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SEMI-GLOBAL SOLUTIONS OF $\bar\partial_b$ WITH L^p $(1 \le p \le \infty)$ BOUNDS ON STRONGLY PSEUDOCONVEX REAL HYPERSURFACES IN $\mathbf{C}^n$ $(n \ge 3)$

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Abstract

Let M be an open subset of a compact strongly pseudoconvex hypersurface $\{\rho = 0\}$ defined by $M = D \times \mathbf{C}^{n-m} \cap \{\rho = 0\}$, where $1 \le m \le n-2$, $D = \{\sigma(z_1, \dots, z_m) < 0\} \subset \mathbf{C}^m$ is strongly pseudoconvex in $\mathbf{C}^m$. For $\bar\partial_b$ closed $(0, q)$ forms f on M, we prove the semi-global existence theorem for $\bar\partial_b$ if $1 \le q \le n-m-2$, or if $q = n-m-1$ and f satisfies an additional "moment condition". Most importantly, the solution operator satisfies L^p estimates for $1 \le p \le \infty$ with $p = 1$ and ∞ included.

1. Introduction and Main Results

Let M be a connected open subset of a compact strongly pseudoconvex hypersurface $\{\rho = 0\}$ in $\mathbf{C}^n$ defined by $M = D \times \mathbf{C}^{n-m} \cap \{\rho = 0\}$, where $1 \le m \le n-2$, $D = \{\sigma(z_1, \dots, z_m) < 0\} \subset \mathbf{C}^m$ is strongly pseudoconvex in $\mathbf{C}^m$. We assume that ρ, $\sigma \in C^3$ are strictly plurisubharmonic in neighborhoods of $\{\rho \le 0\}$ and $\{\sigma \le 0\}$ respectively, and $d\rho \wedge d\sigma \ne 0$ on ∂M. For $\bar\partial_b$ closed $(0, q)$ forms f on M, we study in this paper the semi-global solvability of $\bar\partial_b u = f$ in L^p spaces, $1 \le p \le \infty$. Global solution with L^p estimates for $\bar\partial_b$ on compact strongly pseudoconvex hypersurfaces was obtained by Folland-Stein [**Fol-St**] for $1 \le q < n-1$, and by Henkin [**He**], Skoda [**Sk**] for $1 \le q \le n-1$. Using the explicit integral representation for the solution operator, Henkin [**He**] also gave the first local solution with supnorm estimate (i.e. C^0 estimate). Local solution with L^p estimates, $1 < p < \infty$, was obtained by Shaw [**Sh1**].

Our main results state as follows:

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Theorem 1. *Let M be as in the above. Then there exists a linear operator $\mathcal{L}$ mapping $L^p_{(0,q)}(M)$ to $L^p_{(0,q-1)}(M)$, $1 \le p \le \infty$ with*

$$||\mathcal{L}f||_p \le C||f||_p, \quad \text{where } C \text{ is independent of } p. \tag{1.1}$$

Moreover, $\mathcal{L}f$ solves the equation

$$\bar{\partial}_b u = f \tag{1.2}$$

provided that $f \in L^p_{(0,q)}(M)$ is $\bar{\partial}_b$ closed in distribution sense, and either

(i) $1 \le q \le n-m-2$, *or*

(ii) $q = n-m-1$, $f \in C^1(\bar{M})$ *and for any smooth $\bar{\partial}_b$ closed $(n, m-1)$ form h defined in a neighborhood V_h of ∂M, the condition*

$$\int_{\partial M_\epsilon} f \wedge h = 0 \quad \forall \epsilon > 0 \text{ small} \tag{1.3}$$

holds, where $M_\epsilon = \{\rho = 0\} \cap \{\sigma < -\epsilon\}$ and $\partial M_\epsilon \subset V_h$.

(1.3) is necessary and sufficient for $\bar{\partial}_b$ to have C^1 solution in M at this critical degree.

From this theorem we have

Corollary 1. *Let M be as in Theorem 1. The ranges of the $\bar{\partial}_b$ operator from $L^p_{(0,q-1)}(M)$ to $L^p_{(0,q)}(M)$, $1 \le p \le \infty$, $1 \le q \le n-m-1$ are closed. When $1 \le q \le n-m-2$, they are exactly sets of $\bar{\partial}_b$-closed forms in $L^p_{(0,q)}(M)$, and for $q = n-m-1$, the range is the L^p-closure of $\bar{\partial}_b(C^\infty_{(0,n-m-2)}(\bar{M}))$.*

Corollary 2. *Let M be as in Theorem 1. The $\bar{\partial}_b$-closed $(0,q)$ forms with $C^\infty(\bar{M})$ coefficients, $0 \le q \le n-m-2$, are dense in $\bar{\partial}_b$-closed forms with $L^p(M)$ coefficients, $1 \le p < \infty$.*

Let σ, ρ be defined as above, one may study the $\bar{\partial}_b$ operator on the CR manifold $\tilde{M} = \{\rho = 0\} \cap \{\sigma > 0\}$. Arguments parallel to the proof of Theorem 1 with suitable changes (see Remark 3 in Section 2) give the following:

Theorem 1'. *Let $\tilde{M}$ be as in the above. Then there exists a linear operator $\tilde{\mathcal{L}}$ mapping $L^p_{(0,q)}(\tilde{M})$ to $L^p_{(0,q-1)}(\tilde{M})$, $1 \le p \le \infty$ with*

$$(1.1') \qquad ||\tilde{\mathcal{L}} f||_p \le C ||f||_p, \quad \text{where } C \text{ is independent of } p.$$

Moreover, $\tilde{\mathcal{L}} f$ solves (1.2) provided that $f \in L^p_{(0,q)}(\tilde{M})$ is $\bar{\partial}_b$ closed in distribution sense, and either

(i) *$m \le q \le n-3$, or*

(ii) *$q = n-2$, $f \in C^1(\bar{\tilde{M}})$ and for any $(n,0)$ form h defined in a neighborhood V_h of $\partial\tilde{M}$ with holomorphic coefficient, the moment condition*

$$(1.3') \qquad \int_{\partial \tilde{M}_\epsilon} f \wedge h = 0 \quad \forall \epsilon > 0 \text{ small}$$

holds, where $\tilde{M}_\epsilon = \{\rho = 0\} \cap \{\sigma > \epsilon\}$ and $\partial\tilde{M}_\epsilon \subset V_h$.

(1.3') is necessary and sufficient for $\bar{\partial}_b$ to have C^1 solution in $\tilde{M}$ at this critical degree.

Results parallel to Corollaries 1 and 2 obviously hold, we leave them to readers.

For the critical degree q which is $n-m-1$ for M and $n-2$ for $\tilde{M}$, we have

Corollary 3. *Let M, and $\tilde{M}$ be as in the above.*

(a) *Suppose any smooth $\bar{\partial}_b$-closed $(n, m-1)$ form h defined in a neighborhood of ∂M can be approximated by $\bar{\partial}_b$-closed forms in $C^0(\bar{M})$ on ∂M. Then (1.2) is solvable on M with L^p estimates at degree $q = n-m-1$.*

(b) *Suppose any function holomorphic in a neighborhood of $\partial\tilde{M}$ can be approximated on $\partial\tilde{M}$ by functions holomorphic in $\bar{\tilde{M}}$. Then (1.2) is solvable with L^p estimates for $q = n-2$ on $\tilde{M}$. This condition is satisfied if $\partial\tilde{M}$ is Runge.*

Combine Theorem 1 and Theorem 1' together, we can generalize the special example of [**B**] as follows:

Corollary 4. *Suppose M can also be defined in the form of $\tilde{M}$, in other words, there is a real function $\tilde{\sigma} \in C^3(\mathbf{C}^{\tilde{m}})$, $1 \le \tilde{m} \le n-2$, which is strictly plurisubharmonic such that*

$$M = \tilde{M} = \{\tilde{\sigma} > 0\} \cap \{\rho = 0\}.$$

Then (1.2) is solvable with L^p estimates $1 \le p \le \infty$ for $1 \le q \le n-m-2$ or $\tilde{m} \le q \le n-3$.

Under the same assumption we also have

Corollary 5. *Let M be as in Corollary 4. Suppose in addition that both m and $\tilde{m}$ are greater than 1 with $\partial M = \{\tilde{\sigma} = 0\} \cap \{\sigma = 0\}$, and that $\{\tilde{\sigma} \leq 0\} \cap \{\sigma \leq 0\}$ has a neighborhood system consisting of Runge domains. Then (1.2) is solvable with L^p estimates $1 \leq p \leq \infty$ for $q = n - 2$. In particular, the assertion holds if there exists a decreasing sequence $\{\epsilon_j\}$, $\epsilon_j \to 0$ such that $\{\sigma < \epsilon_j\} \cap \{\tilde{\sigma} < \epsilon_j\}$ are convex for all j.*

This is because Hartogs' theorem gives that every function holomorphic in a neighborhood of ∂M can be extended holomorphically into a neighborhood of $\{\tilde{\sigma} \leq 0\} \cap \{\sigma \leq 0\}$ which is Runge by assumption. It turns out that ∂M is Runge. Corollary 5 now follows from Corollary 3.

