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AN ENDPOINT LITTLEWOOD-PALEY INEQUALITY FOR BVP ASSOCIATED WITH THE LAPLACIAN ON LIPSCHITZ DOMAINS

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Abstract

We prove a commutator inequality of Littlewood-Paley type between partial derivatives and functions of the Laplacian on a Lipschitz domain which gives interior energy estimates for some BVP. It can be seen as an endpoint inequality for a family of energy estimates.

Contents

1. Introduction

Let Ω be a Lipschitz connected domain in $\mathbb{R}^n$. Consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 &\quad \text{in } Q = \Omega \times (0,+\infty), \\ u(0) = f &\quad \text{on } \Omega, \end{aligned}$$

where Δ denotes the Laplacian in Ω with Dirichlet or Neumann boundary condition and u vanishes as t goes to ∞. That is u satisfies either a Dirichlet or Neumann boundary condition on the lateral boundary of Q.

Simple integration by parts gives us the following energy estimate

$$\int_Q |\nabla_x u(X)|^2 \, dX = \frac{1}{2} \int_\Omega |f(x)|^2 \, dx = \frac{1}{2} \|f\|_2^2,$$

where $X = (x,t) \in Q$. If f happens to be smoother then similar weighted estimates of Littlewood-Paley type can be obtained via functional calculus

$$\int_Q |\nabla_x u(X)|^2 \frac{dX}{\eta(X)^s} \le c_s \|f\|_s^2,$$

where $\eta(X) = t$ is the distance of $X = (x,t)$ to the bottom boundary of Q and $\|f\|_s = \|(-\Delta)^{s/2} f\|_2$ which is roughly a semi-norm on $H^s(\Omega)$. This holds for $0 \le s < 1$. A proof will be given later for convenience.

This proof shows that c_s blows up as $s \uparrow 1$. In fact, this inequality does fail for $s = 1$. The question is whether a suitable correction can be made on u to obtain an endpoint estimate. In this case, one imposes f to be in $H^1(\Omega)$, and at first sight, it seems quite natural to compare $\nabla_x u$ with ∇f at least for small time but the integral $\int_0^1 \int_\Omega |e^{t\Delta} \nabla f(x) - \nabla f(x)|^2 \frac{dxdt}{t}$ diverges in general. A better correction is to take instead of ∇f the solution v of the heat equation with initial data ∇f. It turns out as we will show that

$$\int_Q |\nabla_x u(X) - v(X)|^2 \frac{dX}{\eta(X)} \le c \int_\Omega |\nabla f(x)|^2 \, dx.$$

Using semigroup notations, we have for $X = (x,t) \in Q$, $u(X) = (e^{t\Delta} f)(x)$ and $v(X) = (e^{t\Delta} \nabla f)(x)$. This inequality amounts to the commutator inequality

$$\int_0^\infty \int_\Omega |[\nabla, e^{t\Delta}] f(x)|^2 \frac{dxdt}{t} \le c \int_\Omega |\nabla f(x)|^2 \, dx, \tag{1}$$

where $[A,B] = AB - BA$.

A similar phenomenon appears with those harmonic functions on Q solutions of

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} + \Delta u = 0 &\quad \text{in } Q = \Omega \times (0, +\infty),\\ u(0) = f &\quad \text{on } \Omega,\end{aligned}$$

with Dirichlet or Neumann condition on the lateral boundary of Q and u going to 0 at ∞. One has for $s \in [0,1)$,

$$\int_Q \eta(X)^{2(1-s)} |\nabla_x u(X)|^2 \frac{dX}{\eta(X)} \leq c_s \|f\|_s^2,$$

and for $s = 1$, correcting u with the harmonic function v taking values ∇f on Ω we will obtain

$$\int_Q |\nabla_x u(X) - v(X)|^2 \frac{dX}{\eta(X)} \leq c \|\nabla f\|_2^2. \tag{2}$$

This amounts to studying another commutator: $[\nabla_x, e^{-t(-\Delta)^{1/2}}]$.

Both inequalities (1) and (2) can be seen from different viewpoints: they are either endpoint estimates in a family of Littlewood-Paley estimates and or a measure of the defect of the semigroups of not being convolution operators. It should be observed that, if $\Omega = \mathbb{R}^n$, both semigroups are convolution operators and the commutators vanish. We also suspect some connections with the Hodge theory for the Dirichlet or Neumann Laplacian on Ω. Such estimates arose in connection with our work [**2**] on the square roots of second order elliptic divergence operators on Lipschitz domains for which we had to understand this defect precisely.

This paper is organised as follows. In Section 2, we state the main theorem. We explain the result in the case of the upper half-space in Section 3. Then we prove it for special Lipschitz domains in Section 4 and for bounded Lipschitz domains in Section 5. We conclude with other commutators in Section 6.

We want to thank S. Hofmann and A. McIntosh with whom we have discussed these topics.

2. Statement of the main result

As we shall see in Section 6, the commutators introduced above have the same nature as the one defined in terms of the resolvent family $R_t = (1 - t^2\Delta)^{-1}$ for $t > 0$. Specific features of potential theory forces us to use resolvents. Recall that for given $f \in L^2(\Omega)$, $u = R_t f$ is the unique element in V such that

$$\int_\Omega uv + t^2 \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv, \quad \forall\, v \in V,$$

so that

$$\|R_t f\|_2^2 + t^2 \|\nabla R_t f\|_2^2 \le \|f\|_2^2$$

is obtained by letting $v = \overline{u}$. Here and thereafter, ∇ is the n-tuple of partial derivatives $\frac{\partial}{\partial x_j}$, $1 \le j \le n$, defined on V. Also, $V = H_0^1(\Omega)$ in the case of the Dirichlet Laplacian, and $V = H^1(\Omega)$ in the case of the Neumann Laplacian.

Before moving on, let us recall some basic facts for the square root of $-\Delta$. By a straightforward integration by parts, we have

$$\|(-\Delta)^{1/2} f\|_2 = \|\nabla f\|_2, \quad \forall\, f \in V. \tag{3}$$

Since $(-\Delta)^{1/2} V$ is dense in $L^2(\Omega)$, this means that for all $j = 1, \dots, n$, $\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$ extends to a bounded operator in $L^2(\Omega)$ with norm equal to 1. These operators are commonly called the Riesz transforms associated to the Laplacian.

Let $h \in H^1(\Omega)$ with compact support and $f \in (-\Delta)^{1/2} V$. Then

$$\left\langle (-\Delta)^{-1/2} \tfrac{\partial h}{\partial x_j}, f \right\rangle = \left\langle \tfrac{\partial h}{\partial x_j}, (-\Delta)^{-1/2} f \right\rangle = -\left\langle h, \tfrac{\partial}{\partial x_j}(-\Delta)^{-1/2} f \right\rangle \tag{4}$$

where the last equality comes from Green's formula. By the boundedness of $\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$ and density

$$\|(-\Delta)^{-1/2} \tfrac{\partial h}{\partial x_j}\|_2 \le \|h\|_2. \tag{5}$$

Thus, $(-\Delta)^{-1/2} \frac{\partial}{\partial x_j}$ also extends to a bounded operator on $L^2(\Omega)$.

Next, introduce the commutator

$$\mathcal{C}_t = [\nabla, R_t]$$

between the partial derivatives and the resolvent of Δ, that is

$$\mathcal{C}_t f = \left(\frac{\partial}{\partial x_j}(1 - t^2\Delta)^{-1} f - (1 - t^2\Delta)^{-1} \frac{\partial f}{\partial x_j} \right)_{1 \le j \le n}$$

for $f \in V$. By definition $\|R_t \nabla f\|_2 \le \|\nabla f\|_2$ is granted; we also have $\|\nabla R_t f\|_2 \le \|\nabla f\|_2$. Indeed, since functions of Δ commute

$$\begin{aligned}(6)\quad \|\nabla R_t f\|_2 = \|(-\Delta)^{1/2} R_t f\|_2 = \|R_t(-\Delta)^{1/2} f\|_2 \\ \le \|(-\Delta)^{1/2} f\|_2 = \|\nabla f\|_2.\end{aligned}$$

Hence $\|\mathcal{C}_t f\|_2 \le 2\|\nabla f\|_2$. The cancellation in the commutator brings a better result.

