb{T}} p(s)\mu_s\,ds$ exists as a wk* Bochner integral in $\Xi^*_{\mathcal{B}}$ by Theorem 2.6. Hence $\|p*\mu\|_{\Xi^*_{\mathcal{B}}} \le \|p\|_{L^1}\|\mu\|_{\Xi^*_{\mathcal{B}}}$, so (ii) holds. Moreover, (iii) follows from Alaoglu's theorem and Theorem 2.1. Since $\mathcal{B}$ is the closure of the trigonometric polynomials in $\mathcal{B}$ and is closed in $\Xi^*_{\mathcal{B}}$, we deduce from Lemma 2.2 that $(\Xi^*_{\mathcal{B}})_h = \mathcal{B}$. □

For $\mu \in \Xi^*_{\mathcal{B}}$, we have $\sigma_N(\mu) \in \mathcal{B}$ by Lemma 2.2. The next corollary is an immediate consequence of the two previous results and gives a useful characterization of $\Xi^*_{\mathcal{B}}$.

Corollary 2.9. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and let $\mu \in \mathcal{M}$. Then $\mu \in \Xi^*_{\mathcal{B}}$ if and only if $(\sigma_N(\mu))$ is a bounded sequence in $\mathcal{B}$.*

Finally, we can now prove that the correspondence $\mathcal{B} \mapsto \Xi^*_{\mathcal{B}}$ is bijective.

Theorem 2.10. *The maps*

$$\mathcal{B} \mapsto \Xi^*_{\mathcal{B}}$$

($\mathcal{B}$ a homogeneous Banach space on $\mathbb{T}$) and

$$\mathcal{Y} \mapsto \mathcal{Y}_h$$

($\mathcal{Y}$ a prehomogeneous Banach space on $\mathbb{T}$) define a bijective correspondence between the class of homogeneous Banach spaces on $\mathbb{T}$ and the class of prehomogeneous Banach spaces on $\mathbb{T}$.

Proof: By Theorem 2.8, we have $(\Xi^*_{\mathcal{B}})_h = \mathcal{B}$, when $\mathcal{B}$ is a homogeneous Banach space on $\mathbb{T}$. Now, let $\mathcal{Y}$ be a prehomogeneous Banach space on $\mathbb{T}$. For $\mu \in \mathcal{M}$, the previous corollary implies that $\mu \in \Xi^*_{\mathcal{Y}_h}$ if and only if $(\sigma_N(\mu))$ is a bounded sequence in $\mathcal{Y}_h$. However, this condition is also equivalent to $\mu \in \mathcal{Y}$ by Lemma 2.7. Hence $\Xi^*_{\mathcal{Y}_h} = \mathcal{Y}$, which proves the result. □

3. Homogeneous Banach algebras

The simplest example of a homogeneous Banach space on $\mathbb{T}$ which is also a Banach algebra under pointwise multiplication is the algebra $\mathcal{C}$. We have $\mathcal{C}^* = \mathcal{M}$, $\Xi_{\mathcal{C}} = L^1$ and thus

$$\Xi^*_{\mathcal{C}} = L^\infty,$$

so $\Xi^*_{\mathcal{C}}$ is closed under pointwise multiplication. This is true in general.

Theorem 3.1. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and suppose that $\mathcal{B}$ is a Banach algebra under pointwise multiplication. Then*

(a) $\mathcal{B} \subseteq \mathcal{C}$.

(b) $\Xi_{\mathcal{B}}^* \subseteq L^\infty$ *and* $\Xi_{\mathcal{B}}^*$ *is a Banach algebra under pointwise multiplication.*

(c) *If a homogeneous Banach space* $\mathcal{L}$ *on* $\mathbb{T}$ *is a Banach* $\mathcal{B}$*-module under pointwise multiplication, then* $\Xi_{\mathcal{L}}^*$ *is a Banach* $\Xi_{\mathcal{B}}^*$*-module under pointwise multiplication.*

Proof: Let $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} \leq 1$. Then (f^n) is bounded in L^1, so $f \in L^\infty$ with $\|f\|_\infty \leq 1$. Hence $\mathcal{B} \subseteq L^\infty$ and the inclusion is continuous. Moreover, $\sigma_N(f) \to f$ in $\mathcal{B}$ and thus in L^∞ as $N \to \infty$, so we deduce that $f \in \mathcal{C}$. Consequently $\mathcal{B} \subseteq \mathcal{C}$ and thus $\Xi_{\mathcal{B}}^* \subseteq L^\infty$ by Proposition 2.4. Now, let $f \in \Xi_{\mathcal{B}}^*$ and $g \in \Xi_{\mathcal{L}}^*$. Then

$$\|\sigma_N(f)\sigma_N(g)\|_{\mathcal{L}} \leq \|\sigma_N(f)\|_{\mathcal{B}} \cdot \|\sigma_N(g)\|_{\mathcal{L}} \leq \|f\|_{\Xi_{\mathcal{B}}^*} \cdot \|g\|_{\Xi_{\mathcal{L}}^*}$$

for $N \in \mathbb{N}$, so the sequence $(\sigma_N(f)\sigma_N(g))$ has a wk* cluster point $h \in \Xi_{\mathcal{L}}^*$ as $N \to \infty$. Since $(\sigma_N(f)\sigma_N(g))\widehat{\ }(m) \to \widehat{fg}(m)$ as $N \to \infty$ for $m \in \mathbb{Z}$, we deduce that $fg = h \in \Xi_{\mathcal{L}}^*$, with

$$\|fg\|_{\Xi_{\mathcal{L}}^*} \leq \|f\|_{\Xi_{\mathcal{B}}^*} \cdot \|g\|_{\Xi_{\mathcal{L}}^*}.$$