When $m = 1$, the present paper also provides a uniform proof for L^p estimates $1 \leq p \leq \infty$ of local solutions with $p = 1$ and $p = \infty$ included. Moreover, the constant in (1.1) is independent of p. So our results improve [**Sh1**] and give a complete and uniform proof for the existence of a local solution of $\bar{\partial}_b$ with L^p estimates.

The idea of estimation used here is also applied in a forthcoming paper [**C-L3**] to obtain L^p estimates for $\bar{\partial}$-operators in the piecewise smooth pseudoconvex domain $\{\rho < 0\} \cap D \times \mathbf{C}^{n-m}$. The results of this paper can be used to obtain L^p estimates for the solution of $\bar{\partial}_b$ on some special compact piecewise smooth strongly pseudoconvex surfaces. We think this is an interesting direction. It is also interesting to have L^p estimates for $\bar{\partial}_b$ on larger classes of surfaces than those discussed here.

When $q = n - m - 1$ necessary and sufficient conditions for the solvability of $\bar{\partial}_b$ on M are given in [**C-L1**], while in [**C-L2**] we investigate the (non)solvability of $\bar{\partial}_b$ at degrees $q \geq n - m - 1$ on M (respectively, $q \leq m - 1$, on $\tilde{M}$) and we observe much more complicated phenomenon when $m > 1$. The solvability of $\bar{\partial}_b$-operators in various CR-manifolds under various norms have been extensively studied in past few years, see [**Ma-Mi**], [**Ro**], [**Sh3**], [**Sh4**], [**Mi-Sh1**], [**Mi-Sh2**] and references there.

We outline here the plan of the paper. In Section 2, we derive the solution operator for $\bar{\partial}_b$-closed $(0, q)$ form f with coefficients in $C^1(\bar{M})$ (denote by $f \in C^1_{(0,q)}(\bar{M})$). The solution operator consists of two integral operators: one is an integral over M defined by Henkin's kernel $\Omega(\mathfrak{r}, \mathfrak{r}*)$ for $\bar{\partial}_b$ on $\{\rho = 0\}$; the other one is a boundary integral defined by the kernel $\Omega(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})$ which involves not only the Leray sections $\mathfrak{r}$, $\mathfrak{r}*$ of $\{\rho = 0\}$ but also the Leray section $\mathfrak{s}$ for the lower dimensional strongly pseudoconvex domain $\{\sigma < 0\} \subset \mathbf{C}^m$. Section 2 contains definitions of various kernels and their properties. It was proved in [**He**] that the kernel $\Omega(\mathfrak{r}, \mathfrak{r}*)$ defines a linear operator from L^p to L^p (and even better). Thus our main task is to show the kernel $\Omega(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})$ defines an L^p-bounded operator, $1 \le p \le \infty$. We sketch the major difficulties and how we overcome them.

When L^p estimates are concerned, usually we transform by Stokes' theorem the boundary integral of $f \wedge \Omega(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})$ to the integral of $f \wedge \bar\partial\Omega_-(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})$ over M, where $\Omega_-(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})$ is an extension of $\Omega(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})$ to M. By Classical results of Singular Integral Operators it suffices to prove the L^1 norms of $\bar\partial\Omega_-(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})(\zeta, z)$ w.r.t. ζ, z respectively are bounded uniformly in z, ζ respectively. As $\mathfrak{s}$ is the Leray section of a strongly pseudoconvex domain in $\mathbf{C}^m$, $m < n$, the $L^1(M)$-norm of $\bar\partial\Omega_-(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})(\zeta, z)$ grows logarithmically as z approaches boundary. To overcome this difficulty, we first transform part of $\Omega(\mathfrak{r}, \mathfrak{r}*, \mathfrak{s})$ to kernels involving $\mathfrak{r}$, $\mathfrak{r}^*$, $\mathfrak{s}$ and the Bochner-Martinelli kernel form $\Omega(b')$ of $\mathbf{C}^m$. This kind of transformation was used in Sergeev-Henkin, see [**Se-He**] for extensive discussions. The resulting kernels then take the advantages of the following properties of $\Omega(b')$: (i) $\bar\partial\Omega^0(b') = 0$ off the diagonal, where $\Omega^0(b')$ is the component of $\Omega(b')$ of degree 0 in $d\bar z$, (ii) $\int_{0<a<|\zeta'|<b} \Omega^0(b') = 0$, (iii) apparently its order of singularity is $2m-1$ without using any coordinate transformation. This is done in Section 2 under the title Reductions.

Let $\mathcal{K}$ denote an arbitrary component of kernels resulting from the above transformation. To estimate the L^1 norm of $\bar\partial\mathcal{K}$, the most difficult part lies in the fact that there are points where $\partial\rho/\partial z_j = 0$, $\forall\, j > m$ (they are characteristic points when $m = 1$). This is because $\sum_{j>m} |\partial\rho/\partial z_j|$ is essentially related to the jacobian of the coordinate transformation that will make the L^1 norm of $\bar\partial\mathcal{K}$ converge. By a closer observation of the integrals to be estimated, we see that for most critical terms, their integrands in a certain sense contain factors of the jacobians of the needed coordinate transformations in their numerators. Therefore, we decompose the domains of integration V by comparing the sizes of jacobians with, say $|\zeta' - z'|^\tau$, $0 < \tau < 1$ to be determined, for $\zeta \in \mathbf{C}^n$ we write $\zeta = (\zeta', \zeta'')$ with $\zeta' \in \mathbf{C}^m$ and $\zeta'' \in \mathbf{C}^{n-m}$. To illustrate, e.g. we let $V = V_1 \cup V_2$, where

$$V_1 = \left\{ \zeta \in V : \sum_{j>m} |\partial\rho/\partial z_j| \le |\zeta' - z'|^\tau \right\},$$

$$V_2 = \left\{ \zeta \in V : \sum_{j>m} |\partial\rho/\partial z_j| > \frac{1}{2}|\zeta' - z'|^\tau \right\}.$$

Then in V_1 the order of singularity is reduced by τ, so with suitably chosen τ, the integral over V_1 is finite, while in V_2 we are able to perform coordinate transformation with controllable jacobian. On the other hand, to guarantee that coordinate transformations are of finite multiplicity, we incorporate the technique first introduced by Range-Siu [**Ra-Siu**]. We remark that the strong pseudoconvexity also plays an important role, see (3.15) and Section 4. These estimates are summarized as three key lemmas stated in Section 3 and proved in Section 4.

Using these lemmas we prove the L^1 norm of $\bar{\partial}\mathcal{K}$ is bounded by a constant independent of p. Thus assertions of Theorem 1 hold for $\bar{\partial}_b$-closed $f \in C^1_{(0,q)}(\bar{M})$. We then apply the classical mollification procedure of K. O. Friedrichs to complete the proof of Theorem 1. These are contained in Section 3.

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2. Notations, the Solution Operator and Reductions

Throughout the paper, the constants C, c denote positive numbers which may vary from time to time.

We adopt the convention that $\mathbf{C}^n = \{(x_1 + \sqrt{-1}x_2, \dots, x_{2j-1} + \sqrt{-1}x_{2j}, \dots, x_{2n-1} + \sqrt{-1}x_{2n})\} \cong \{(x_1, \dots, x_{2n})\} = \mathbf{R}^{2n}$. In this paper, we often write $\zeta \in \mathbf{C}^n$ as (ζ', ζ'') where $\zeta' \in \mathbf{C}^m$ and $\zeta'' \in \mathbf{C}^{n-m}$, (respectively, $x = (x', x'')$ where $x' \in \mathbf{R}^{2m}$ and $x'' \in \mathbf{R}^{2(n-m)}$). Similarly, for differential forms we write $df = (d'f, d''f)$, and $\partial f = (\partial' f, \partial'' f)$, where $d'\cdot$, $d''\cdot$ denote respectively the differentials with repect to the first $2m$ variables and those to the last $2(n-m)$ variables, likewise for $\partial'\cdot$ and $\partial''\cdot$. Also we have $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$, $d\zeta' = d\zeta_1 \wedge \cdots \wedge d\zeta_m$ and $d\zeta'' = d\zeta_{m+1} \wedge \cdots \wedge d\zeta_n$. We denote by $|f| = (\sum |f_I|^2)^{\frac{1}{2}}$ the length of a q-form $f = \sum f_I \, dx^I$. For fixed $a' \in \mathbf{C}^m$ the notation $\Gamma_{a'}$ denotes the fibre $\{(a', \zeta'') \in M\}$ of M over a'.