Theorem 1. *We have*

$$(7) \qquad \int_0^\infty \int_\Omega |\mathcal{C}_t f(x)|^2 \frac{dxdt}{t} \le c \int_\Omega |\nabla f(x)|^2 \, dx, \quad f \in V.$$

Here and from now on $V = H_0^1(\Omega)$ for a Dirichlet boundary condition and $V = H^1(\Omega)$ for a Neumann boundary condition.

Remark. The constant c depends only on the Lipschitz character of Ω when Ω is a special Lipschitz domain (see Section 4). When Ω is a bounded Lipschitz domain, the proof gives a right hand side of the form $c(\int_\Omega |\nabla f(x)|^2 \, dx + d^{-2} \int_\Omega |f(x)|^2 \, dx)$ where d has the homogeneity of a length and c depends only on the Lipschitz character of Ω. On gets rid of $\int_\Omega |f(x)|^2 \, dx$ by Poincaré's inequality, but this makes the constant in (7) depend on other geometrical parameters of the domain. Still, it remains invariant under rigid motion between domains.

3. The case of the upper half-space

It is interesting to examine in some detail the case of a flat boundary ($\Omega = \mathbb{R}^n_+$) since the reflection principle yields a kernel representation of the commutator.

Let $E(x)$ be the fundamental solution of $1 - \Delta$ on $\mathbb{R}^n$, which vanishes at ∞. Hence $E_t(x) = \frac{1}{t^n} E(\frac{x}{t})$ is the fundamental solution of $1 - t^2\Delta$ and the kernel of R_t is given by

$$E_t(x-y) \pm E_t(x-y^*),$$

where the $-$ sign is for Dirichlet boundary condition and the kernel is called $G_t(x,y)$ and the $+$ sign is for Neumann boundary condition and the kernel is called $N_t(x,y)$. Here, $y^* = (y_1, \ldots, y_{n-1}, -y_n)$ denotes the reflection of y across the boundary in the vertical direction.

Since R_t is of convolution type in the first $n-1$ variables, it commutes with $\frac{\partial}{\partial x_j}$ for $1 \leq j \leq n-1$. When $j = n$, easy computations give a kernel representation of $\mathcal{C}_t f$ in terms of $\frac{\partial f}{\partial x_n}$. More precisely, for $f \in C_0^1(\mathbb{R}^n_+)$,

$$\text{(8a)} \quad \int_{\mathbb{R}^n_+} \big(G_t(x,y)\frac{\partial f(y)}{\partial y_n} - \frac{\partial G_t(x,y)}{\partial x_n} f(y)\big)\, dy = -2\int_{\mathbb{R}^n_+} E_t(x-y^*)\frac{\partial f(y)}{\partial y_n}\, dy$$

and for $f \in C_0^1(\overline{\mathbb{R}^n_+})$,

$$\text{(8b)} \quad \int_{\mathbb{R}^n_+} \big(N_t(x,y)\frac{\partial f(y)}{\partial y_n} - \frac{\partial N_t(x,y)}{\partial x_n} f(y)\big)\, dy = +2\int_{\mathbb{R}^n_+} E_t(x-y^*)\frac{\partial f(y)}{\partial y_n}\, dy.$$

Note that (8a) would not hold if f had a non vanishing trace on the boundary.

Hence (7) in both cases follows from an inequality of the type

$$\int_0^\infty \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_+} E_t(x-y^*)u(y)\, dy \right|^2 \frac{dxdt}{t} \leq C \int_{\mathbb{R}^n_+} |u(y)|^2\, dy.$$

As we shall see in Section 4.3, this is basically a consequence of Hardy inequality together with the classical estimates for E which we now recall (see [**7**]).

Lemma 2. *For all $x \neq 0$, $E(x) > 0$, and there exists non-negative constants such that*

$$c_1 |x|^{2-n} e^{-\alpha_1 |x|} \leq E(x) \leq c_2 |x|^{2-n} e^{-\alpha_2 |x|}$$

with $|x|^{2-n}$ replaced by $\ln(2 + |x|^{-1})$ when $n = 2$.

In the case of a general Lipschitz domain, kernel representation will also be our basic tool together with Green formula.

4. Special Lipschitz domains

Assume that Ω is the open set above a Lipschitz graph. Let us introduce some notations. We are given a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\|\nabla\phi\|_\infty = M < \infty$. The Lipschitz character of Ω is this number M. Then $\Omega = \{(x', x_n) \in \mathbb{R}^n;\, x_n > \phi(x')\}$. If $x = (x', x_n) \in \Omega$ then $\bar{x} = (x', \phi(x'))$ is its vertical projection on $\partial\Omega$ and $x^* = (x', 2\phi(x') - x_n)$ is its vertical reflection across $\partial\Omega$. We shall consistently use the notation $\bar{x}$ to denote a point on $\partial\Omega$. It is worth noticing that $x \to x^*$ is a bilipschitz transformation from Ω onto ${}^c\Omega$ with jacobian determinant equal to 1. Finally, σ denotes surface measure on $\partial\Omega$ and $N(x)$ the exterior unit normal.

We first assume that Ω is smooth, that is with a C^∞ boundary, in order to make the computations rigourous. The limiting argument is done in Section 4.4. Of course, we only use quantitatively the Lipschitz character of Ω.

4.1. Dirichlet boundary condition.

Let $G_t(x, y)$ be the Green's function of $I - t^2\Delta$ on Ω. It is defined for $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ and $x \neq y$. Elliptic boundary regularity tells us that G_t is C^∞ where it is defined. Moreover, for fixed $y \in \Omega$, G_t and $\nabla_x G_t$ decay exponentially fast as $|x| \to \infty$, $x \in \overline{\Omega}$.

Define a function $H_t(x, y)$ by

$$G_t(x, y) = E_t(x - y) - H_t(x, y), \quad (x, y) \in \overline{\Omega} \times \overline{\Omega},\ x \neq y, \tag{9}$$

where $E_t(x)$ is the fundamental solution of $I - t^2\Delta$ on $\mathbb{R}^n$ as introduced above. In other words, for fixed $y \in \Omega$, $H_t(\cdot, y)$ satisfies

$$\begin{cases} (I - t^2\Delta_x)H_t(x, y) = 0, & \text{in } \Omega, \\ H_t(x, y) = E_t(x - y), & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Also $H_t(x, y)$ and $\nabla_x H_t(x, y)$ have exponential decay as $|x| \to \infty$, $x \in \overline{\Omega}$.

We now come to the proof of (7). By density, it suffices to take $f \in C_0^1(\Omega)$. Denote by E_t and H_t the operators on $C_0^1(\Omega)$ associated with the kernels above. Since E_t is a convolution operator, $[\nabla, E_t] = 0$ when acting on $C_0^1(\Omega)$. Thus, we have

$$\mathcal{C}_t f = -\nabla H_t f + H_t \nabla f, \quad f \in C_0^1(\Omega).$$

The difference no longer plays any role and (7) will follow from

$$I_1 = \int_0^\infty \int_\Omega |H_t \nabla f(x)|^2 \frac{dxdt}{t} \le c \int_\Omega |\nabla f(x)|^2 \, dx \tag{11a}$$

$$I_2 = \int_0^\infty \int_\Omega |\nabla H_t f(x)|^2 \frac{dxdt}{t} \le c \int_\Omega |\nabla f(x)|^2 \, dx. \tag{11b}$$

Pointwise interior estimates on $H_t(x,y)$ give us a direct proof of (11a). The proof of (11b) is more involved and use instead boundary representation and Rellich inequalities in the spirit of [**5**].

We begin with controlling I_1. Let us make a simple but useful geometric observation whose proof is left to the reader.

Lemma 3. *There is a constant $c(M) > 1$ ($c(M) = \sqrt{1+M^2} + M$ works) such that*

$$c(M)^{-1}|\bar{x} - y^*| \le |\bar{x} - y| \le c(M)|\bar{x} - y^*|, \quad \bar{x} \in \partial\Omega,\, y \in \Omega. \tag{12}$$

The key estimate is the following

Lemma 4. *There are numbers $c > 0$ and $a \ge 1$ depending only on M such that*

$$0 \le H_t(x,y) \le cE_{at}(x - y^*), \quad x, y \in \Omega,\, t > 0. \tag{13}$$

Proof: Let us observe that $0 \le H_t(x,y)$ follows from the minimum principle, since $E_t(\bar{x} - y)$ is non-negative everywhere for $\bar{x} \in \partial\Omega$.