Considering $\mathcal{B}$ as a Banach $\mathcal{B}$-module, this shows that $\Xi_{\mathcal{B}}^*$ is a Banach algebra under pointwise multiplication, and the result follows. ☐

It is well known that the dual space $\mathcal{B}^*$ of a commutative Banach algebra $\mathcal{B}$ is a commutative Banach $\mathcal{B}$-module under the action $\langle a, b\varphi\rangle = \langle ab, \varphi\rangle$ $(a, b \in \mathcal{B}, \varphi \in \mathcal{B}^*)$. The fact that L^1 is a Banach L^∞-module thus states that $\Xi_{\mathcal{C}}$ is a $\Xi_{\mathcal{C}}^*$-submodule of $\Xi_{\mathcal{C}}^{**}$, and we shall see that this holds in general.

Proposition 3.2. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ which is also a Banach algebra under pointwise multiplication. Then $\Xi_{\mathcal{B}}$ is a closed $\Xi_{\mathcal{B}}^*$-submodule of $\Xi_{\mathcal{B}}^{**}$.*

Proof: Let $\rho : L^1 \to \Xi_{\mathcal{B}}$ be as in Proposition 2.4 (with $\mathcal{B}_2 = \mathcal{C}$). For $f \in L^1$ and $g \in \Xi_{\mathcal{B}}^*$, it is easily verified that $g\rho(f) = \rho^{**}(\rho^*(g)f)$ in $\Xi_{\mathcal{B}}^{**}$. Since $\rho^*(g)f \in L^1$, we thus have $g\rho(f) \in \Xi_{\mathcal{B}}$, and the result follows since ρ has dense range in $\Xi_{\mathcal{B}}$. ☐

As a standard corollary, we obtain the following.

Corollary 3.3. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ which is also a Banach algebra under pointwise multiplication. Then multiplication is separately* wk* *continuous in* $\Xi_{\mathcal{B}}^*$.

Proof: Let (f_i) be a net in $\Xi_{\mathcal{B}}^*$ that converges wk* to 0, and let $g \in \Xi_{\mathcal{B}}^*$ and $\varphi \in \Xi_{\mathcal{B}}$. Then $g\varphi \in \Xi_{\mathcal{B}}$ by the previous proposition, so we deduce that

$$\langle \varphi, gf_i \rangle = \langle gf_i, \varphi \rangle = \langle f_i, g\varphi \rangle = \langle g\varphi, f_i \rangle \to 0$$

as required. □

Remark. Multiplication need not be wk* continuous in $\Xi_{\mathcal{B}}^*$. In L^∞, we have $\alpha^n \to 0$ wk* as $|n| \to \infty$, but $\alpha^n \alpha^{-n} = 1$ for $n \in \mathbb{N}$.

We say that a homogeneous Banach space $\mathcal{B}$ on $\mathbb{T}$ is a homogeneous Banach algebra on $\mathbb{T}$ if $\mathcal{B}$ is a Banach algebra under pointwise multiplication and the character space of $\mathcal{B}$ is $\mathbb{T}$. We shall use the inclusion $\Xi_{\mathcal{B}}^* \subseteq L^\infty$ to show that $\mathcal{C}$ is the only homogeneous Banach algebra on $\mathbb{T}$ in which every closed ideal has an bounded approximate identity with a common bound. The following result as well as its proof is similar to [**4**, Lemma XVII.2.1].

Proposition 3.4. *Let $\mathcal{Y}$ be a Banach algebra continuously embedded in L^∞ and suppose that there exists a constant C such that $1_V \in \mathcal{Y}$ with $\|1_V\|_{\mathcal{Y}} \leq C$ for every open set $V \subseteq \mathbb{T}$. Then $\mathcal{C} \subseteq \mathcal{Y}$ and $\|\cdot\|_{\mathcal{Y}}$ is equivalent to the uniform norm on $\mathcal{C}$.*

Proof: Let $V_1, \dots, V_N$ be pairwise disjoint open sets in $\mathbb{T}$, let $a_1, \dots, a_N$ be complex numbers and consider the function

$$f = \sum_{n=1}^{N} a_n 1_{V_n}.$$

Then $\|f\|_\infty = \sup_{1 \leq n \leq N} |a_n|$. For $\varphi \in \mathcal{Y}^*$ with $\|\varphi\|_{\mathcal{Y}^*} \leq 1$, we have

$$|\langle f, \varphi \rangle| \leq \sum_{n=1}^{N} |a_n| \cdot |\langle 1_{V_n}, \varphi \rangle| \leq \|f\|_\infty \sum_{n=1}^{N} |\langle 1_{V_n}, \varphi \rangle|.$$

Denoting the real and imaginary part of φ by φ_1 and φ_2, we have

$$\begin{aligned}\sum_{n=1}^{N} |\langle 1_{V_n}, \varphi \rangle| \leq & \sum_{\langle 1_{V_n}, \varphi_1 \rangle \geq 0} \langle 1_{V_n}, \varphi_1 \rangle - \sum_{\langle 1_{V_n}, \varphi_1 \rangle < 0} \langle 1_{V_n}, \varphi_1 \rangle \\ & + \sum_{\langle 1_{V_n}, \varphi_2 \rangle \geq 0} \langle 1_{V_n}, \varphi_2 \rangle - \sum_{\langle 1_{V_n}, \varphi_2 \rangle < 0} \langle 1_{V_n}, \varphi_2 \rangle \\ = & \langle 1_{W_{1+}}, \varphi_1 \rangle - \langle 1_{W_{1-}}, \varphi_1 \rangle + \langle 1_{W_{2+}}, \varphi_2 \rangle - \langle 1_{W_{2-}}, \varphi_2 \rangle,\end{aligned}$$