We denote by $x^{\hat{j}} := \{x_1, \dots, \hat{x}_j, \dots, x_{2n}\}$, the coordinate system obtained from that of $\mathbf{R}^{2n}$ by deleting the coordinate function x_j; and we denote by $dx^{\hat{j}} := dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{2n}$, the corresponding $2n-1$ form.

The notation $B_\delta(z)$ denotes the Euclidean ball centered at $z \in \mathbf{C}^n$ with radius δ. "$A \lesssim B$" means the quotient $|A|/|B|$ is a nonvanishing function bounded from above.

We recall that M is an open subset of the real hypersurface $\{\rho = 0\}$. Let $\mathbf{v}_M$ be the measure on M induced by the Lebesgue measure in the

Euclidean space. Let ν be the 1-form dual to the unit normal at points of $\{\rho = 0\}$, explicitly, $\nu = |d\rho|^{-1}\, d\rho = |d\rho|^{-1}(\sum_1^n \partial\rho/\partial z_j\, dz_j + \partial\rho/\partial\bar z_j\, d\bar z_j)$. Then $d\mathbf{v}_M$ is the $(2n-1)$-form such that $\nu \wedge d\mathbf{v}_M = (-2i)^{-n} dz \wedge d\bar z = dx_1 \wedge\ dx_2 \wedge \cdots \wedge\ dx_{2n} = d\mathbf{v}_{\mathbf{R}^{2n}}$ as forms in $\mathbf{R}^{2n}$.

It is well-known that any differential form defined in a neighborhood of M in $\mathbf{R}^{2n}$ can be uniquely written on M as the sum of the component tangential to M and that normal to M. In particular, if the differential form g is of degree $2n-1$, its tangential component can be expressed in terms of $d\mathbf{v}_M$, i.e. $g = g_M\, d\mathbf{v}_M + d\rho \wedge R$, where R is a $(2n-2)$-form, and it is clear that it is g_M that matters for integrations over M. In view of the above, the function g_M can be determined by

$$g_M\, d\mathbf{v}_{\mathbf{R}^{2n}} = |d\rho|^{-1}\, d\rho \wedge g. \tag{2.2}$$

$g_M\, d\mathbf{v}_M$ is referred as *the component of g tangential to M*.

In arguments followed we will use the general area and coarea formula given in [**Si**], see also [**F**]. We also need consider various maps g from l-dimensional submanifolds $\mathcal{M} \subset \mathbf{R}^N$ to $\mathbf{R}^k$. When $l \le k$ the generalized jacobian is defined by $Jg(x) = \{\det(dg_x)^* \circ (dg_x)\}^{\frac{1}{2}}$, and $dg_x : T_xM \to \mathbf{R}^k$ denotes the induced linear map. And when $l > k$ the generalized jacobian is defined by $J^*g(x) = \{\det(dg_x) \circ (dg_x)^*\}^{\frac{1}{2}}$. In particular, we note that

Remark 1. If $\mathcal{M} = \{\varrho = 0\}$ is a real hypersurface in $\mathbf{R}^N$, and g is a map from $\mathcal{M}$ to $\mathbf{R}^N$ such that one of the component of g, say g_1 is ϱ, then $Jg = |dg_1 \wedge \cdots \wedge dg_N|/|d\varrho|$.

The Solution Operator.

We introduce the following notations and exterior calculus developed by Harvey and Polking [**H-P**]:

Let $E^1, \dots, E^\alpha$ be a collection of n-tuples of C^2 functions in $(\zeta, z) \in \mathbf{C}^n \times \mathbf{C}^n$, following Harvey-Polking we define

$$\Omega(E^1, \dots, E^\alpha) = \frac{\langle E^1, d\zeta\rangle}{\langle E^1, \zeta - z\rangle} \wedge \cdots \wedge \frac{\langle E^\alpha, d\zeta\rangle}{\langle E^\alpha, \zeta - z\rangle} \wedge \sum_{\lambda_1 + \cdots + \lambda_\alpha = n-\alpha} \left(\frac{\langle \bar\partial_{\zeta,z} E^1, d\zeta\rangle}{\langle E^1, \zeta - z\rangle}\right)^{\lambda_1} \wedge \cdots \wedge \left(\frac{\langle \bar\partial_{\zeta,z} E^\alpha, d\zeta\rangle}{\langle E^\alpha, \zeta - z\rangle}\right)^{\lambda_\alpha} \tag{2.3}$$

where $\langle x, y\rangle = \sum x_i y_i$ for vectors x, y in $\mathbf{C}^n$ and $d\zeta$ here is understood to be the n-vector $(d\zeta_1, \dots, d\zeta_n)$. Then Ω is C^1 away from the singular set

$A = \bigcup_1^\alpha \{(\zeta, z),\ \langle E^j, \zeta - z\rangle = 0\}$. We can rewrite Ω as $\Omega(E^1, \dots, E^\alpha) = \sum_0^{n-1} \Omega^q(E^1, \dots, E^\alpha)$, where Ω^q is the sum of components of Ω which are of degree q in $d\bar{z}$. Outside the singular set A we have the following identity:

$$\bar{\partial}_{\zeta,z}\Omega(E^1, \dots, E^\alpha) = \sum_j (-1)^j \Omega(E^1, \dots, \widehat{E^j}, \dots, E^\alpha). \tag{2.4}$$

To construct the kernel we need following results of [**For**]:

For any strongly pseudoconvex domain $X = \{\varrho < 0\} \subset \mathbf{C}^N$, where $\varrho \in C^k$, $k \geq 2$, is strictly plurisubharmonic in a neighborhood of $\bar{X}$, there exists $\epsilon > 0$ and $H(\zeta, z) \in C^{k-1}(X_\epsilon \times X_\epsilon)$, $X_\epsilon = \{z \in \mathbf{C}^n,\ \varrho(z) < \epsilon\}$ satisfying

(2.5) $H(\zeta, \cdot)$ is holomorphic in X_ϵ,

(2.6) $\exists\, h_j(\zeta, z) \in C^{k-1}(X_\epsilon \times X_\epsilon)$, $j = 1, \dots, N$, holomorphic in z, such that

$$H(\zeta, z) = \sum_1^N h_j(\zeta, z)(\zeta_j - z_j),$$

(2.7) $\exists\, c > 0$, such that $\forall\, z \in \bar{X}$, $\zeta \in \bar{X}$

$$2\,\mathrm{Re}\, H(\zeta, z) \geq \varrho(\zeta) - \varrho(z) + c|\zeta - z|^2,$$

(2.8) $d_\zeta H(\zeta, z)|_{z=\zeta} = \partial\varrho(\zeta)$.

For the strongly pseudoconvex domain $\{\rho < 0\} \subset \mathbf{C}^n$, let $\mathfrak{r}(\zeta, z) = (\mathfrak{r}_1, \dots, \mathfrak{r}_n)$, $\mathfrak{r}_j$, $j = 1, \dots, n$ be the n-tuple function and its components corresponding to (2.6). We use $\mathfrak{s}'(\zeta', z')$, $\mathfrak{s}_j$, $j = 1, \dots, m$ to denote those for $\{\sigma < 0\} \subset \mathbf{C}^m$. We denote by $\mathfrak{s}(\zeta, z)$ the map $(\mathfrak{s}_1(\zeta', z'), \dots, \mathfrak{s}_m(\zeta', z'), 0, \dots, 0)$. Let $\mathfrak{r}^*(\zeta, z) = (\mathfrak{r}_1^*(\zeta, z), \dots, \mathfrak{r}_n^*(\zeta, z))$, where $\mathfrak{r}_j^*(\zeta, z) = -\mathfrak{r}_j(z, \zeta)$. Thus $\mathfrak{r}$, $\mathfrak{r}^*$ and $\mathfrak{s}$ are C^2 in a neighborhood of $M \times M$. We remark that (2.7) implies there exists $c > 0$ such that

$$\begin{aligned} -\sigma(\zeta) + \mathrm{Re}\langle \mathfrak{s}, \zeta - z\rangle &\geq -\sigma(\zeta)/2 - \sigma(z)/2 \\ &\quad + c|\zeta - z|^2 \geq c|\zeta - z|^2, \quad \text{for } \zeta,\ z \in \bar{D}, \end{aligned} \tag{2.9}$$

$$\mathrm{Re}\langle \mathfrak{r}, \zeta - z\rangle \geq c|\zeta - z|^2, \qquad \text{for } \zeta,\ z \in \bar{M}, \tag{2.10}$$

$$\mathrm{Re}\langle \mathfrak{r}^*, \zeta - z\rangle \geq c|\zeta - z|^2, \qquad \text{for } \zeta,\ z \in \bar{M}. \tag{2.11}$$

We define $\Omega(\mathfrak{r},\mathfrak{r}^*)$, $\Omega(\mathfrak{r},\mathfrak{s})$, $\Omega(\mathfrak{r}^*,\mathfrak{s})$, and $\Omega(\mathfrak{r},\mathfrak{r}^*,\mathfrak{s})$, according to formula (2.3). In view of the fact that $\mathfrak{s}$ depends only on ζ', z', we see that the exponents of $\frac{\langle \bar\partial_\zeta \mathfrak{s}, d\zeta\rangle}{\langle \mathfrak{s}, \zeta - z\rangle}$ in $\Omega(\mathfrak{r},\mathfrak{s})$, $\Omega(\mathfrak{r}^*,\mathfrak{s})$ and $\Omega(\mathfrak{r},\mathfrak{r}^*,\mathfrak{s})$ must be $\le m-1$.