Let $\alpha_2 < \alpha_1$ be the constants of Lemma 2 and set $a = c(M)\alpha_1/\alpha_2$ where $c(M)$ is the constant in (12). Then, there exists a constant $c > 0$ such that

$$E_t(\bar{x} - y) \le cE_{at}(\bar{x} - y^*), \quad \bar{x} \in \partial\Omega,\, y \in \Omega,\, t > 0.$$

Hence, the function $u(x) = cE_{at}(x - y^*) - H_t(x,y)$ is positive on $\partial\Omega$.

Now, using (10) we also have

$$u(x) - t^2\Delta u(x) = c\frac{a^2 - 1}{a^2} E_{at}(x - y^*)$$

so that $u - t^2\Delta u \ge 0$ in Ω since $a \ge 1$. It follows from the minimum principle that u is positive on Ω. ■

The estimate (11a) is now a simple consequence of Lemma 5 whose proof is postponed for the moment.

Lemma 5. *Define on $\mathbb{R}^n$, $w(x) = |x|^{-\beta} e^{-\alpha|x|}$, where $\alpha > 0$ and $\beta < n-1$. Define $w_t(x) = t^{-n} w(x/t)$ and $A_t f(x) = \int_\Omega w_t(x-y^*) f(y)\, dy$ for $f \in L^2(\Omega)$. Then,*

$$\int_0^\infty \int_\Omega |A_t f(x)|^2 \frac{dxdt}{t} \leq c \int_\Omega |f(x)|^2\, dx,$$

where c depends only on M, α, β and n.

We now turn to proving (11b). First we reduce things to boundary integrals.

Lemma 6. *Let $u \in C^2(\overline{\Omega})$ be a real valued function satisfying $(I - t^2\Delta)u = 0$ in Ω. Assume furthermore that u and ∇u have rapid decay at ∞. Then*

$$\int_\Omega |\nabla u|^2 + \frac{1}{t^2} \int_\Omega |u|^2 \leq \sqrt{\int_{\partial\Omega} t |\nabla u(\bar{x}) \cdot N(\bar{x})|^2\, d\sigma(\bar{x}) \int_{\partial\Omega} t^{-1} |u(\bar{x})|^2\, d\sigma(\bar{x})}.$$

The proof is classical. Integrate by parts $0 = \int_{\Omega_R} u(I - t^2\Delta)u$ using Green's theorem on $\Omega_R = \Omega \cap B_R(0)$ where $B_R(0)$ is the ball of radius R centered at 0 to get (for R large enough)

$$t^{-2} \int_{\Omega_R} u^2 + \int_{\Omega_R} |\nabla u|^2 = \int_{\partial\Omega_R} (\nabla u \cdot N)\, u\, d\sigma.$$

Then apply Cauchy-Schwarz inequality and let R tends to ∞. ∎

For $f \in C_0^1(\Omega)$ the qualitative properties of $H_t(x,y)$ imply that for each $t > 0$, $u = H_t f$ satisfies the hypothesis of Lemma 6. Hence, we have $I_2 \leq \sqrt{I_3 I_4}$, where

$$I_3 = \int_0^\infty \int_{\partial\Omega} |H_t f(\bar{x})|^2 \frac{d\sigma(\bar{x})\, dt}{t^2},$$

and

$$I_4 = \int_0^\infty \int_{\partial\Omega} |\nabla H_t f(\bar{x}) \cdot N(\bar{x})|^2\, d\sigma(\bar{x})\, dt.$$

The integral I_3 is the easiest to deal with. Since f vanishes on $\partial\Omega$, we can write

$$f(y) = \int_{\bar{y}_n}^{y_n} \frac{\partial f}{\partial y_n}(\bar{y} + u e_n)\, du,$$

where $e_n = (0, \dots, 0, 1)$ so that we obtain, using Fubini's theorem,

$$H_t f(\bar{x}) = \int_\Omega \widetilde{H}_t(\bar{x}, y) \frac{\partial f}{\partial y_n}(y)\, dy,$$

where

$$\widetilde{H}_t(\bar{x}, y) = \int_{y_n}^\infty H_t(\bar{x}, \bar{y} + u e_n)\, du.$$

Next, using Lemma 4 and Lemma 3, we obtain

$$|\widetilde{H}_t(\bar{x}, y)| \le c t^{-2} |\bar{x} - y|^{3-n}\, e^{-\frac{\alpha|\bar{x}-y|}{t}}$$

when $n \ge 4$ and similar estimates when $n = 2$ and $n = 3$. We conclude applying the following lemma to the operator $\frac{H_t}{t}$ whose proof is in Section 4.3.

Lemma 7. *Define on $\mathbb{R}^n$, $w(x) = |x|^{-\beta} e^{-\alpha|x|}$, where $\alpha > 0$ and $\beta < n-1$. Define $w_t(x) = t^{-n} w(x/t)$ and $B_t f(x) = \int_\Omega w_t(x-y) f(y)\, dy$ for $f \in L^2(\Omega)$. Then,*

$$\int_0^\infty \int_{\partial\Omega} |B_t f(\bar{x})|^2\, d\sigma(\bar{x})\, dt \le c \int_\Omega |f(x)|^2\, dx,$$

where c depends only on M, α, β and n.

Of course, one cannot do the same thing with $\nabla_x H_t(x, y)$ as we do not have enough information on the normal component (recall that we do not want to use quantitatively the smoothness assumption on Ω). This is where we use the Rellich identity.

Lemma 8. *Let $u \in C^2(\overline{\Omega})$ be a real valued function satisfying $(I - t^2\Delta)u = 0$ in Ω. Assume furthermore that u and ∇u have rapid decay at ∞. Then*

$$\int_{\partial\Omega} |\nabla u(\bar{x}) \cdot N(\bar{x})|^2\, d\sigma(\bar{x}) \le c \int_{\partial\Omega} |\nabla_T u(\bar{x})|^2\, d\sigma(\bar{x}),$$

where ∇_T denotes the tangential gradient at the boundary, and c depends only on M.

Proof: Let $e \in \mathbb{R}^n$. Observe that in Ω we have

$$\operatorname{div}\left(|\nabla u|^2 e - 2(\nabla u \cdot e)\nabla u\right) = -2(\nabla u \cdot e)\Delta u = -2(\nabla u \cdot e)\frac{u}{t^2} = -\frac{\nabla u^2 \cdot e}{t^2}.$$

Hence, by Stokes theorem (as in Lemma 6 do it on Ω_R and let R tend to ∞) we obtain

$$\int_{\partial\Omega} \big(|\nabla u|^2 (e \cdot N) - 2(\nabla u \cdot e)(\nabla u \cdot N)\big)\, d\sigma = \int_{\partial\Omega} \frac{u^2}{t^2}(e \cdot N)\, d\sigma.$$

The tangential gradient is defined by $\nabla_T u = \nabla u - (\nabla u \cdot N)N$ so that it is orthogonal to the normal derivative. Hence $|\nabla u|^2 = |\nabla_T u|^2 + |\nabla u \cdot N|^2$ and $(\nabla u \cdot e)(\nabla u \cdot N) = (\nabla_T u \cdot e)(\nabla u \cdot N) + |\nabla u \cdot N|^2 (e \cdot N)$. Hence

$$\int_{\partial\Omega} |\nabla u \cdot N|^2 (e \cdot N)\, d\sigma = \int_{\partial\Omega} |\nabla_T u|^2 (e \cdot N)\, d\sigma \\ - 2\int_{\partial\Omega} (\nabla_T u \cdot e)(\nabla u \cdot N)\, d\sigma - \int_{\partial\Omega} \frac{u^2}{t^2}(e \cdot N)\, d\sigma.$$

Note that all integrals converge thanks to the assumptions. Now choose $e = (0, \dots, 0, -1)$. Since Ω is a special Lipschitz domain, there exists $\alpha > 0$ depending on M only such that $1 \geq e \cdot N \geq \alpha$ a.e. on $\partial\Omega$. Using this and $|2\int_{\partial\Omega} (\nabla_T u \cdot e)(\nabla u \cdot N)\, d\sigma| \leq \frac{\alpha}{2}\int_{\partial\Omega} |\nabla u \cdot N|^2\, d\sigma + \frac{2}{\alpha}\int_{\partial\Omega} |\nabla_T u|^2\, d\sigma$, we conclude that

$$\frac{\alpha}{2}\int_{\partial\Omega} |\nabla u \cdot N|^2\, d\sigma \leq \left(1 + \frac{2}{\alpha}\right)\int_{\partial\Omega} |\nabla_T u|^2\, d\sigma$$

because $-\int_{\partial\Omega} \frac{u^2}{t^2}(e \cdot N)\, d\sigma \leq 0$. ■

Remark. As we were finishing writing this paper, we learned that Ancona proved the Rellich identity on bounded Lipschitz domains for functions in the domain of the Laplacian, *i.e.* $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$. See [**1**] for details. The weak version given here is enough for our needs.