where W_{1+}, W_{1-}, W_{2+} and W_{2-} are open sets. Hence

$$|\langle f, \varphi \rangle| \leq 4C\|f\|_\infty,$$

so we deduce that

$$\|f\|_{\mathcal{Y}} \le 4C\|f\|_\infty.$$

Consequently the two norms are equivalent on

$$\mathrm{span}\{1_V : V \subseteq \mathbb{T} \text{ is open}\}.$$

Let $f \in \mathcal{C}$ with $0 \le f \le 1$. For $N \in \mathbb{N}$, let

$$U_{Nn} = \{t \in \mathbb{T} : f(t) > n/N\} \quad (n = 0, \dots, N-1),$$

and let

$$f_N = \frac{1}{N}\sum_{n=0}^{N-1} 1_{U_{Nn}}.$$

Then $\|f - f_N\|_\infty \le 1/N$, so we deduce that

$$\mathcal{C} \subseteq \overline{\mathrm{span}}\{1_V : V \subseteq \mathbb{T} \text{ is open}\},$$

and the result follows. □

Theorem 3.5. *The algebra $\mathcal{C}$ is the only homogeneous Banach algebra on $\mathbb{T}$ in which there exists a constant C such that every closed ideal has a approximate identity bounded by C.*

Proof: Let $\mathcal{B}$ be homogeneous Banach algebra on $\mathbb{T}$ and suppose that there exists a constant C such that every closed ideal in $\mathcal{B}$ has a bounded approximate identity bounded by C. Let $E \subseteq \mathbb{T}$ be a closed set and let (f_n) be an approximate identity bounded by C for the closed ideal $\{f \in \mathcal{B} : f = 0 \text{ on } E\}$. Then $f_n \to 1$ uniformly on compact sets in $\mathbb{T} \setminus E$ and (f_n) is uniformly bounded, so

$$\widehat{f_n}(m) \to \widehat{1_{\mathbb{T}\setminus E}}(m) \quad (m \in \mathbb{Z})$$

as $n \to \infty$. On the other hand, let f be a wk* cluster point in $\Xi^*_{\mathcal{B}}$ of the sequence (f_n) and let (f_{n_i}) be a subnet of (f_n) which converges wk* to f. Then

$$\widehat{f_{n_i}}(m) \to \widehat{f}(m) \quad (m \in \mathbb{Z}),$$

so we deduce that $1_{\mathbb{T}\setminus E} = f \in \Xi^*_{\mathcal{B}}$ with $\|1_{\mathbb{T}\setminus E}\|_{\Xi^*_{\mathcal{B}}} \le C$. Theorem 3.1 and the previous proposition thus imply that the norm $\|\cdot\|_{\mathcal{B}}$ is equivalent to the uniform norm on $\mathcal{B}$, so $\mathcal{B}$ is a closed subalgebra of $\mathcal{C}$. Since $\mathcal{B}$ is a homogeneous Banach algebra on $\mathbb{T}$, it thus follows that $\mathcal{B} = \mathcal{C}$. □

We do not know whether there exists a homogeneous Banach algebra on $\mathbb{T}$ other than $\mathcal{C}$ in which every closed ideal has a bounded approximate identity.

4. Multipliers

Let X_1 and X_2 be Banach spaces continuously embedded in $\mathcal{M}$. We say that a linear operator $T : X_1 \to X_2$ is a multiplier if there exists a sequence $(\widehat{T}(n))$ such that

$$\widehat{T\mu} = \widehat{T}\widehat{\mu} \quad (\mu \in X_1).$$

Such an operator is automatically continuous. We denote the space of multipliers from X_1 to X_2 by (X_1, X_2). (See [**5**] or [**14**] for general information on multipliers and for descriptions of (X_1, X_2) for various choices of X_1 and X_2.) For homogeneous Banach spaces, we have the following well-known result. Observe that every homogeneous Banach space $\mathcal{B}$ on $\mathbb{T}$ is a Banach $\mathcal{M}$-module for the convolution product since $\mu * f = (1/(2\pi)) \int_{\mathbb{T}} f_s \, d\mu(s) \in \mathcal{B}$ for $\mu \in \mathcal{M}$ and $f \in \mathcal{B}$.

Proposition 4.1. *For homogeneous Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ on $\mathbb{T}$ and a linear operator $T : \mathcal{B}_1 \to \mathcal{B}_2$, the following conditions are equivalent.*

(a) $T \in (\mathcal{B}_1, \mathcal{B}_2)$.
(b) $T(\mu * f) = \mu * Tf \quad (\mu \in \mathcal{M},\ f \in \mathcal{B}_1)$.
(c) $TR_s = R_s T \quad (s \in \mathbb{T})$.

For the homogeneous Banach spaces $\mathcal{B}$ considered in this section, we assume, for simplicity, that, for every $n \in \mathbb{Z}$, there exists $f \in \mathcal{B}$ such that $\widehat{f}(n) \neq 0$.