For any $(0,q)$ form f on M whose coefficients are C^1 up to $\bar M$, following Shaw [**Sh1**] (see also Henkin [**He**]), we have for $z \in M$,

$$(2.12)\quad (-1)^q f(z) = \bar\partial_b \int_M f \wedge \Omega(\mathfrak{r},\mathfrak{r}^*) + \int_M \bar\partial_b f \wedge \Omega(\mathfrak{r},\mathfrak{r}^*) - \int_{\partial M} f \wedge \Omega(\mathfrak{r},\mathfrak{r}^*).$$

In fact, in (2.12) M can be any open subset of $\{\rho = 0\}$ with C^2 boundary.

Using the identity (2.4), we can rewrite the boundary integral in (2.12) as

$$(2.13)\quad \int_{\partial M} f \wedge \Omega(\mathfrak{r},\mathfrak{s}) - \int_{\partial M} f \wedge \Omega(\mathfrak{r}^*,\mathfrak{s}) - \int_{\partial M} f \wedge \bar\partial_\zeta \Omega(\mathfrak{r},\mathfrak{r}^*,\mathfrak{s}) + (-1)^{q+1}\bar\partial_b \int_{\partial M} f \wedge \Omega(\mathfrak{r},\mathfrak{r}^*,\mathfrak{s}).$$

Since $\Omega(\mathfrak{r},\mathfrak{s})$ is an $(n,n-2)$ form in ζ, $f \wedge \Omega(\mathfrak{r},\mathfrak{s})$ is an $(n,n-2+q)$ form and $q \ge 1$, so the integral against ∂M must be null by type consideration. As for $\int_{\partial M} f \wedge \Omega(\mathfrak{r}^*,\mathfrak{s})$, we note that $f \wedge \Omega(\mathfrak{r}^*,\mathfrak{s})$ is an $(n,n-2)$ form in ζ only when $\beta = n-2-q$, where β is the exponent of $\frac{\langle \bar\partial_\zeta \mathfrak{s}, d\zeta\rangle}{\langle \mathfrak{s}, \zeta - z\rangle}$ in $\Omega(\mathfrak{r}^*,\mathfrak{s})$. But as $\beta \le m-1$, $\int_{\partial M} f \wedge \Omega(\mathfrak{r}^*,\mathfrak{s})$ is zero if $q \le n-m-2$ again by type consideration. If $q = n-m-1$, then $\Omega(\mathfrak{r}^*,\mathfrak{s})$ is a $\bar\partial_b$-closed form in a neighborhood of ∂M, thus for $(0,n-m-1)$ form f that satisfies (1.3) $\int_{\partial M} f \wedge \Omega(\mathfrak{r}^*,\mathfrak{s})$ is null. We therefore arrive at the following homotopy formula for $(0,q)$ forms f with $1 \le q \le n-m-2$, or $q = n-m-1$ and f satisfies (1.3).

$$(2.14)\quad \begin{aligned} (-1)^q f(z) &= \int_M \bar\partial_b f \wedge \Omega^q(\mathfrak{r},\mathfrak{r}^*)(\zeta,z) \\ &\quad + (-1)^{q+1}\int_{\partial M} \bar\partial_b f \wedge \Omega^q(\mathfrak{r},\mathfrak{r}^*,\mathfrak{s})(\zeta,z) \\ &\quad + \bar\partial_b \left\{ \int_M f \wedge \Omega^{q-1}(\mathfrak{r},\mathfrak{r}^*)(\zeta,z) \right. \\ &\quad \left. + (-1)^q \int_{\partial M} f \wedge \Omega^{q-1}(\mathfrak{r},\mathfrak{r}^*,\mathfrak{s})(\zeta,z) \right\} \\ &= S^q(\bar\partial_b f) + \bar\partial_b S^{q-1} f. \end{aligned}$$

An operator $\mathcal{L}$ which equals S^{q-1} for $\bar\partial_b$-closed $f \in C^1_{(0,q)}(\bar M)$ apparently solves (1.2). The term $\int_M f \wedge \Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*)$ admits L^p estimates for $1 \le p \le \infty$ as $\Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*)$ defines an operator of weak type $(1, 2n/2n-1)$. (See [**He**].) The main task of this paper is to estimate the term $\int_{\partial M} f \wedge \Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})$. As explained in the introduction we will have to transform part of $\Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})$ to kernel forms involving the Bochner-Martinelli kernel form of $\mathbf{C}^m$ which are easier to estimate.

Remark 2. Webster proved in [**W**] that $\int_M f \wedge \Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*)$ is in $C^1(M)$ for $f \in C^1(\bar M)$. The term $\int_{\partial M} f \wedge \Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})$ is obviously in $C^1(M)$, as ρ, σ are C^3 in a neighborhood of M. Hence the operator $\mathcal{L}$ sends $\bar\partial_b$-closed $f \in C^1(\bar M)$ to forms with coefficients in $C^1(M)$.

Remark 3. We observe that with $\mathfrak{s}(\zeta, z)$ replaced by $\mathfrak{s}^*(\zeta, z) = -\mathfrak{s}(z, \zeta)$ Theorem 1' will be proved by arguments parallel to the proof of Theorem 1.

Reductions.

Let $R_a = \{z = (z', z'') \in M,\ d(z', \partial D) < a\}$ and χ_a be the characteristic function of R_a. In view of (2.9)-(2.11), it is easy to see that it suffices to compute the L^p norm of the function $\chi_d \int_{\partial M} f \wedge \Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})$, where d is a fixed constant to be determined.

We write out $\Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})$, $1 \le q \le n-2$ as follows:

$$\Omega^{q-1}(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s}) = \sum_{1 \le \alpha \le m} \theta^\alpha(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^\alpha(\mathfrak{s}),$$

where

$$\theta^\gamma(\mathfrak{r}, \mathfrak{r}^*) = \frac{\langle \mathfrak{r}, d\zeta\rangle}{\langle \mathfrak{r}, \zeta - z\rangle} \wedge \frac{\langle \mathfrak{r}^*, d\zeta\rangle}{\langle \mathfrak{r}^*, \zeta - z\rangle} \wedge \left(\frac{\langle \bar\partial_\zeta \mathfrak{r}, d\zeta\rangle}{\langle \mathfrak{r}, \zeta - z\rangle}\right)^{n-q-\gamma-1} \wedge \left(\frac{\langle \bar\partial_z \mathfrak{r}^*, d\zeta\rangle}{\langle \mathfrak{r}^*, \zeta - z\rangle}\right)^{q-1}$$

$$\omega^\beta(\mathfrak{s}) = \frac{\langle \mathfrak{s}, d\zeta\rangle}{\langle \mathfrak{s}, \zeta - z\rangle} \wedge \left(\frac{\langle \bar\partial_\zeta \mathfrak{s}, d\zeta\rangle}{\langle \mathfrak{s}, \zeta - z\rangle}\right)^{\beta-1} = \frac{\langle \mathfrak{s}', d\zeta'\rangle}{\langle \mathfrak{s}', \zeta' - z'\rangle} \wedge \left(\frac{\langle \bar\partial'_\zeta \mathfrak{s}', d\zeta'\rangle}{\langle \mathfrak{s}', \zeta' - z'\rangle}\right)^{\beta-1}.$$

Let $\omega^\alpha_-(\mathfrak{s})$ be the kernel obtained by replacing $\langle \mathfrak{s}, \zeta - z\rangle$ in $\omega^\alpha(\mathfrak{s})$ with $\langle \mathfrak{s}, \zeta - z\rangle - \sigma(\zeta')$. Let $\theta^\alpha_\lambda(\mathfrak{r}, \mathfrak{r}^*)$ be the kernel obtained by replacing in $\theta^\alpha(\mathfrak{r}, \mathfrak{r}^*)$ the denominators $\langle \mathfrak{r}, \zeta - z\rangle$ and $\langle \mathfrak{r}^*, \zeta - z\rangle$ respectively with $\langle \mathfrak{r}, \zeta - z\rangle + \lambda$ and $\langle \mathfrak{r}^*, \zeta - z\rangle + \lambda$ where λ is a small positive number. Then (2.9) implies $\theta^\alpha_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s})$ has no singularities on $R_{2d} \times R_{2d}$, $\forall\, d$, $0 < d \le d_0$, for some suitably chosen d_0.