Since f has compact support, this lemma applies to $u = H_t f$ for each $t > 0$ thanks to the qualitative properties of $H_t(x, y)$. Integrating with respect to t gives us

$$I_4 \leq c\int_0^\infty \int_{\partial\Omega} t^2 |\nabla_T H_t f(\bar{x})|^2\, d\sigma(\bar{x})\, dt.$$

By definition of H_t, we have $\nabla_T H_t f(\bar{x}) = -\nabla_T E_t f(\bar{x})$ on $\partial\Omega$, so that it suffices to prove

$$I_5 = \int_0^\infty \int_{\partial\Omega} t^2 |\nabla E_t f(\bar{x})|^2\, d\sigma(\bar{x})\, dt \leq c\int_\Omega |\nabla f|^2, \quad f \in C_0^1(\Omega),$$

where now the gradient is taken over all directions. Using the same technique as for I_3, we have

$$\nabla E_t f(\bar{x}) = \int_\Omega \widetilde{E}_t(\bar{x}, y) \frac{\partial f}{\partial y_n}(y)\, dy,$$

where

$$\widetilde{E}_t(\bar{x}, y) = \int_{y_n}^{\infty} \nabla E_t(\bar{x} - \bar{y} - u e_n)\, du.$$

It is well-known that $|\nabla E(x)| \le c|x|^{1-n} e^{-\alpha|x|}$, so that an easy calculation gives

$$|\widetilde{E}_t(\bar{x}, y)| \le c t^{-2} |\bar{x} - y|^{2-n}\, e^{-\alpha|\bar{x}-y|/t},$$

with the usual change if $n = 2$. We apply again Lemma 7. The proof of Theorem 1 in this special case is complete.

Remark. We owe Steve Hofmann another argument to estimate I_2 which avoids the use of Rellich identity. The idea is to prove the identity,

$$(14)\quad \int_0^\infty \int_\Omega \left(|\nabla u_t|^2 + \frac{|u_t|^2}{t^2} \right) \frac{dxdt}{t} = \int_\Omega \int_\Omega K(x,y) \frac{\partial f}{\partial y_n}(y) \frac{\partial f}{\partial y_n}(x)\, dx\, dy$$

whenever $u_t = H_t f$, where $|K(x,y)| \le c|x - y^*|^{-n}$ for some number c depending only on M.

Indeed, an application of Hardy's inequality yields

$$\left| \int_\Omega \int_\Omega |x - y^*|^{-n} g(y) g(x)\, dx\, dy \right| \le c \int_\Omega |g|^2$$

for some number c depending only on M. Thus, (14) implies $I_2 \le c\|\partial f/\partial y_n\|_2^2$, which is even more precise than (11b).

Let us prove (14). Fix $t > 0$. We start out with the similar integration by parts:

$$\int_\Omega \left(|\nabla u_t|^2 + \frac{|u_t|^2}{t^2} \right) = \int_{\partial\Omega} \nabla u_t(\bar{x}) \cdot N(\bar{x}) u_t(\bar{x})\, d\sigma(\bar{x}).$$

Now, replace in the normal derivative $u_t = H_t f$ by $E_t f - R_t f$ to obtain

$$\begin{aligned}\int_{\partial\Omega} \nabla u_t(\bar{x}) \cdot N(\bar{x}) u_t(\bar{x})\, d\sigma(\bar{x}) &= \int_{\partial\Omega}\int_\Omega \nabla_x E_t(\bar{x} - y) \cdot N(\bar{x}) f(y) u_t(\bar{x})\, d\sigma(\bar{x})\, dy \\ &\quad - \int_{\partial\Omega}\int_\Omega \nabla_x G_t(\bar{x}, y) \cdot N(\bar{x}) f(y) u_t(\bar{x})\, d\sigma(\bar{x})\, dy.\end{aligned}$$

The key observation comes from Green's theorem which gives us that for all $y \in \Omega$,

$$\int_{\partial\Omega} \nabla_x G_t(\bar{x}, y) \cdot N(\bar{x}) u_t(\bar{x})\, d\sigma(\bar{x}) = -\frac{u_t(y)}{t^2}.$$

Hence, by Fubini's theorem,

$$\int_{\partial\Omega} \int_{\Omega} \nabla_x G_t(\bar{x}, y) \cdot N(\bar{x}) f(y) u_t(\bar{x})\, d\sigma(\bar{x})\, dy = -\int_{\Omega} f(y) \frac{u_t(y)}{t^2}\, dy.$$

Once this is done it suffices to replace u_t by $H_t f$ and to express as before f in terms of $\partial f / \partial_{y_n}$ since f vanishes on $\partial\Omega$. Then integrate in t to obtain an explicit but quite messy expression for $K(x, y)$. The control of its size uses Lemma 2, Lemma 3, Lemma 4 and elementary calculations which we leave to the interested reader.

4.2. Neumann boundary condition.

We now work with the Neumann Laplacian. We prove

$$\int_0^\infty \int_\Omega |\mathcal{C}_t f(x)|^2 \frac{dxdt}{t} \le c \int_\Omega |\nabla f|^2, \quad f \in C_0^1(\overline{\Omega}) \tag{15}$$

by following a similar strategy. This is enough since $C_0^1(\overline{\Omega})$ is a dense subspace of $V = H^1(\Omega)$.

Define a function $F_t(x, y)$ on $\Omega \times \Omega$ by

$$N_t(x, y) = E_t(x - y) + E_t(x - y^*) + F_t(x, y), \tag{16}$$

where $N_t(x, y)$ is the Neumann function of $I - t^2\Delta$ on Ω. In other words, for fixed $y \in \Omega$, $F_t(\cdot, y)$ satisfies

$$\begin{cases} (I - t^2\Delta_x) F_t(x, y) = 0, & \text{in } \Omega, \\ \nabla_x F_t(\bar{x}, y) \cdot N(\bar{x}) = -\nabla_x (E_t(\bar{x} - y) + E_t(\bar{x} - y^*)) \cdot N(\bar{x}), & \text{on } \partial\Omega. \end{cases} \tag{17}$$

This is a simple reformulation of the definition of $N_t(x, y)$. Since Ω is smooth, $N_t(x, y)$ extends smoothly to $\overline{\Omega} \times \overline{\Omega}$ away from the diagonal and for fixed $y \in \Omega$, $N_t(x, y)$ and $\nabla_x N_t(x, y)$ decay rapidly to 0 as $|x| \to \infty$ with $x \in \overline{\Omega}$. The same properties hold for $F_t(x, y)$ as well. Let us list further properties of $F_t(x, y)$.

Lemma 9.

(i) *For some constants $c, a > 0$ depending only on M, we have*

$$|F_t(x,y)| \le cE_{at}(x-y^*), \quad t>0,\, x\in\overline{\Omega},\, y\in\Omega.$$

(ii) *For all $x \in \overline{\Omega}$ and $t>0$,*

$$\int_\Omega F_t(x,y)\,dy = 0.$$

Remark. The mean value property is of crucial importance; it is the reason of our choice for $F_t(x,y)$.

Proof: To see (i), recall by (12) that $|\bar{x}-y| \sim |\bar{x}-y^*|$ for $\bar{x}\in\partial\Omega$ and $y\in\Omega$. Since Ω has the extension property, we have the estimate (see [**3**]),

$$|N_t(\bar{x},y)| \le ct^{-2}|\bar{x}-y|^{2-n}e^{-\frac{\alpha|\bar{x}-y|}{t}}, \quad t>0,\, \bar{x}\in\partial\Omega,\, y\in\Omega,$$

where c depends only on M, while α is independent of Ω. Hence, we have

$$|F_t(\bar{x},y)| \le cE_{at}(\bar{x}-y^*), \quad t>0,\, \bar{x}\in\partial\Omega,\, y\in\Omega,$$

for some $c>0$ and $a>1$. We conclude as in the proof of Lemma 4 applying the minimum principle to the functions $cE_{at}(x-y^*)\pm F_t(x,y)$.