Proposition 4.2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be homogeneous Banach spaces on $\mathbb{T}$. Every $T \in (\mathcal{B}_1, \mathcal{B}_2)$ extends uniquely by* wk* *continuity to $\widetilde{T} \in (\Xi^*_{\mathcal{B}_1}, \Xi^*_{\mathcal{B}_2})$. Conversely, every $S \in (\Xi^*_{\mathcal{B}_1}, \Xi^*_{\mathcal{B}_2})$ maps $\mathcal{B}_1$ into $\mathcal{B}_2$, so $S|_{\mathcal{B}_1} \in (\mathcal{B}_1, \mathcal{B}_2)$. Moreover, $(\mathcal{B}_1, \mathcal{B}_2) = (\mathcal{B}_1, \Xi^*_{\mathcal{B}_2})$.*

Proof: For $T \in (\mathcal{B}_1, \mathcal{B}_2)$ and $n \in \mathbb{Z}$, we have

$$\langle f, T^*\xi_n \rangle = \widehat{T}(n)\widehat{f}(n) = \langle f, \widehat{T}(n)\xi_n \rangle \quad (f \in \mathcal{B}_1),$$

so $T^*\xi_n = \widehat{T}(n)\xi_n$. Hence $T^*(\Xi_{\mathcal{B}_2}) \subseteq \Xi_{\mathcal{B}_1}$ and $\widetilde{T} = (T^*|_{\Xi_{\mathcal{B}_2}})^*$ is the required extension. For $S \in (\Xi^*_{\mathcal{B}_1}, \Xi^*_{\mathcal{B}_2})$, we have $S\alpha^n = \widehat{S}(n)\alpha^n$ $(n \in \mathbb{Z})$, so $S(\mathcal{B}_1) \subseteq \mathcal{B}_2$. The same argument proves the last statement. $\square$

Remark. For linear operators $S : \Xi^*_{\mathcal{B}_1} \to \Xi^*_{\mathcal{B}_2}$, we still have (a)⇔(b)⇒(c) in Proposition 4.1, but not necessarily (c)⇒(b); see, for example, [**19**].

Figà-Talamanca and Gaudry ([**6**], [**7**]) have shown that (L^p, L^q) $(1 \le p, q < \infty)$ is a dual space. We shall use a result of Rieffel ([**18**]) to show that this holds for homogeneous Banach spaces in general. For Banach spaces X and Y, denote their projective tensor product by $X \otimes_\gamma Y$. When X and Y are Banach L^1-modules, let

$$K = \overline{\operatorname{span}}\{fx \otimes y - x \otimes fy : f \in L^1,\, x \in X,\, y \in Y\}$$

and let

$$X \otimes_{L^1} Y = (X \otimes_\gamma Y)/K.$$

Then every $\rho \in X \otimes_{L^1} Y$ has an expansion

$$\rho = \sum_{n=1}^{\infty} x_n \otimes y_n$$

with $\sum_{n=1}^{\infty} \|x_n\|_X \cdot \|y_n\|_Y < \infty$, and

$$\|\rho\|_{X \otimes_{L^1} Y} = \inf\left\{ \sum_{n=1}^{\infty} \|x_n\|_X \cdot \|y_n\|_Y : \rho = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}$$

is the norm on $X \otimes_{L^1} Y$.

Theorem 4.3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be homogeneous Banach spaces on $\mathbb{T}$ and define*

$$\Phi : (\mathcal{B}_1, \mathcal{B}_2) \to (\mathcal{B}_1 \otimes_{L^1} \Xi_{\mathcal{B}_2})^*$$

by

$$\langle f \otimes \varphi, \Phi(T) \rangle = \langle Tf, \varphi \rangle \quad (f \in \mathcal{B}_1,\, \varphi \in \Xi_{\mathcal{B}_2},\, T \in (\mathcal{B}_1, \mathcal{B}_2)).$$

Then Φ is an isometric isomorphism, and Φ is a homeomorphism between the closed unit balls equipped with the weak operator and the wk* *topology respectively.*

Proof: It follows from [**18**] that the map $\widetilde{\Phi} : (\mathcal{B}_1, \Xi^*_{\mathcal{B}_2}) \to (\mathcal{B}_1 \otimes_{L^1} \Xi_{\mathcal{B}_2})^*$ defined by $\langle f \otimes \varphi, \widetilde{\Phi}(T) \rangle = \langle \varphi, Tf \rangle$ for $f \in \mathcal{B}_1$, $\varphi \in \Xi_{\mathcal{B}_2}$ and $T \in (\mathcal{B}_1, \Xi^*_{\mathcal{B}_2})$ is an isometric isomorphism, so Φ is an isometric isomorphism by the previous proposition. Let (T_i) be a net in the closed unit ball of $(\mathcal{B}_1, \mathcal{B}_2)$ which converges to T in the weak operator topology. Then

$$\langle f \otimes \varphi, \Phi(T_i) \rangle \to \langle f \otimes \varphi, \Phi(T) \rangle \quad (f \in \mathcal{B}_1,\, \varphi \in \Xi_{\mathcal{B}_2}),$$

and it follows that $\Phi(T_i) \to \Phi(T)$ wk*. From [**2**, Proposition IX.5.5] (the proof given there carries over to Banach spaces), we deduce that the closed unit ball of $(\mathcal{B}_1, \mathcal{B}_2)$ is compact in the weak operator topology, so the result follows. □

We shall now show that the abstract space $\mathcal{B}_1 \otimes_{L^1} \Xi_{\mathcal{B}_2}$ can be identified with a sequence space. For $f \in \mathcal{B}_1$ and $\varphi \in \Xi_{\mathcal{B}_2}$, define $f*\varphi$ as a sequence by $\widehat{f*\varphi} = \widehat{f}\widehat{\varphi}$, and let

$$\mathcal{B}_1 * \Xi_{\mathcal{B}_2} = \left\{ \rho = \sum_{n=1}^{\infty} f_n * \varphi_n : \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{B}_1} \cdot \|\varphi_n\|_{\Xi_{\mathcal{B}_2}} < \infty \right\}$$

equipped with norm

$$\|\rho\|_{\mathcal{B}_1 * \Xi_{\mathcal{B}_2}} = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{B}_1} \cdot \|\varphi_n\|_{\Xi_{\mathcal{B}_2}} : \rho = \sum_{n=1}^{\infty} f_n * \varphi_n \right\}.$$

For L^p spaces, the following was proved in [**18**, Theorem 3.3].