We note that when $\alpha = m$, $\omega^m(\mathfrak{s})(z', \zeta')$ is the Leray kernel associated to the section $\mathfrak{s}'$ for $\{\sigma < 0\} \subset \mathbf{C}^m$, the $L^1(\{\sigma < 0\})$ norm of $\bar{\partial}\omega_-^m(\mathfrak{s})$ grows logarithmically as z approaches the boundary (cf. [**Se-He**] for related arguments). This is the term that needs to be transformed. We thus investigate separately the case $1 \leq \alpha \leq m-1$ and the case $\alpha = m$.

For fixed $z = (z', z'') \in R_d$ and $f \in C^1_{(0,q)}(\bar{M})$, we write for $1 \leq \alpha < m$

$$\begin{aligned}\int_{\partial M} f(\zeta) \wedge \theta^\alpha(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^\alpha(\mathfrak{s})(\zeta, z) &= \lim_{\lambda \to 0} \int_{\partial M} f(\zeta) \wedge \theta^\alpha_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s})(\zeta, z)\\ &= \lim_{\lambda \to 0} \int_{\partial R_{2d} \setminus \partial M} f(\zeta) \wedge \theta^\alpha_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s})(\zeta, z)\\ &\quad + \lim_{\lambda \to 0} \int_{R_{2d}} \bar{\partial}_b(f(\zeta) \wedge \theta^\alpha_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s}))(\zeta, z)\\ &= \lim_{\lambda \to 0} \{F^1_\lambda(z) + F^2_\lambda(z)\}.\end{aligned}$$

For the sake of simplicity, *we denote by* $\mathbf{S}^\alpha_\lambda$ *both the form* $\bar{\partial}_b(\theta^\alpha_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s}))$ *and the operator it defines.* It is easy to see that F^1_λ is in L^p with its L^p norm bounded by the L^p norm of f multiplied by a constant independent of λ.

When $\alpha = m$, $\omega^m(\mathfrak{s}) = \dfrac{\langle \mathfrak{s}, d\zeta \rangle}{\langle \mathfrak{s}, \zeta - z \rangle} \wedge (\dfrac{\langle \bar{\partial}_\zeta \mathfrak{s}, d\zeta \rangle}{\langle \mathfrak{s}, \zeta - z \rangle})^{m-1}$ is a double form in $\mathbf{C}^m$. Denote by b' the Bochner-Martinelli section in $\mathbf{C}^m$, (2.4) gives the following identity:

$$\omega^m(\mathfrak{s}) = \frac{\langle b', d\zeta' \rangle}{\langle b', \zeta' - z' \rangle} \wedge (\frac{\langle \bar{\partial}_{\zeta', z'} b', d\zeta' \rangle}{\langle b', \zeta' - z' \rangle})^{m-1} - \bar{\partial}_{\zeta', z'} \Omega(b', \mathfrak{s}')$$

off the singular set $\{(\zeta', z'), \zeta' = z'\}$.

Taking into consideration types of differential forms under integration, we rewrite, for $\bar{\partial}_b$-closed form f, the integral $\int_{\partial M} f(\zeta) \wedge \theta^m(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega^m(\mathfrak{s})(\zeta, z)$ as follows:

$$\int_{\partial M} f(\zeta) \wedge \theta^m(\mathfrak{r}, \mathfrak{r}^*) \wedge \Omega^0(b')(\zeta, z) + (-1)^q \int_{\partial M} f(\zeta) \wedge \bar{\partial}_\zeta \theta^m(\mathfrak{r}, \mathfrak{r}^*) \wedge \Omega^0(\mathfrak{s}', b')(\zeta, z),$$

since $\omega^m(\mathfrak{s})$ is $d\bar{z}$ free.

Let $\Omega^0_-(\mathfrak{s}',b')$ be the kernel obtained by replacing $\langle\mathfrak{s},\zeta-z\rangle$ in $\Omega^0(\mathfrak{s}',b')$ with $\langle\mathfrak{s},\zeta-z\rangle-\sigma(\zeta')$. Applying Stokes' Theorem to the above integrals, let $G_\epsilon=R_{2d}\setminus\{|\zeta'-z'|<\epsilon\}$ and $F_\epsilon=M\cap\{|\zeta'-z'|=\epsilon\}$, we obtain

$$\begin{aligned}\int_{\partial M} f(\zeta)\wedge&\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\omega^m(\mathfrak{s})\\ &=\int_{\partial R_{2d}\setminus\partial M} f(\zeta)\wedge\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b')\\ &\quad+(-1)^q\int_{\partial R_{2d}\setminus\partial M} f(\zeta)\wedge\bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0_-(\mathfrak{s}',b')\\ &\quad+\lim_{\epsilon\to0}\Big\{(-1)^q\int_{G_\epsilon} f(\zeta)\wedge\bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b')\\ &\quad+\int_{F_\epsilon} f(\zeta)\wedge\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b')\\ &\quad+(-1)^{q-1}\int_{G_\epsilon} f(\zeta)\wedge\bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\bar\partial_{\zeta'}\Omega^0_-(\mathfrak{s}',b')\\ &\quad+(-1)^q\int_{F_\epsilon} f(\zeta)\wedge\bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0_-(\mathfrak{s}',b')\Big\}.\end{aligned}$$

Let ϕ be the characteristic function of $[0,1]\subset\mathbf{R}$ and $\psi_\epsilon(\zeta,z)=1-\phi(\frac{|\zeta'-z'|}{\epsilon})$. *Denote by* $\mathbf{S}^m_\epsilon(b')$ *and* $\mathbf{S}^m_\epsilon(\mathfrak{s}_-,b')$ *respectively both the operators defined by* $\psi_\epsilon\bar\partial_\zeta(\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b'))(\zeta,z)$ *and* $\psi_\epsilon\bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\bar\partial_{\zeta'}\Omega^0_-(\mathfrak{s}',b')(\zeta,z)$, *and the forms themselves.*

As for integrations over F_ϵ, we first remark that if $\partial''\rho\neq0$ on $\Gamma_{z'}$, then $\lim_{\epsilon\to0}\int_{F_\epsilon} f(\zeta)\wedge\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b')$ gives the residue $\int_{\Gamma_{z'}} f(z',\zeta'')\wedge\theta^m(\mathfrak{r},\mathfrak{r}^*)((z',\zeta''),z)$. For the general case let $\phi_\epsilon=\epsilon^{-2m}\phi(\frac{|\zeta'-z'|}{\epsilon})$, then

$$\begin{aligned}\int_{F_\epsilon} f(\zeta)\wedge&\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b')\\ &=\epsilon^{-2m}\lim_{\lambda\to0}\int_{F_\epsilon} f(\zeta)\wedge\theta^m_\lambda(\mathfrak{r},\mathfrak{r}^*)\wedge\langle b',d\zeta'\rangle\wedge\langle d\bar\zeta',d\zeta'\rangle^{m-1}\\ &=\lim_{\lambda\to0}(-1)^q\int_M f(\zeta)\wedge\phi_\epsilon\bar\partial_\zeta\theta^m_\lambda(\mathfrak{r},\mathfrak{r}^*)\wedge\langle b',d\zeta'\rangle\wedge\langle d\bar\zeta',d\zeta'\rangle^{m-1}\\ &\quad-\lim_{\lambda\to0}(-1)^q\int_M f(\zeta)\wedge\phi_\epsilon\theta^m_\lambda(\mathfrak{r},\mathfrak{r}^*)\wedge\langle d\bar\zeta',d\zeta'\rangle^m\end{aligned}$$

and

$$(-1)^{m-1} \int_{F_\epsilon} f(\zeta) \wedge \bar{\partial}_\zeta \theta^m(\mathfrak{r}, \mathfrak{r}^*) \wedge \Omega^0_-(\mathfrak{s}', b')$$

$$= (-1)^{m-1} \sum_{1 \le \alpha \le m-1} \epsilon^{-2\alpha} \lim_{\lambda \to 0} \int_{F_\epsilon} f(\zeta) \wedge \bar{\partial}_\zeta \theta^m_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \omega_-^{m-\alpha}(\mathfrak{s}) \wedge \langle b', d\zeta' \rangle \wedge \langle d\bar{\zeta}', d\zeta' \rangle^{\alpha-1}$$

$$= \sum_{1 \le \alpha \le m-1} \epsilon^{2m-2\alpha} \lim_{\lambda \to 0} \int_M f(\zeta) \wedge \phi_\epsilon \bar{\partial}_\zeta \theta^m_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \bar{\partial}_{\zeta'}(\omega_-^{m-\alpha}(\mathfrak{s}) \wedge \langle b', d\zeta' \rangle \wedge \langle d\bar{\zeta}', d\zeta' \rangle^{\alpha-1}).$$