To prove (ii), it suffices to remark that by change of variable $y\mapsto y^*$

$$\int_\Omega E_t(x-y)+E_t(x-y^*)\,dy = \int_{\mathbb{R}^n} E_t(x-y)\,dy = 1 \tag{18}$$

and that $\int_\Omega N_t(x,y)\,dy = 1$ from the construction of $N_t(x,y)$. The lemma is proved. ■

Lemma 10. *For $f\in C_0^1(\overline{\Omega})$ and $x\in\Omega$, we have*

$$\begin{aligned}\mathcal{C}_t f(x) = 2\int_\Omega E_t(x-y^*)\frac{\partial f}{\partial y_n}(y)\widetilde{N}(\bar{y})\,dy \\ - \int_\Omega F_t(x,y)\nabla f(y)\,dy + \int_\Omega \nabla_x F_t(x,y) f(y)\,dy,\end{aligned}$$

where

$$\widetilde{N}(\bar{y}) = \sqrt{1+|\nabla\phi(y')|^2}N(\bar{y}), \quad \bar{y}=(y',\phi(y')).$$

Proof: Denote by E_t, $E_{t,*}$ and F_t the operators defined on $C_0^1(\overline{\Omega})$ associated with the kernels in (16). Writing

$$\mathcal{C}_t f(x) = [\nabla, E_t]f(x) + [\nabla, E_{t,*}]f(x) - F_t \nabla f(x) + \nabla F_t f(x),$$

we have to show

$$[\nabla, E_t]f(x) + [\nabla, E_{t,*}]f(x) = 2\int_\Omega E_t(x-y^*)\frac{\partial f}{\partial y_n}(y)\widetilde{N}(\bar{y})\,dy.$$

Note that although E_t is a convolution operator, $[\nabla, E_t]$ is not 0 when acting on smooth functions that do not vanish on the boundary of Ω. Indeed, integrating by parts via Green's theorem (with the classical way of taking care of the singularity at x for $E_t(x-y)$ and using $\nabla_x(E_t(x-y)) = -\nabla_y(E_t(x-y))$) yield

$$[\nabla, E_t]f(x) = -\int_{\partial\Omega} E_t(x-\bar{y})f(\bar{y})N(\bar{y})\,d\sigma(\bar{y}).$$

Now,

$$E_{t,*}f(x) = \int_\Omega E_t(x-y^*)f(y)\,dy = \int_{^c\Omega} E_t(x-y)f(y^*)\,dy,$$

and since N points into ${}^c\Omega$, we have

$$\begin{aligned}\nabla_x E_{t,*}f(x) &= -\int_{^c\Omega} \nabla_y(E_t(x-y))f(y^*)\,dy\\ &= +\int_{\partial\Omega} E_t(x-\bar{y})f(\bar{y})N(\bar{y})\,d\sigma(\bar{y}) + \int_{^c\Omega} E_t(x-y)\nabla_y(f(y^*))\,dy.\end{aligned}$$

Using the change of variables $y \to y^*$ one can see that

$$\int_{^c\Omega} E_t(x-y)\nabla_y(f(y^*))\,dy = \int_\Omega E_t(x-y^*)J(\bar{y})\nabla f(y)\,dy$$

where $J(\bar{y})$ is the jacobian matrix of $y \to y^*$, which depends only on $\bar{y}$. A straightforward computation shows that

$$J(\bar{y})\nabla f(y) - \nabla f(y) = 2\frac{\partial f}{\partial y_n}(y)\widetilde{N}(\bar{y}),$$

where $\widetilde{N}(\bar{y})$ is defined in the statement. Thus,

$$\begin{aligned}[\nabla, E_{t,*}]f(x) = +\int_{\partial\Omega} &E_t(x-\bar{y})\,f(\bar{y})\,N(\bar{y})\,d\sigma(\bar{y})\\ &+ 2\int_\Omega E_t(x-y^*)\frac{\partial f}{\partial y_n}(y)\widetilde{N}(\bar{y})\,dy.\end{aligned}$$

Putting altogether these equalities proves the lemma. ■

We now prove (15). Using Lemma 5 and the interior estimates on $E_t(x-y^*)$ and $F_t(x,y)$, we obtain good control on both terms $\int_\Omega E_t(x-y^*)\frac{\partial f}{\partial y_n}(y)\widetilde{N}(\bar{y})\,dy$ and $\int_\Omega F_t(x,y)\nabla f(y)\,dy$. By Lemma 10, it remains to look at the last term, namely $\int_\Omega \nabla_x F_t(x,y)f(y)\,dy$. We use again boundary integrals. By the qualitative properties of F_t and the fact that f has compact support, we easily see that $u = F_t f$ satisfies the hypotheses of Lemma 6 and we have

$$2\int_\Omega |\nabla F_t f|^2 \le \int_{\partial\Omega} t|\nabla F_t f(\bar{x})\cdot N(\bar{x})|^2\,d\sigma(\bar{x}) + \int_{\partial\Omega} t^{-1}|F_t f(\bar{x})|^2\,d\sigma(\bar{x}).$$

Thus, it is enough to show that

$$J_1 = \int_0^\infty \int_{\partial\Omega} |F_t f(\bar{x})|^2 \frac{d\sigma(\bar{x})dt}{t^2} \le c\int_\Omega |\nabla f|^2 \tag{19a}$$

and

$$J_2 = \int_0^\infty \int_{\partial\Omega} |\nabla F_t f(\bar{x})\cdot N(\bar{x})|^2\,d\sigma(\bar{x})\,dt \le c\int_\Omega |\nabla f|^2. \tag{19b}$$

To estimate J_1, we use the mean value property (ii) in Lemma 9 to write

$$F_t f(\bar{x}) = \int_\Omega F_t(\bar{x},y)(f(y)-f(\bar{x}))\,dy.$$

Since

$$f(y)-f(\bar{x}) = f(\bar{y})-f(\bar{x}) + \int_{\bar{y}_n}^{y_n} \frac{\partial f}{\partial y_n}(\bar{y}+ue_n)\,du,$$

we obtain, using Fubini's theorem,

$$F_t f(\bar{x}) = \int_{\partial\Omega} F_t(\bar{x},\bar{y})(f(\bar{y})-f(\bar{x}))\,d\sigma(\bar{y}) + \int_\Omega \widetilde{F}_t(\bar{x},y)\frac{\partial f}{\partial y_n}(y)\,dy, \tag{20}$$

where

$$\widetilde{F}_t(\bar{x},y) = \int_{y_n}^\infty F_t(\bar{x},\bar{y}+ue_n)\,du.$$

The term with $\widetilde{F}_t(\bar{x},y)$ is handled similarly as the one with $\widetilde{H}_t(x,y)$ in the Dirichlet case, by using Lemma 7.

The first integral in (20) is of a new type and can be estimated by using the following result.

Lemma 11. *Define on* $\mathbb{R}^n$, $w(x) = |x|^{-\beta} e^{-\alpha|x|}$, *where* $\alpha > 0$ *and* $\beta < n-1$. *Define* $w_t(x) = t^{-n} w(x/t)$ *and* $C_t f(\bar{x}) = \int_{\partial\Omega} w_t(\bar{y}-\bar{x})(f(\bar{y}) - f(\bar{x}))\, d\sigma(\bar{y})$ *for* $f \in C_0^1(\partial\Omega)$. *Then,*

$$\int_0^\infty \int_{\partial\Omega} |C_t f(\bar{x})|^2 \, d\sigma(\bar{x})\, dt \leq c \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(\bar{y}) - f(\bar{x})|^2}{|\bar{y} - \bar{x}|^n} \, d\sigma(\bar{x})\, d\sigma(\bar{y}), \tag{21}$$

where c *depends only on* M, α, β *and* n.

Admit this lemma for the moment. Then, it is classical that the integral in the right hand side of (21) is equivalent to the square of the norm of f in the homogeneous Sobolev space $\dot{H}^{1/2}(\partial\Omega)$, and the trace theorem, therefore, implies that

$$\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(\bar{y}) - f(\bar{x})|^2}{|\bar{y} - \bar{x}|^n} \, d\sigma(\bar{x})\, d\sigma(\bar{y}) \leq c \int_\Omega |\nabla f|^2.$$

See [**6**]. This yields the desired control of J_1 and (19a) is proved.