Proposition 4.4. *There exists an isometric isomorphism* $\Psi : \mathcal{B}_1 \otimes_{L^1} \Xi_{\mathcal{B}_2} \to \mathcal{B}_1 * \Xi_{\mathcal{B}_2}$ *such that* $\Psi(f \otimes \varphi) = f * \varphi$ $(f \in \mathcal{B}_1,\ \varphi \in \Xi_{\mathcal{B}_2})$.

Proof: The map $\psi : \mathcal{B}_1 \times \Xi_{\mathcal{B}_2} \to \mathcal{B}_1 * \Xi_{\mathcal{B}_2}$ defined by $\psi(f,\varphi) = f*\varphi$ $(f \in \mathcal{B}_1,\ \varphi \in \Xi_{\mathcal{B}_2})$ is bilinear and continuous, so there is a continuous, linear map $\widetilde{\psi} : \mathcal{B}_1 \otimes_\gamma \Xi_{\mathcal{B}_2} \to \mathcal{B}_1 * \Xi_{\mathcal{B}_2}$ with $\widetilde{\psi}(f \otimes \varphi) = f * \varphi$ $(f \in \mathcal{B}_1,\ \varphi \in \Xi_{\mathcal{B}_2})$. Moreover, $\widetilde{\psi}$ is surjective and $K \subseteq \ker \widetilde{\psi}$. For $f \in \mathcal{B}_1$ and $\varphi \in \Xi_{\mathcal{B}_2}$, we have $f * \varphi = (1/(2\pi)) \int_{\mathbb{T}} f(s)\varphi_s \, ds \in \Xi_{\mathcal{B}_2}$, so it follows from the proof of [**18**, Theorem 3.3] that $\ker \widetilde{\psi} \subseteq K$, which finishes the proof. □

Every trigonometric polynomial p defines a multiplier $T_p \in (\mathcal{B}_1, \mathcal{B}_2)$ by $T_p f = p*f$ $(f \in \mathcal{B}_1)$, which we identify with p. We have the following density results.

Proposition 4.5.

(a) *For homogeneous Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ on $\mathbb{T}$, the unit ball of the trigonometric polynomials is strongly dense in the unit ball of $(\mathcal{B}_1, \mathcal{B}_2)$.*

(b) *For a homogeneous Banach space $\mathcal{B}$ on $\mathbb{T}$, the span of $\{R_s : s \in \mathbb{T}\}$ is strongly dense in $(\mathcal{B}, \mathcal{B})$.*

Proof: (a) For $T \in (\mathcal{B}_1, \mathcal{B}_2)$, let

$$\sigma_N(T) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{T}(n)\alpha^n \quad (N \in \mathbb{N}).$$

Then $\sigma_N(T)f = \sigma_N(Tf) \to Tf$ in $\mathcal{B}_2$ as $N \to \infty$ for $f \in \mathcal{B}_1$ and $\|\sigma_N(T)\|_{(\mathcal{B}_1,\mathcal{B}_2)} \le \|T\|_{(\mathcal{B}_1,\mathcal{B}_2)}$ $(N \in \mathbb{N})$.

(b) For $n \in \mathbb{Z}$ and $f \in \mathcal{B}$, the integral

$$\alpha^n * f = \frac{1}{2\pi} \int_{\mathbb{T}} R_s f \cdot e^{ins}\, ds$$

exists as a Bochner integral, so α^n belongs to the strong closure of $\operatorname{span}\{R_s : s \in \mathbb{T}\}$, and the conclusion thus follows from (a). □

Finally, we shall see that $(\mathcal{B}, \mathcal{B})$ is the dual space of a homogeneous Banach space of continuous functions on $\mathbb{T}$. Let $\mathcal{W}$ be the Wiener algebra of absolutely convergent Fourier series on $\mathbb{T}$.

Corollary 4.6. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and let $A_{\mathcal{B}} = \mathcal{B} * \Xi_{\mathcal{B}}$. Then $A_{\mathcal{B}}$ is a homogeneous Banach space on $\mathbb{T}$ with $\mathcal{W} \subseteq A_{\mathcal{B}} \subseteq \mathcal{C}$ and $(\mathcal{B}, \mathcal{B}) = A_{\mathcal{B}}^*$.*

Proof: For $f \in \mathcal{B}$ and $\varphi \in \Xi_{\mathcal{B}}$, we have

$$(f*\varphi)(s) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n)\widehat{\varphi}(n)e^{ins} = \langle R_{-s}f, \varphi\rangle \quad (s \in \mathbb{T}),$$

so $A_{\mathcal{B}}$ is a homogeneous Banach space on $\mathbb{T}$ with $A_{\mathcal{B}} \subseteq \mathcal{C}$. For $n \in \mathbb{Z}$ and $\varepsilon > 0$, choose $f \in \mathcal{B}$, $f \neq 0$ such that

$$|\widehat{f}(n)| = |\langle f, \alpha^n\rangle| \geq (1-\varepsilon)\|f\|_{\mathcal{B}} \cdot \|\alpha^n\|_{\mathcal{B}^*}.$$

Since $f * \alpha^n = \widehat{f}(n)\alpha^n$, we thus have

$$\|\alpha^n\|_{A_{\mathcal{B}}} \leq \frac{\|f\|_{\mathcal{B}} \cdot \|\alpha^n\|_{\mathcal{B}^*}}{|\widehat{f}(n)|} \leq \frac{1}{1-\varepsilon}.$$

Hence $\|\alpha^n\|_{A_{\mathcal{B}}} = 1$, so $\mathcal{W} \subseteq A_{\mathcal{B}}$. The last statement is just a reformulation of Theorem 4.3 and Proposition 4.4. □

5. Examples

In this section, we shall determine $\Xi_{\mathcal{B}}^*$ for some examples of homogeneous Banach spaces $\mathcal{B}$ on $\mathbb{T}$. We have already seen that $\Xi_{L^1}^* = \mathcal{M}$ and that $\Xi_{\mathcal{C}}^* = L^\infty$.