Denote by $\mathbf{T}_{\epsilon,\lambda}(b')$ *and* $\mathbf{T}^\alpha_{\epsilon,\lambda}(\mathfrak{s}_-, b')$ *respectively the forms* $\phi_\epsilon \bar{\partial}_\zeta \theta^m_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \langle b', d\zeta' \rangle \wedge \langle d\bar{\zeta}', d\zeta' \rangle^{m-1}$, $\epsilon^{2m-2\alpha} \phi_\epsilon \bar{\partial}_\zeta \theta^m_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \bar{\partial}_{\zeta'}(\omega_-^{m-\alpha}(\mathfrak{s}) \wedge \langle b', d\zeta' \rangle \wedge \langle d\bar{\zeta}', d\zeta' \rangle^{\alpha-1})$*, and the operators defined by them.* And *let* $\mathbf{T}_{\epsilon,\lambda}$ *be the form* $\phi_\epsilon \theta^m_\lambda(\mathfrak{r}, \mathfrak{r}^*) \wedge \langle d\bar{\zeta}', d\zeta' \rangle^m$ *and the operator corresponding to it.*

3. Estimation of Kernels and Proof of Theorem 1

Let $\delta > 0$ be a small number to be determined and let $d > 0$ be such that $d < \frac{\delta}{3}$. Fix an arbitrary $z \in R_d$, let $V = B_\delta(z) \cap \{a \le |\zeta' - z'| < b\} \cap R_{2d}$, where $0 \le a < b < 1$.

Let $1 \le q$, $m \le n-2$, and let

$$r = \mathrm{Im}\langle \mathfrak{r}, \zeta - z \rangle, \quad s = \mathrm{Im}\langle \mathfrak{s}, \zeta - z \rangle.$$

The following three lemmas are key to the estimation of this paper.

Lemma 1. *Let* $m \ge 2$ *be an integer. For each of the following integrals, there exists a positive number* $e < 1$ *such that the following hold:*

$$(3.1) \quad \int_V \frac{|\partial\rho \wedge \partial\sigma|^2 d\mathbf{v}_M}{(|r| + |\zeta - z|^2)^{n-q-p+\frac{1}{2}} |\zeta - z|^{2q} (|\sigma| + |s| + |\zeta' - z'|^2)^p} \le C b^e |\log b| \quad \text{for } 2 \le p \le m$$

$$(3.2) \quad \int_V \frac{|d\rho \wedge \partial\rho \wedge \partial\sigma| d\mathbf{v}_M}{(|r| + |\zeta - z|^2)^{n-q-p} |\zeta - z|^{2q} (|\sigma| + |s| + |\zeta' - z'|^2)^p} \le C b^e |\log b| \quad \text{for } 1 \le p \le m-1$$

$$\int_V \frac{d\mathbf{v}_M}{(|r|+|\zeta-z|^2)^{n-q-p-\frac{1}{2}}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^p} \leq Cb^e|\log b| \quad \text{for } 1\leq p\leq m-1 \tag{3.3}$$

where C is a constant independent of z.

Lemma 2. *Let $m\geq 2$ be an integer. For each of the following integrals, there exists a positive number $e<1$ such that the following hold:*

$$\int_V \frac{|\partial''\rho\wedge\partial\rho\wedge\partial\sigma\wedge\bar{\partial}\sigma|d\mathbf{v}_M}{(|r|+|\zeta-z|^2)^{n-q-m}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^2|\zeta'-z'|^{2m-3}} \leq Cb^e|\log b| \tag{3.4}$$

$$\int_V \frac{|\partial''\rho\wedge\partial\rho\wedge\partial\sigma|d\mathbf{v}_M}{(|r|+|\zeta-z|^2)^{n-q-m}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)|\zeta'-z'|^{2m-2}} \leq Cb^e|\log b| \tag{3.5}$$

$$\int_V \frac{|\partial''\rho\wedge\partial\rho|d\mathbf{v}_M}{(|r|+|\zeta-z|^2)^{n-q-m}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)|\zeta'-z'|^{2m-3}} \leq Cb^e|\log b| \tag{3.6}$$

where C is a constant independent of z.

Lemma 3. *Let $m\geq 1$ be an integer. The following estimates hold:*

$$\int_V \frac{|\partial''\rho|d\mathbf{v}_M}{(|r|+|\zeta-z|^2)^{n-q-m}|\zeta-z|^{2q}|\zeta'-z'|^{2m-1}} \leq \begin{cases} C(b^e|\log b|-a^e|\log a|), & \text{if } a>0 \\ Cb^e|\log b|, & \text{if } a=0 \end{cases} \tag{3.7}$$

$$\int_V \frac{|\partial''\rho|^2 d\mathbf{v}_M}{(|r|+|\zeta-z|^2)^{n-q-m-\frac{1}{2}}|\zeta-z|^{2q}|\zeta'-z'|^{2m-1}} \leq Cb \tag{3.8}$$

where $e<1$, C are postive constants independent of z.

Proofs of Lemmas 1-3 are postponed to Section 4.

In the following, we fix an arbitrary monomial $(0,q)$ form $d\bar\zeta^J$, let $K_i(\zeta,z)$, $i=1,2,3$ be coefficients of the following (n,n) forms:

$$K_1(\zeta,z) \sim d\rho \wedge d\bar\zeta^J \wedge \bar\partial_\zeta(\theta^\alpha_\lambda(\mathfrak{r},\mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s}))(\zeta,z) \quad 1 \le \alpha \le m-1$$

$$K_2(\zeta,z) \sim d\rho \wedge d\bar\zeta^J \wedge \bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*) \wedge \Omega^0(b')(\zeta,z)$$

$$K_3(\zeta,z) \sim d\rho \wedge d\bar\zeta^J \wedge \bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*) \wedge \bar\partial_{\zeta'}\Omega^0_-(\mathfrak{s}',b')(\zeta,z).$$

These are tangential parts corresponding respectively to kernel forms $\mathbf{S}^\alpha_\lambda$, $\mathbf{S}^m_\epsilon(b')$, and $\mathbf{S}^m_\epsilon(\mathfrak{s}_-,b')$ in the solution operator. We note that if $m=1$, then K_1 and K_3 are null.

As remarked in Section 2, it suffices to prove the integrability of the K_i's over R_{2d}. And the L^p boundedness of these operators will follow from classical theorem on integral operators once we prove the following:

$$\int_{R_{2d}} |K_i(\zeta,z)|\, d\mathbf{v}(\zeta) \le C \quad \text{for almost all } z \in R_d$$

$$\int_{R_d} |K_i(\zeta,z)|\, d\mathbf{v}(z) \le C \quad \text{for almost all } \zeta \in R_{2d}.$$

For fixed $z \in R_d$, we divide the region of integration into two parts; $R_{2d} \cap \{|\zeta'-z'|<d\}$ and $R_{2d}\setminus\{|\zeta'-z'|<d\}$. For $\zeta \in R_{2d}\setminus\{|\zeta'-z'|<d\}$, (2.9)-(2.11) implies that functions in the denominators of $|K_i|$'s are bounded away from zero by constants independent of λ, ζ, z. Hence integrals of $|K_i|$ over $R_{2d}\setminus\ \{|\zeta'-z'|<d\}$ is bounded by constants depending only on M and d. We need only calculate integrals over $R_{2d} \cap \{|\zeta'-z'|<d\}$.