We now turn to the control of J_2. The explicit value of the normal derivative of $F_t(x,y)$ at the boundary given by (17) implies

$$\nabla F_t f(\bar{x}) \cdot N(\bar{x}) = -\nabla(E_t + E_{t,*}) f(\bar{x}) \cdot N(\bar{x}).$$

Now, observe that as in (18)

$$\int_\Omega (\nabla E_t)(x-y) + (\nabla E_t)(x-y^*)\, dy = \nabla_x \int_{\mathbb{R}^n} E_t(x-y)\, dy = \nabla_x 1 = 0.$$

Hence, we can proceed as before and obtain for $(\nabla F_t f)(\bar{x}) \cdot N(\bar{x})$ a representation similar to (20) where $\widetilde{F}_t(\bar{x}, y)$ is replaced by

$$F_t^\sharp(\bar{x}, y) = -\int_{y_n}^\infty \{(\nabla E_t)(\bar{x} - \bar{y} - ue_n) + (\nabla E_t)(\bar{x} - \bar{y} + ue_n))\}\, du.$$

Since $\bar{x} \in \partial\Omega$, we know that $|\bar{x} - \bar{y} - ue_n| \sim |\bar{x} - \bar{y} + ue_n|$ so that both terms can be treated similarly using Lemma 11. Further details are left to the reader. This proves (19b) and with it (15).

4.3. Proofs of technical lemmas.

For simplicity, we assume that $\Omega = \mathbb{R}^n_+$. If not the case, we can pull back Ω to $\mathbb{R}^n_+$ via the transformation $F : (x', x_n) \mapsto (x', \phi(x') + x_n)$ which is a bilipschitz homeomorphism from $\mathbb{R}^n_+$ onto Ω and with jacobian determinant equal to 1. With $\Omega = \mathbb{R}^n_+$, we have $x^* = (x', -x_n)$ when $x = (x', x_n)$ and the boundary is identified with $\mathbb{R}^{n-1}$ via $\bar{x} = (x', 0)$.

Proof of Lemma 5: It is easy to obtain from the definition of w that

$$w_t(x - y^*) \le \frac{c}{t^n} \widetilde{w}\left(\frac{x' - y'}{t}\right) \exp\left(-\frac{\alpha(x_n + y_n)}{t}\right)$$

where $\widetilde{w}(x') = |x'|^{-\beta} e^{-\alpha|x'|} \in L^1(\mathbb{R}^{n-1})$. Hence, Young's inequality and the definition of A_t give us

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} |A_t f(x', x_n)|^2 \, dx' \\
&\qquad \le e^{-\frac{2\alpha x_n}{t}} \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^{n-1}} \frac{c}{t^{n-1}} \widetilde{w}\left(\frac{x' - y'}{t}\right) f_t(y') \, dy' \right|^2 dx' \\
&\qquad \le c e^{-\frac{2\alpha x_n}{t}} \int_{\mathbb{R}^{n-1}} |f_t(x')|^2 \, dx',
\end{aligned}$$

where $f_t(x') = \frac{1}{t} \int_0^\infty e^{-\frac{\alpha u}{t}} f(x', u) \, du$. Thus, integrating with respect to x_n yields

$$\int_{\mathbb{R}^n_+} |A_t f(x)|^2 \, dx \le c \int_{\mathbb{R}^{n-1}} \frac{1}{t} \left| \int_0^\infty e^{-\frac{\alpha u}{t}} f(x', u) \, du \right|^2 dx'.$$

Next, we expand the square and integrate with respect to $\frac{dt}{t}$ to obtain a bound

$$c \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_0^\infty \frac{f(x', u) \overline{f(x', v)}}{u + v} \, du \, dv \, dx'$$

which, by Hardy's bilinear inequality ([**4**, p. 229]), is controlled by

$$c\pi \int_{\mathbb{R}^{n-1}} \int_0^\infty |f(x', u)|^2 \, du \, dx'.$$

The proof is complete. ■

Proof of Lemma 7: Using the same setting and notation as in the previous proof, we have

$$|B_t f(x',0)| \leq \frac{c}{t^{n-1}} \int_{\mathbb{R}^{n-1}} \widetilde{w}\left(\frac{x'-y'}{t}\right) |f_t(y')|\,dy'.$$

Since $\widetilde{w} \in L^1(\mathbb{R}^{n-1})$, we obtain

$$\int_{\mathbb{R}^{n-1}} |B_t f(x',0)|^2\,dx' \leq c \int_{\mathbb{R}^{n-1}} |f_t(x')|^2\,dx'.$$

Now, integrate against dt and finish the argument as in the preceding proof. ■

Proof of Lemma 11: We write $\bar{x} = (x',0)$. By Schwarz inequality, since $w \in L^1(\mathbb{R}^{n-1})$

$$|C_t f(x',0))|^2 \leq \frac{c}{t^{n+1}} \int_{\mathbb{R}^{n-1}} w\left(\frac{y'-x'}{t}\right) |f(y')-f(x')|^2\,dy'.$$

Now, $\int_0^\infty w(\frac{u}{t})\frac{dt}{t^{n+1}} = \frac{c}{u^n}$, hence

$$\int_0^\infty |C_t f(x',0)|^2\,dt \leq c \int_{\mathbb{R}^{n-1}} \frac{|f(y')-f(x')|^2}{|y'-x'|^n}\,dy'$$

and the conclusion follows immediately by integrating with respect to x'. ■

4.4. Limiting argument.

Let us use a particular change of variable introduced by Kenig and Stein to approximate a special Lipschitz domains by smooth ones with a uniform character.

Let $g \in C_0^\infty(\mathbb{R}^{n-1})$ be even with $g \geq 0$ and $\int g = 1$. Let $g_t(x) = t^{-n+1}g(x/t)$. Then, choosing $c > 0$ large enough ($c \geq 2M\int g(y)|y|\,dy$ works), the mapping

$$F(x',x_n) = (x', cx_n + (g_{x_n} * \phi)(x'))$$

is a bilipchitz change of variable between $\mathbb{R}^n_+$ and Ω. Now $\Omega_k = \{F(x',x_n);\ x'_n \geq 2^{-k}\}$ for $k = 1,2,\ldots$, is a C^∞ domain with Lipschitz character bounded by M. Remark also that $\Omega_k \uparrow \Omega$.

We begin with the case of a Dirichlet boundary condition. Let us see that if (7) holds for all Ω_k with a constant that depends only on M then it holds for Ω.

Denote by R_t^k and $\mathcal{C}_t^k$ respectively the resolvent of the Dirichlet Laplacian on Ω_k and the associated commutator. Let $f \in C_0^1(\Omega)$, then $f \in C_0^1(\Omega_k)$ for k large enough and $\mathcal{C}_t^k f$ is well-defined on Ω_k; extend it by 0 outside.

Lemma 12. *With the above notation, for all $t>0$, $\mathcal{C}_t^k f$ converges to $\mathcal{C}_t f$ in $L^2(\Omega)$ as k tends to ∞.*

Admitting this lemma, an application of Fatou lemma yields (7) on Ω for such an f and a density argument concludes the proof.

The proof of Lemma 12 follows from a classical fact which we recall for convenience. Let $g \in L^2(\Omega)$ and define $u_k = R_t^k g$ and $u = R_t g$. This means that $u \in H_0^1(\Omega)$ is the unique solution of

$$\int_\Omega u\, v + t^2 \int_\Omega \nabla u \cdot \nabla v = \int_\Omega g v, \quad \forall\, v \in H_0^1(\Omega),$$

and $u_k \in H_0^1(\Omega_k)$ is the unique solution of

$$\int_{\Omega_k} u_k\, v + t^2 \int_{\Omega_k} \nabla u_k \cdot \nabla v = \int_{\Omega_k} g v, \quad \forall\, v \in H_0^1(\Omega_k).$$

Extend u_k to be 0 outside of Ω_k then (u_k) is bounded in $H_0^1(\Omega)$, so it has a weakly convergent subsequence, which, by taking weak limits in the variational formulation (with v having compact support in Ω) must converge to u. Thus (u_k) converges weakly to u in $H_0^1(\Omega)$. Now expanding the squares and using the equations, one finds

$$\int_\Omega |u-u_k|^2 + t^2 \int_\Omega |\nabla(u-u_k)|^2 = \operatorname{Re} \int_\Omega g(\overline{u} - \overline{u}_k).$$

Letting k tend to ∞ proves that u_k converges strongly to u in $H_0^1(\Omega)$.