We denote the space of left-continuous functions of bounded variation by $\mathcal{BVC}_l$. For $f \in \mathcal{BVC}_l$, there exists a unique measure $\mu(f)$ with $\widehat{\mu(f)}(0) = 0$ and $c \in \mathbb{C}$ such that

$$f(t) = \mu(f)([0,t)) + c \quad (t \in \mathbb{T}).$$

Moreover, $f \in \mathcal{BVC}_l$ is absolutely continuous on $\mathbb{T}$ if and only if $\mu(f) \in L^1$, and in this case $f' = \mu(f)$ a.e. The space of absolutely continuous functions on $\mathbb{T}$ is denoted by $\mathcal{AC}$.

This generalizes to (pre)homogeneous Banach spaces on $\mathbb{T}$. For a homogeneous Banach space $\mathcal{B}$ on $\mathbb{T}$, let $\mathcal{B}^1$ be the space of functions $f \in \mathcal{AC}$ for which $\mu(f) \in \mathcal{B}$ and equip $\mathcal{B}^1$ with the norm $\|f\|_{\mathcal{B}^1} = \|f'\|_{\mathcal{B}}+|f(0)|$ ($f \in \mathcal{B}^1$). Similarly, for a prehomogeneous Banach space $\mathcal{Y}$ on $\mathbb{T}$, we say that a function $f \in \mathcal{BVC}_l$ belongs to $\mathcal{Y}^1$ if $\mu(f) \in \mathcal{Y}$, and we norm $\mathcal{Y}^1$ by $\|f\|_{\mathcal{Y}^1} = \|\mu(f)\|_{\mathcal{Y}} + |f(0)|$ ($f \in \mathcal{Y}^1$). Then $\mathcal{B}^1$ is a homogeneous Banach space on $\mathbb{T}$ and $\mathcal{Y}^1$ is a prehomogeneous Banach space on $\mathbb{T}$. The following result is easily proved.

Proposition 5.1. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$. Then*

$$\Xi^*_{\mathcal{B}^1} = (\Xi^*_{\mathcal{B}})^1.$$

It immediately follows that

$$\Xi^*_{\mathcal{AC}} = \Xi^*_{(L^1)^1} = \left(\Xi^*_{L^1}\right)^1 = (\mathcal{M})^1 = \mathcal{BVC}_l.$$

We now turn to other examples.

Dual spaces. The following result is not surprising, considering that the main motivation for the construction of the space $\Xi^*_{\mathcal{B}}$ was to embed $\mathcal{B}$ in a "small" dual space.

Lemma 5.2. *Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$. Suppose that there is a Banach space $\mathcal{Z}$ such that $\mathcal{Z}^* = \mathcal{B}$ and such that $\xi_n \in \mathcal{Z}$ for $n \in \mathbb{Z}$. Then $\Xi_{\mathcal{B}} = \mathcal{Z}$ and thus $\Xi^*_{\mathcal{B}} = \mathcal{B}$.*

Proof: Observe that $\mathcal{Z}$ is a closed subspace of $\mathcal{Z}^{**} = \mathcal{B}^*$. Also,

$$\Xi_{\mathcal{B}} = \overline{\operatorname{span}\{\xi_n : n \in \mathbb{Z}\}}^{\mathcal{B}^*} = \overline{\operatorname{span}\{\xi_n : n \in \mathbb{Z}\}}^{\mathcal{Z}},$$

so $\Xi_{\mathcal{B}}$ is a closed subspace of $\mathcal{Z}$. If $f \in \mathcal{Z}^* = \mathcal{B}$ and $f \perp \Xi_{\mathcal{B}}$, then $\widehat{f}(n) = 0$ for $n \in \mathbb{Z}$ and thus $f = 0$. It thus follows from the Hahn-Banach theorem that $\Xi_{\mathcal{B}} = \mathcal{Z}$. □

The result applies, in particular, to the spaces L^p with $1 < p < \infty$ and to the Wiener algebra $\mathcal{W} \sim l^1(\mathbb{Z})$.

Lipschitz algebras. For $0 < \gamma \leq 1$, let Λ_γ be the Lipschitz algebra of functions f on $\mathbb{T}$ for which there exists a constant C such that

$$|f(t) - f(s)| \leq C|t - s|^\gamma$$

for $s, t \in \mathbb{T}$. Normed by $\|f\|_{\Lambda_\gamma} = \|f\|_\infty + \sup\{|f(t) - f(s)| \cdot |t - s|^{-\gamma} : t, s \in \mathbb{T}, t \neq s\}$, it is well known that Λ_γ is a Banach algebra. For $0 < \gamma < 1$, let λ_γ be the closed subalgebra of Λ_γ of functions satisfying

$$|f(t) - f(s)| = o(|t - s|^\gamma)$$

uniformly as $|t-s| \to 0$. Then λ_γ is a homogeneous Banach algebra on $\mathbb{T}$ ([**16**]). Moreover, it follows from [**3**] that the map $\Psi : \lambda_\gamma^{**} \to \Lambda_\gamma$ defined by

$$\Psi(F)(t) = \langle \delta_t, F\rangle \quad (F \in \lambda_\gamma^{**},\, t \in \mathbb{T})$$

is an isomorphism, where $\delta_t \in \lambda_\gamma^*$ denotes the point evalution functional at t.