In view of the definition of $\mathfrak{r}^*$ we have that $\mathfrak{r}(\zeta,z)+\mathfrak{r}^*(\zeta,z) = O(|\zeta-z|)$ for ζ near z, this and (2.8) imply

$$\begin{aligned}\langle \mathfrak{r}, d\zeta\rangle \wedge \langle \mathfrak{r}^*, d\zeta\rangle &= \begin{cases} \langle \mathfrak{r}, d\zeta\rangle \wedge \langle \mathfrak{r}+\mathfrak{r}^*, d\zeta\rangle \\ \langle \mathfrak{r}+\mathfrak{r}^*, d\zeta\rangle \wedge \langle \mathfrak{r}^*, d\zeta\rangle \end{cases} \\ &= \begin{cases} (\partial\rho|_\zeta) \wedge O^1(|\zeta-z|)) \\ O^1(|\zeta-z|)) \wedge (-\partial\rho|_z) \end{cases}\end{aligned} \tag{3.9}$$

where $O^1(x)$ denotes a 1-form whose coefficients are $O(x)$ functions. *We choose δ so that the above holds for $|\zeta-z|<\delta$.*

Estimation of $K_1(\zeta, z)$.

A straightforward calculation shows that $|K_1(\zeta,z)|$ is bounded by the sum of following functions:

$$\frac{|A_1 d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge\langle\mathfrak{s},d\zeta'\rangle\wedge\bar\partial\sigma\wedge d\zeta^{\hat{i_1},\hat{i_2}}\wedge d\bar\zeta^{\hat{l_1},\hat{l_2}}|}{(|r|+|\zeta-z|^2)^{(n-q-\alpha)}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{(\alpha+1)}} \tag{3.10}$$

$$\frac{|A_2 d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge\langle\mathfrak{s},d\zeta'\rangle\wedge d\zeta^{\hat{i_1},\hat{i_2}}\wedge d\bar\zeta^{\hat{j}}|}{(|r|+|\zeta-z|^2)^{(n-q-\alpha)}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{\alpha}} \tag{3.11}$$

$$\frac{|A_3 d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge\langle\mathfrak{s},d\zeta'\rangle\wedge d\zeta^{\hat{i_1},\hat{i_2}}\wedge d\bar\zeta^{\hat{j}}|}{(|r|+|\zeta-z|^2)^{(n-q-\alpha+1)}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{\alpha}} \tag{3.12}$$

$$\frac{|A_4 d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge d\zeta^{\hat{i}}\wedge d\bar\zeta^{\hat{j}}|}{(|r|+|\zeta-z|^2)^{(n-q-\alpha)}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{\alpha}} \tag{3.13}$$

$$\frac{|A_5 d\rho\wedge\langle\mathfrak{s},d\zeta'\rangle\wedge d\zeta^{\hat{i}}\wedge d\bar\zeta^{\hat{j}}|}{(|r|+|\zeta-z|^2)^{(n-q-\alpha)}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{\alpha}} \tag{3.14}$$

where $1\le\alpha\le m-1$, $i_1<i_2$, $d\zeta^{\hat{i_1},\hat{i_2}}=d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_{i_1}}\wedge\cdots\wedge\widehat{d\zeta_{i_2}}\wedge\cdots\wedge d\zeta_n$ denotes the $(n-2,0)$ form in $\mathbf{C}^n$ with indice i_1,i_2 being deleted, likewise for $d\zeta^{\hat{i}}$, $d\bar\zeta^{\hat{l_1},\hat{l_2}}$ and $d\bar\zeta^{\hat{j}}$, and A_j, $j=1,\dots,5$ are functions continuous on $\bar M$. Specifically, A_1 contains coefficients of $\mathfrak{r}+\mathfrak{r}^*$ and $\bar\partial\langle\mathfrak{s},\zeta'-z'\rangle$ as factors; A_2 contains the coefficient of $\bar\partial(\mathfrak{r}+\mathfrak{r}^*)$; A_3 contains coefficients of $\bar\partial\langle\mathfrak{r},\zeta-z\rangle$ and $\mathfrak{r}+\mathfrak{r}^*$; A_4 contains those of $\bar\partial\mathfrak{s}$ and $\mathfrak{r}+\mathfrak{r}^*$, while A_5 contains those of $\bar\partial\mathfrak{r}$ and $\mathfrak{r}+\mathfrak{r}^*$. Thus when $|\zeta-z|<\delta$, A_1, A_4, A_5 are $O(|\zeta-z|)$ functions, while $A_2=O(1)$ and $A_3=O(|\zeta-z|^2)$ by (3.9).

In view of (2.9)-(2.11) and (3.9), we see that Lemma 1 implies that (3.10)-(3.14) are integrable over the subregion $\{|\zeta'-z'|<d\}\cap B_\delta(z)$.

As for points in $W=R_{2d}\cap\{|\zeta'-z'|<d\}\cap\{|\zeta-z|>\delta\}$, (2.10) and (2.11) imply that $|r|+|\zeta-z|^2$, $|\zeta-z|^2$ are bounded away from zero by constants indepedent of ζ, z. We cover W by two parts:

$$W_1=\{\zeta\in W:|\partial''\rho(\zeta)|\le|\zeta'-z'|^{\frac12}\}$$

$$W_2=\{\zeta\in V:|\partial''\rho(\zeta)|>\frac12|\zeta'-z'|^{\frac12}\}.$$

For points ζ with $|\partial''\rho(\zeta)|\le|\zeta'-z'|^t$, $0<t\le1$, the fact that $\mathrm{Re}\langle\partial\rho,\zeta-z\rangle\ge c|\zeta-z|^2$ implies

$$|\zeta''-z''|\le c|\zeta'-z'|^{\frac{t}{2}} \tag{3.15}$$

for some constant c independent of ζ, z.

Thus

$$|\zeta - z| \le c|\zeta' - z'|^{\frac{t}{2}}. \tag{3.15'}$$

Denote by $\mathcal{D}$ the denominator of (3.10), then for $\zeta \in W_1$ (3.15') with $t = \frac{1}{2}$ and (2.9)-(2.11) give

$$\mathcal{D}^{-1} \lesssim |\zeta - z|^{-(2n-2\alpha+8)}(|\zeta' - z'|^2)^{-\alpha} \lesssim (|\zeta' - z'|^2)^{-\alpha}.$$

Since $\alpha \le m-1$, it is easy to see that the integration of (3.10) over W_1 is bounded by a constant independent of z.

In case $\{|\partial''\rho| > \frac{1}{2}|\zeta' - z'|^{\frac{1}{2}}\}$, we use general coarea formula applied to the projection map π from $\mathbf{C}^n$ to $\mathbf{C}^m$. Elementary calculation shows that $J^*\pi = |d''\rho|$. Thus the integral of (3.10) over W_2 is bounded by

$$C\int_{\pi(W_2)} (|\sigma| + |s| + |\zeta' - z'|^2)^{-(\alpha+1)}|\zeta' - z'|^{-\frac{1}{2}}, \quad 1 \le \alpha \le m-1.$$

The standard coordinate transformation for $\{\sigma < 0\} \subset \mathbf{C}^m$ as defined in [**Ra**, Chapter V, Lemma 3.4], denoted here as $h'_{z'}$ and an elementary integration give a finite bound for this integral which is independent of z.

Similarly, integrals of (3.11)-(3.14) over W are bounded by constants independent of λ, z. In summary $\int_{R_{2d}\cap\{|\zeta'-z'|<d\}} |K_1(\zeta, z)|\, d\mathbf{v}(\zeta) < C$, C is independent of λ, z.

Estimation of $K_2(\zeta, z)$.

A straightforward calculation shows that $|K_2(\zeta, z)|$ is bounded by the sum of the following functions:

$$\frac{|B_1 d\rho \wedge \langle \mathfrak{r}, d\zeta\rangle \wedge d\zeta^{\hat{i}} \wedge d\bar{\zeta}^{\hat{j}}|}{(|r| + |\zeta - z|^2)^{(n-q-m)}|\zeta - z|^{2q}|\zeta' - z'|^{2m-1}} \tag{3.16}$$

$$\frac{|B_2 d\rho \wedge \langle \mathfrak{r}, d\zeta\rangle \wedge d\zeta^{\hat{i}} \wedge d\bar{\zeta}^{\hat{j}}|}{(|r| + |\zeta - z|^2)^{(n-q-m+1)}|\zeta - z|^{2q}|\zeta' - z'|^{2m-1}} \tag{3.17}$$

$$\frac{|B_3 d\rho \wedge d\zeta \wedge d\bar{\zeta}^{\hat{j}}|}{(|r| + |\zeta - z|^2)^{(n-q-m)}|\zeta - z|^{2q}|\zeta' - z'|^{2m-1}} \tag{3.18}$$

where $i > m$ by type consideration, and $B_1, \ldots, B_3$ are functions continuous on M, with the property that when $|\zeta - z| < \delta$, $B_1 = O(1)$, $B_2 = O(|\zeta - z|^2)$ and $B_3 = O(|\zeta - z|)$.