Now $\mathcal{C}_t^k f = R_t^k \nabla f - \nabla R_t^k f$ and the above fact applied with $g = f$ and $g = \nabla f$ proves the claim. ∎

We now turn to the case of a Neumann boundary condition. This time we approximate Ω from outside (applying the same construction to ${}^c\Omega$) so that $\Omega_k \downarrow \Omega$. Recall that $H^1(\Omega)$ is the space of restrictions to Ω of functions in $H^1(\mathbb{R}^n)$ and similarly for Ω_k replacing Ω. Let $f \in H^1(\mathbb{R}^n)$. Thus with evident notations $\mathcal{C}_t^k f$ and $\mathcal{C}_t f$ are well-defined on Ω_k and Ω respectively.

Lemma 13. *With the above notation, for all $t>0$, $\mathcal{C}_t^k f$ converges to $\mathcal{C}_t f$ in $L^2(\Omega)$ as k tends to ∞.*

Admitting this lemma, let $f \in H^1(\Omega)$ be extended in such a way that it belongs to $H^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |\nabla f|^2 \le c(M) \int_\Omega |\nabla f|^2$. By (7) on Ω_k we have

$$\int_0^\infty \int_{\Omega_k} |\mathcal{C}_t^k(f)|^2 \frac{dxdt}{t} \le c(M) \int_{\Omega_k} |\nabla f|^2 \le c(M) \int_\Omega |\nabla f|^2$$

and (7) on Ω follows from Fatou lemma and the above lemma.

To prove Lemma 13, it is enough to consider resolvents. Let $g \in L^2(\mathbb{R}^n)$, then $u = R_t g$ is the unique solution in $H^1(\Omega)$ of

$$\int_\Omega u\, v + t^2 \int_\Omega \nabla u \cdot \nabla v = \int_\Omega gv, \quad \forall\, v \in H^1(\Omega),$$

while $u_k = R_t^k g$ is the unique solution in $H^1(\Omega_k)$ of

$$\int_{\Omega_k} u_k\, v + t^2 \int_{\Omega_k} \nabla u_k \cdot \nabla v = \int_{\Omega_k} gv, \quad \forall\, v \in H^1(\Omega_k).$$

Recall also that $\int_{\Omega_k} |u_k|^2 + t^2 |\nabla u_k|^2 \le \int_{\Omega_k} |g|^2 \le \|g\|_2^2$. If $v \in H^1(\mathbb{R}^n)$ then

$$\begin{aligned}\int_\Omega (u - u_k)\, v + t^2 \int_\Omega \nabla(u - u_k) \cdot \nabla v &= \int_{\Omega_k - \Omega} gv - u_k\, v - t^2 \nabla u_k \cdot \nabla v \\ &\le 2\|g\|_2 \bigg(\int_{\Omega_k - \Omega} |v|^2 + t^2 |\nabla v|^2 \bigg)^{1/2}\end{aligned}$$

and this tends to 0 by dominated convergence. Thus u_k converges weakly to u in $H^1(\Omega)$.

Strong convergence now follows from the inequality

$$\int_{\Omega_k} |u - u_k|^2 + t^2 \int_{\Omega_k} |\nabla(u - u_k)|^2 \le \operatorname{Re} \int_\Omega g(\overline{u} - \overline{u}_k) + \operatorname{Re} \int_{\Omega_k - \Omega} g \overline{u}_k$$

obtained from the defining equations for u_k and u, the last integral being controlled by

$$\bigg(\int_{\Omega_k - \Omega} |g|^2 \bigg)^{1/2} \|u_k\|_2 \le \bigg(\int_{\Omega_k - \Omega} |g|^2 \bigg)^{1/2} \|g\|_2$$

which tends to 0 as $k \to \infty$ by dominated convergence. ∎

5. Bounded Lipschitz domains

We now assume that Ω is a bounded and connected Lipschitz domain with Lipschitz character M. Before going into details, let us remark that it is enough to obtain (7) with a right hand side equal to $c \int_\Omega (|f|^2 + |\nabla f|^2)$. Indeed if $f \in V = H_0^1(\Omega)$ (Dirichlet) Poincaré inequality yields $\int_\Omega |f|^2 \le C \int_\Omega |\nabla f|^2$. If $f \in V = H^1(\Omega)$ (Neumann), then it should be observed that the commutator annihilates constants so that $\mathcal{C}_t f = \mathcal{C}_t(f - m)$, hence the Poincaré-Wirtinger inequality $\int_\Omega |f - m|^2 \le C \int_\Omega |\nabla f|^2$ for $m = |\Omega|^{-1} \int_\Omega f$ yields the desired result. See the remark in Section 2 for the behavior of the constants.

Since Ω is a bounded Lipschitz domain, there exists a finite number of $C_0^\infty(\mathbb{R}^n)$ functions, $\chi_1, \ldots, \chi_s$, with the following properties:

1. $\sum_{1\le k\le s} \chi_k(x) = 1, \quad x \in \Omega$;
2. For each k, there exists $\widetilde{\Omega}_k$, image of a special Lipschitz domain under an orthogonal transformation in $\mathbb{R}^n$ such that $\operatorname{Supp}\chi_k \cap \Omega \subset \widetilde{\Omega}_k \cap \Omega$;
3. There exist open neighborhoods O_k, P_k of $\operatorname{Supp}\chi_k$ in $\Omega \cap \widetilde{\Omega}_k$ such that $\overline{O}_k \subset P_k$, $\Omega \cap \overline{P}_k \subset \widetilde{\Omega}_k \cap \overline{P}_k$ and $\partial\Omega \cap \overline{P}_k = \partial\widetilde{\Omega}_k \cap \overline{P}_k$.

The Lipschitz character M of Ω is, by definition, the infimum of the numbers $\sup \widetilde{M_k}$, where $\widetilde{M_k}$ is the Lipschitz character of $\widetilde{\Omega}_k$, taken over all decompositions of Ω in such a way. From now on the letter c denotes constants that depend only on M.

Since there are a finite number of sets, there is $d > 0$ such that $d(O_k, {}^cP_k) \ge d$ and $d(\operatorname{Supp}\chi_k, {}^cO_k) \ge d$ for all k. Denote by $\eta_k \in C_0^\infty(P_k)$ a real function such that $\eta_k = 1$ on $\overline{O}_k$. This distance d depends on the chosen partition: the largest possible value has an intrinsic geometrical meaning, that is not related to the Lipschitz character.

For large scales (ie $t > 1$) we have

Lemma 14. *For all $f \in V$*

$$\int_d^\infty \int_\Omega |R_t \nabla f(x)|^2 + |\nabla R_t f(x)|^2 \frac{dxdt}{t} \le \frac{n+1}{2d^2} \int_\Omega |f|^2.$$

Proof: This follows immediatly from $\|\nabla R_t f\|_2 \le t^{-1}\|f\|_2$ and

$$\begin{aligned}\|R_t \nabla f\|_2 &= \|R_t(-\Delta)^{1/2}(-\Delta)^{-1/2}\nabla f\|_2 \\ &\le t^{-1}\|(-\Delta)^{-1/2}\nabla f\|_2 \le \sqrt{n}t^{-1}\|f\|_2. \ \blacksquare\end{aligned}$$

From Lemma 14, it is enough to prove that

$$\int_0^d \int_\Omega |\mathcal{C}_t f(x)|^2 \frac{dxdt}{t} \le c\left(\int_\Omega |\nabla f|^2 + \frac{1}{d^2}\int_\Omega |f|^2\right).$$

Let us discuss first the case of a Dirichlet boundary condition.

Let $f \in H_0^1(\Omega)$ and write $f = \sum f_k$ with $f_k = f\chi_k \in V_k = H_0^1(\Omega) \cap H_0^1(\widetilde{\Omega}_k)$. Write

$$\begin{aligned}\mathcal{C}_t f &= \sum_{1\le k\le s} (1-\eta_k)\mathcal{C}_t f_k + \sum_{1\le k\le s} \eta_k(\mathcal{C}_t f_k - \mathcal{C}_t^k f_k) + \sum_{1\le k\le s} \eta_k \mathcal{C}_t^k f_k \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III},\end{aligned}$$

where $\mathcal{C}_t^k$ is the commutator defined on $\widetilde{\Omega}_k$ using the resolvent which we denote by R_t^k.