Now, suppose that $F \in \lambda_\gamma^{**}$ with $F \perp \Xi_{\lambda_\gamma}$ and let $m \in \mathbb{Z}$. Then

$$\widehat{\Psi(F)}(m) = \frac{1}{2\pi}\int_{\mathbb{T}} \langle \delta_t, F\rangle e^{-imt}\,dt = \frac{1}{2\pi}\left\langle \int_{\mathbb{T}} \delta_t e^{-imt}\,dt, F\right\rangle,$$

where $\int_{\mathbb{T}} \delta_t e^{-imt}\,dt$ exists as a wk* Bochner integral in λ_γ^*. Since

$$\begin{aligned}\frac{1}{2\pi}\left\langle \alpha^n, \int_{\mathbb{T}} \delta_t e^{-imt}\,dt\right\rangle &= \frac{1}{2\pi}\int_{\mathbb{T}} \langle \alpha^n, \delta_t\rangle e^{-imt}\,dt \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} e^{i(n-m)t}\,dt = \langle \alpha^n, \xi_m\rangle\end{aligned}$$

for $n \in \mathbb{Z}$, we deduce that $(1/2\pi)\int_{\mathbb{T}} \delta_t e^{-imt}\,dt = \xi_m$, so

$$\widehat{\Psi(F)}(m) = \langle \xi_m, F\rangle = 0.$$

Hence $\Psi(F) = 0$ and thus $F = 0$, so we deduce that $\Xi_{\lambda_\gamma} = \lambda_\gamma^*$ and thus

$$\Xi_{\lambda_\gamma}^* = \Lambda_\gamma.$$

Moreover, $\operatorname{span}\{\delta_t : t \in \mathbb{T}\}$ is dense in λ_γ^* ([**3**, Lemma 2.6]), so we deduce that a sequence (f_n) in $\Lambda_\gamma (= \Xi_{\lambda_\gamma}^*)$ converges wk* to 0 if and only if it is bounded in Λ_γ and converges pointwise to 0 on $\mathbb{T}$.

To complete our discussion of Lipschitz algebras, we mention that

$$\Xi_{\mathcal{C}^1}^* = (\Xi_{\mathcal{C}}^*)^1 = (L^\infty)^1 = \Lambda_1$$

by Proposition 5.1.

The Pisier algebra. Let $(\xi_n(\omega))$ be a sequence of independent, normal, complex random variables defined on some probability space Ω. For $f \in L^2$, consider the Gaussian Fourier series

$$f_\omega(t) = \sum_{n=-\infty}^{\infty} \xi_n(\omega)\widehat{f}(n)e^{int} \quad (t \in \mathbb{T},\, \omega \in \Omega).$$

We say that $f \in \mathcal{C}$ almost surely if $f_\omega \in \mathcal{C}$ for almost every $\omega \in \Omega$. The space $\mathcal{C}_{\text{a.s.}}$ of such functions is a homogeneous Banach space on $\mathbb{T}$ equipped with the norm

$$\|f\|_{\text{a.s.}} = \int_\Omega \|f_\omega\|_\infty\,d\omega \quad (f \in \mathcal{C}_{\text{a.s.}}).$$

Pisier proved that

$$\mathcal{P} = \mathcal{C}_{\text{a.s.}} \cap \mathcal{C}$$

is closed under pointwise multiplication and that it is a homogeneous Banach algebra on $\mathbb{T}$ (the so-called Pisier algebra) equipped with the norm

$$\|f\|_{\mathcal{P}} = \|f\|_\infty + \|f\|_{\text{a.s.}} \quad (f \in \mathcal{P}).$$

(For this and other results, see, for instance, [**17**] or [**12**].)

It follows from [**15**, Corollary VI.1.5] that $\mathcal{C}_{\text{a.s.}} = \mathcal{Z}^*$ for some Banach space $\mathcal{Z}$ with $\xi_n \in \mathcal{Z}$ for $n \in \mathbb{N}$. Hence

$$\Xi^*_{\mathcal{P}} \subseteq \Xi^*_{\mathcal{C}_{\text{a.s.}}} \cap \Xi^*_{\mathcal{C}} = \mathcal{C}_{\text{a.s.}} \cap L^\infty$$

by Proposition 2.4 and Lemma 5.2. Conversely, for $f \in \mathcal{C}_{\text{a.s.}} \cap L^\infty$, the sequence $(\sigma_N(f))$ is bounded in $\mathcal{C}_{\text{a.s.}}$ and $\mathcal{C}$, so it follows from Corollary 2.9 that $f \in \Xi^*_{\mathcal{P}}$. Hence

$$\Xi^*_{\mathcal{P}} = \mathcal{C}_{\text{a.s.}} \cap L^\infty.$$

Finally, with an obvious notation, we have $\mathcal{C}_{\text{a.s.}} = L^\infty_{\text{a.s.}}$ ([**12**, p. 58]), so we obtain the more symmetric expresion

$$\Xi^*_{\mathcal{C}_{\text{a.s.}} \cap \mathcal{C}} = L^\infty_{\text{a.s.}} \cap L^\infty.$$