Analysis of III. By (7) on $\widetilde{\Omega}_k$ (this inequality is clearly invariant under orthogonal transformations) we have for each k

$$\int_0^d \int_\Omega |\eta_k(x)\mathcal{C}_t^k f_k(x)|^2 \frac{dxdt}{t} \le c\int_\Omega |\nabla f_k|^2 \le c\left(\int_\Omega |\nabla f|^2 + \frac{1}{d^2}\int_\Omega |f|^2\right).$$

Analysis of II. We use a Cacciopoli inequality for each term in the sum. We remark that

$$\eta_k(x)(\mathcal{C}_t f_k(x) - \mathcal{C}_t^k f_k(x)) = \eta_k(x)\big[(\nabla(R_t - R_t^k))f_k(x) - ((R_t - R_t^k)\nabla f_k)(x)\big].$$

If $g \in L^2$ satisfies $\operatorname{Supp} g \subset \operatorname{Supp}\chi_k$ and $u_k = (R_t - R_t^k)g$ we have

$$\int u_k\, v + t^2 \int \nabla u_k \cdot \nabla v = 0, \quad \forall\, v \in V_k.$$

Inserting $v = \overline{u}_k\, \eta_k^2$ yields

$$\int |u_k|^2\eta_k^2 + t^2\int |\nabla u_k|^2 \eta_k^2 = -2t^2 \int (\eta_k \nabla u_k)\cdot(\overline{u}_k \nabla \eta_k)$$

hence

$$\int |u_k|^2\eta_k^2 + \frac{t^2}{2}\int |\nabla u_k|^2\eta_k^2 \le 2t^2\int |u_k\nabla\eta_k|^2.$$

For $g = f_k$, we obtain

$$\int |\nabla(R_t - R_t^k)f_k|^2\eta_k^2 \le 4\int |(R_t - R_t^k)f_k|^2 |\nabla\eta_k|^2.$$

Now, by the maximum principle, the kernels of R_t and R_t^k are bounded by $at^{-2}|x-y|^{2-n}e^{-(\alpha|x-y|/t)}$ for some $a<\infty$ and $\alpha>0$ depending only on dimension (usual change if $n=2$). Since $|x-y|\geq d$ for $x\in \operatorname{Supp}\nabla\eta_k$ and $y\in\operatorname{Supp} f_k$ and $t\leq d$, is easy to deduce a bound of the form

$$\int |(R_t-R_t^k)f_k\nabla\eta_k|^2 \leq a/d^2 e^{-\alpha d/t}\|f_k\|_2^2$$

for other values of a and α. Similarly, for $g=\nabla f_k$, one obtains

$$\begin{aligned}\int |(R_t-R_t^k)\nabla f_k|^2|\eta_k|^2 &\leq 2t^2\int |(R_t-R_t^k)\nabla f_k|^2|\nabla\eta_k|^2\\ &\leq ae^{-\alpha d/t}\|\nabla f_k\|_2^2.\end{aligned}$$

It is then easy to get $\int_0^d\int_\Omega |\mathrm{II}|^2\frac{dxdt}{t}\leq c\int_\Omega(\frac{|f|^2}{d^2}+|\nabla f|^2)$.

Analysis of I. If $g\in L^2$ with $\operatorname{Supp} g\subset\operatorname{Supp}\chi_k$ and $u=R_tg\in H_0^1(\Omega)$ we have

$$\int u\,v+t^2\int\nabla u\cdot\nabla v=\int g\,v,\quad \forall\, v\in H_0^1(\Omega).$$

Inserting $v=\overline{u}(1-\eta_k)^2$ yields a null right hand side because $\eta_k=1$ on the support of g. We conclude using the same argument as before and the fact that $d(\operatorname{Supp} g,\operatorname{Supp}(1-\eta_k))\geq d$. Further details are left to the reader.

A similar analysis can be done under Neumann boundary condition (Here, the kernels of R_t and R_t^k have estimates with constants that depend on M). The proof of Theorem 1 is complete. ■

Remark. Minor modifications of the argument shows that the result in Theorem 1 is valid (with an extra term $c\|f\|_2^2$ on the right hand side of (7)) on all strongly Lipschitz domains, which are those connected open sets in $\mathbb{R}^n$ whose boundary is covered by finitely many parts of (rotated) Lipschitz graphs, possibly one of those parts being infinite.

6. Other commutators

Using functional calculus we can consider other commutators such as the ones in (1) and (2). The argument is identitical for the Dirichlet and Neumann Laplacians.

Proposition 15. *Let φ be a complex bounded continuous function on $[0,\infty)$ with $\varphi(r)=\varphi(0)+0(r^s)$ at 0 and $|r^s\varphi(r)|$ bounded for some $s>0$. Then*

$$\int_0^\infty\int_\Omega \big|[\nabla,\varphi(-t^2\Delta)]f(x)\big|^2\frac{dxdt}{t}\le c\int_\Omega|\nabla f(x)|^2\,dx,\quad f\in V. \tag{22}$$

Proof: Assume $\varphi(0)\neq 0$, otherwise apply the next argument to φ. There is no loss of generality to set $\varphi(0)=1$. Let $\psi(r)=\varphi(r)-(1+r)^{-1}$. Then $|\psi(r)|\le cr^{\inf(s,1)}$ for $r\le 1$ and $|\psi(r)|\le cr^{-\inf(s,1)}$ for $r\ge 1$. Hence,

$$\int_0^\infty\int_\Omega|\psi(-t^2\Delta)u(x)|^2\frac{dxdt}{t}\le C\int_\Omega|u(x)|^2\,dx$$

where by the Borel functional calculus, C is the norm of the self-adjoint operator $\int_0^\infty\psi(-t^2\Delta)\overline{\psi}(-t^2\Delta)\frac{dt}{t}$. By the spectral theorem,

$$C=\sup_{r>0}\int_0^\infty|\psi(t^2r)|^2\frac{dt}{t}=\frac{1}{2}\int_0^\infty|\psi(t)|^2\frac{dt}{t}$$

which is easily seen to be finite.

We have therefore, for $f\in V$

$$\int_0^\infty\int_\Omega|\psi(-t^2\Delta)\nabla f(x)|^2\frac{dxdt}{t}\le C\int_\Omega|\nabla f(x)|^2\,dx$$

and since functions of Δ commute,

$$\begin{aligned}\int_0^\infty\int_\Omega&|\nabla\psi(-t^2\Delta)f(x)|^2\frac{dxdt}{t}\\&=\int_0^\infty\int_\Omega|(-\Delta)^{1/2}\psi(-t^2\Delta)f(x)|^2\frac{dxdt}{t}\\&=\int_0^\infty\int_\Omega|\psi(-t^2\Delta)(-\Delta)^{1/2}f(x)|^2\frac{dxdt}{t}\\&\le C\int_\Omega|(-\Delta)^{1/2}f(x)|^2\,dx=C\int_\Omega|\nabla f(x)|^2\,dx.\end{aligned}$$

Hence, (22) follows from (7). ■

Let us go back to the case $0\le s<1$ mentioned in the introduction and finish by proving

$$\int_Q|\nabla_x u(X)|^2\frac{dX}{\eta(X)^s}\le c_s\|f\|_s^2,$$

where in semigroup notations, $u(X) = (e^{t\Delta} f)(x)$ for $X = (x, t) \in Q$. Hence

$$\begin{aligned}\int_Q |\nabla_x u(X)|^2 \frac{dX}{\eta(X)^s} &= \int_0^\infty \int_\Omega t^{1-s} |(-\Delta)^{1/2} e^{t\Delta} f(x)|^2 \frac{dxdt}{t} \\ &= \int_0^\infty \int_\Omega |\psi_s(t\Delta)(-\Delta)^{s/2} f(x)|^2 \frac{dxdt}{t} \\ &\leq c_s \int_\Omega |(-\Delta)^{s/2} f(x)|^2 \, dx,\end{aligned}$$

where we have set

$$\psi_s(r) = r^{1-s/2} e^{-r}$$

and used as in the above proof the Littlewood-Paley estimate

$$\int_0^\infty \int_\Omega |\psi_s(t\Delta) g(x)|^2 \frac{dxdt}{t} \leq c_s \int_\Omega |g(x)|^2 \, dx$$

with

$$c_s = \int_0^\infty r^{1-s} e^{-2r} \tfrac{dr}{r}.$$

Note that c_s is independent of Ω.

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