Uniformly convergent Fourier series. For $\mu \in \mathcal{M}$ and $N \in \mathbb{N}$, let

$$S_N(\mu) = \sum_{n=-N}^{N} \widehat{\mu}(n)\alpha^n$$

be the partial sums of μ, and let

$$\mathcal{U} = \{f \in \mathcal{C} : \|S_N(f) - f\|_\infty \to 0 \text{ as } N \to \infty\}$$

be the space of uniformly convergent Fourier series on $\mathbb{T}$ equipped with the norm

$$\|f\|_{\mathcal{U}} = \sup_N \|S_N(f)\|_\infty \quad (f \in \mathcal{U}).$$

It is easily seen that $\mathcal{U}$ is a homogeneous Banach space on $\mathbb{T}$. For $\mu \in \Xi^*_{\mathcal{U}}$, we have $\widehat{\sigma_M(\mu)}(n) \to \widehat{\mu}(n)$ as $M \to \infty$ for $n \in \mathbb{Z}$ and thus $\|S_N(\sigma_M(\mu)) - S_N(\mu)\|_\infty \to 0$ as $M \to \infty$ for $N \in \mathbb{N}$. Hence

$$\sup_N \|S_N(\mu)\|_\infty \le \sup_{M,N} \|S_N(\sigma_M(\mu))\|_\infty = \sup_M \|\sigma_M(\mu)\|_{\mathcal{U}} < \infty.$$

In particular, $\mu \in L^\infty$. Conversely, let $f \in L^\infty$ with $\sup_N \|S_N(f)\|_\infty < \infty$. Then $(S_N(f))$ is bounded in $\mathcal{U}$ and $\widehat{S_N(f)}(m) \to \widehat{f}(m)$ as $N \to \infty$ for $m \in \mathbb{Z}$. Hence $S_N(f) \to f$ wk* in $\Xi^*_{\mathcal{U}}$ as $N \to \infty$, so we deduce that

$$\Xi^*_{\mathcal{U}} = \{f \in L^\infty : \sup_N \|S_N(f)\|_\infty < \infty\},$$

the space of uniformly bounded Fourier series on $\mathbb{T}$.

Spaces of analytic functions. As mentioned earlier, the definition of a homogeneous Banach space $\mathcal{B}$ includes the case where $\widehat{f}(n) = 0$ for every $f \in \mathcal{B}$ for some values of n and in particular spaces of analytic functions. For instance, the Hardy space $\mathcal{H}^1$ on the open unit disc $\mathbb{D}$ is a homogeneous Banach space on $\mathbb{T}$. Moreover, by the F. and M. Riesz theorem ([**11**, p. 47]), we have

$$\Xi^*_{\mathcal{H}^1} \subseteq \{\mu \in \mathcal{M} : \widehat{\mu}(n) = 0 \text{ for every } n < 0\} = \mathcal{H}^1,$$

so

$$\Xi^*_{\mathcal{H}^1} = \mathcal{H}^1.$$

This also follows from Lemma 5.2, since $\mathcal{H}^1 = (\mathcal{C}/\mathcal{A}_0)^*$, where $\mathcal{A}$ is the disc algebra of functions analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ and $\mathcal{A}_0 = \{f \in \mathcal{A} : f(0) = 0\}$. For the disc algebra, we have $\mathcal{A}^* = \mathcal{M}/\mathcal{A}^\perp = \mathcal{M}/\mathcal{H}^1_0$ (where $\mathcal{H}^1_0 = \{f \in \mathcal{H}^1 : f(0) = 0\}$). Hence $\Xi_{\mathcal{A}} = L^1/\mathcal{H}^1_0$ and thus

$$\Xi^*_{\mathcal{A}} = \mathcal{H}^\infty.$$

Finally, we consider the space

$$\mathcal{BMOA} = \mathcal{BMO} \cap \mathcal{H}^1$$

of functions in $\mathcal{H}^1$ which are of bounded mean oscillation on $\mathbb{T}$, and the closed subspace $\mathcal{VMOA}$ of functions of vanishing mean oscillation on $\mathbb{T}$ (see, for example, [**8**] for the definitions). Fefferman's duality theorem ([**1**, Corollary 8.1] or [**8**, Theorem VI.4.4]) states that

$$\mathcal{BMOA} = (\mathcal{H}^1)^*.$$

Hence conditions (i) and (iii) in the definition of a prehomogeneous Banach space on $\mathbb{T}$ are satisfied for $\mathcal{BMOA}$. Moreover, for $g \in \mathcal{BMOA}$, we have $g_s \to g$ wk* in $\mathcal{BMOA}$ as $s \to 0$, so it follows from the proof of Theorem 2.6 that

$$f * g = \frac{1}{2\pi} \int_{\mathbb{T}} f(s) g_s \, ds$$

exists as a wk* Bochner integral in $\mathcal{BMOA}$ for $f \in L^1$. Hence

$$\|f * g\|_{\mathcal{BMOA}} \le \|f\|_{L^1} \|g\|_{\mathcal{BMOA}},$$

so we deduce that $\mathcal{BMOA}$ is a prehomogeneous Banach space on $\mathbb{T}$. Also,

$$(\mathcal{BMOA})_h = \mathcal{VMOA},$$

by [**8**, Theorem VI.5.1], so it follows from Theorem 2.10 that $\mathcal{VMOA}$ is a homogeneous Banach space on $\mathbb{T}$ with

$$\Xi^*_{\mathcal{VMOA}} = \mathcal{BMOA}.$$

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Primera versió rebuda el 21 de gener de 1999,
darrera versió rebuda el 17 de juny de 1999.