We proceed as in estimating the integral of $|K_1(\zeta,z)|$. For fixed $z\in R_d$, the integrability of (3.16)-(3.18) over $\{|\zeta'-z'|<d\}\cap B_\delta(z)$ follows from (3.7) of Lemma 3. And integrations over W are calculated by methods similar to those for (3.10)-(3.14). Thus $\int_{R_{2d}}|K_2(\zeta,z)|\,d\mathbf{v}(\zeta)$ is bounded by a constant independent of z.

Estimation of $K_3(\zeta,z)$.

$|K_3(\zeta,z)|$ is bounded by the sum of the following functions:

(3.19)
$$\frac{|C_1 d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge\langle\mathfrak{s},d\zeta'\rangle\wedge\bar\partial\sigma\wedge d\zeta^{\hat{i_1},\hat{i_2}}\wedge d\bar\zeta^{\hat{l_1},\hat{l_2}}|}{(|r|+|\zeta-z|^2)^{n-q-m}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{m-\alpha+1}|\zeta'-z'|^{2\alpha-1}}$$

(3.20)
$$\frac{|C_2 d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge\langle\mathfrak{s},d\zeta'\rangle\wedge d\zeta^{\hat{i_1},\hat{i_2}}\wedge d\bar\zeta^{\hat{j}}|}{(|r|+|\zeta-z|^2)^{n-q-m}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{m-\alpha}|\zeta'-z'|^{2\alpha}}$$

(3.21)
$$\frac{|C_3 d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge d\zeta^{\hat{i}}\wedge d\bar\zeta^{\hat{j}}|}{(|r|+|\zeta-z|^2)^{n-q-m}|\zeta-z|^{2q}(|\sigma|+|s|+|\zeta'-z'|^2)^{m-\alpha}|\zeta'-z'|^{2\alpha-1}}$$

where $1\le\alpha\le m-1$, and $i_1\le m$, $i_2>m$, $i>m$ by type considerations. And $C_l=O(1)$, $l=1,\dots,3$ are functions continuous on M.

For fixed $z\in R_d$, the integrability of (3.19)-(3.21) over $\{|\zeta'-z'|<d\}\cap B_\delta(z)$ follows from Lemma 2. For the integral of $|K_3|$ over W, we argue as in the proof of integrability for $K_1(\zeta,z)$ and get that it is bounded by a constant independent of z.

We note that all estimates are valid if we drop the subscript λ. On the other hand, it is obvious that for each fixed z and for almost all ζ, we have

$$\lim_{\lambda\to 0}|d\bar\zeta^J\wedge\mathbf{S}^\alpha_\lambda(\zeta,z)|=|d\bar\zeta^J\wedge\bar\partial_\zeta(\theta^\alpha(\mathfrak{r},\mathfrak{r}^*)\wedge\omega^\alpha_-(\mathfrak{s}))(\zeta,z)|$$

$$\lim_{\epsilon\to 0}|d\bar\zeta^J\wedge\mathbf{S}^m_\epsilon(b')(\zeta,z)|=|d\bar\zeta^J\wedge\bar\partial_\zeta(\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b'))(\zeta,z)|$$

$$\lim_{\epsilon\to 0}|d\bar\zeta^J\wedge\mathbf{S}^m_\epsilon(\mathfrak{s}_-,b')(\zeta,z)|=|d\bar\zeta^J\wedge\bar\partial_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\bar\partial_{\zeta'}\Omega^0_-(\mathfrak{s}',b')(\zeta,z)|.$$

Lebesgue convergence theorem gives that the coefficient of the (n, n) form $d\rho \wedge d\bar{\zeta}^J \wedge \mathbf{S}^\alpha_\lambda$ converges in $L^1(M)$ to the coefficient of $d\rho \wedge d\bar{\zeta}^J \wedge \bar{\partial}_\zeta(\theta^\alpha(\mathfrak{r},\mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s}))$, likewise, the coefficient of $d\rho \wedge d\bar{\zeta}^J \wedge \mathbf{S}^m_\epsilon(b')$ converges to that of $d\rho \wedge d\bar{\zeta}^J \wedge \bar{\partial}_\zeta(\theta^m(\mathfrak{r},\mathfrak{r}^*) \wedge \Omega^0(b'))$ in $L^1(M)$ and the coefficient of $d\rho \wedge d\bar{\zeta}^J \wedge \mathbf{S}^m_\epsilon(\mathfrak{s}_-, b')$ converges in $L^1(M)$ to that of $d\rho \wedge d\bar{\zeta}^J \wedge \bar{\partial}_\zeta \theta^m(\mathfrak{r},\mathfrak{r}^*) \wedge \bar{\partial}_{\zeta'} \Omega^0_-(\mathfrak{s}', b')$. *We denote by* $\mathbf{S}^\alpha$, $\mathbf{S}^m(b')$ *and* $\mathbf{S}^m(\mathfrak{s}_-, b')$ *operators defined respectively by forms* $\bar{\partial}_\zeta(\theta^\alpha(\mathfrak{r},\mathfrak{r}^*) \wedge \omega^\alpha_-(\mathfrak{s}))$, $\bar{\partial}_\zeta(\theta^m(\mathfrak{r},\mathfrak{r}^*) \wedge \Omega^0(b'))$ *and* $\bar{\partial}_\zeta \theta^m(\mathfrak{r},\mathfrak{r}^*) \wedge \bar{\partial}_{\zeta'} \Omega^0_-(\mathfrak{s}', b')$.

Estimations of $\mathbf{T}_{\epsilon,\lambda}(b')$, $\mathbf{T}^\alpha_{\epsilon,\lambda}(\mathfrak{s}_-, b')$ and $\mathbf{T}_{\epsilon,\lambda}$.

As for $\mathbf{T}_{\epsilon,\lambda}(b')$, we recall that

$$\mathbf{T}_{\epsilon,\lambda}(b') = \epsilon^{-2m} \phi\left(\frac{|\zeta' - z'|}{\epsilon}\right) \bar{\partial}_\zeta \theta^m_\lambda(\mathfrak{r},\mathfrak{r}^*) \wedge \langle b', d\zeta'\rangle \wedge \langle d\bar{\zeta}', d\zeta'\rangle^{m-1}$$

and

$$|d\rho \wedge d\bar{\zeta}^J \wedge \mathbf{T}_{\epsilon,\lambda}(b')| \le \left| d\rho \wedge \phi\left(\frac{|\zeta' - z'|}{\epsilon}\right) d\bar{\zeta}^J \wedge \bar{\partial}_\zeta \theta^m(\mathfrak{r}_\lambda, \mathfrak{r}^*_\lambda) \wedge \Omega^0(b')\right|.$$

(3.7) of Lemma 3 shows that the integral of $|d\rho \wedge d\bar{\zeta}^J \wedge \mathbf{T}_{\epsilon,\lambda}(b')|$ is bounded by a constant of the order $\epsilon^\tau |\log \epsilon|$ for some appropriately chosen $\tau < 1$ (τ independent of z, λ). As ϵ goes to zero the integral diminishes to zero, we conclude that

$$\lim_{\epsilon \to 0} \lim_{\lambda \to 0} \int |d\rho \wedge d\bar{\zeta}^J \wedge \mathbf{T}_{\epsilon,\lambda}(b')| = 0. \tag{3.22}$$

Similarly, since

$$\mathbf{T}^\alpha_{\epsilon,\lambda}(\mathfrak{s}_-, b') = \epsilon^{-2\alpha} \phi\left(\frac{|\zeta' - z'|}{\epsilon}\right) \bar{\partial}_\zeta \theta^m_\lambda(\mathfrak{r},\mathfrak{r}^*) \wedge \bar{\partial}_{\zeta'}(\omega^{m-\alpha}_-(\mathfrak{s}) \wedge \langle b', d\zeta'\rangle \wedge \langle d\bar{\zeta}', d\zeta'\rangle^{\alpha-1}),$$

we have

$$|d\rho \wedge d\bar{\zeta}^J \wedge \mathbf{T}^\alpha_{\epsilon,\lambda}(\mathfrak{s}_-, b')| \le \left| d\rho \wedge d\bar{\zeta}^J \wedge \phi\left(\frac{|\zeta' - z'|}{\epsilon}\right) \bar{\partial}_\zeta \theta^m_\lambda(\mathfrak{r},\mathfrak{r}^*) \wedge \bar{\partial}_{\zeta'}\left(\omega^{m-\alpha}_-(\mathfrak{s}) \wedge \frac{\langle b', d\zeta'\rangle}{\langle b', \zeta' - z'\rangle} \wedge \left(\frac{\langle d\bar{\zeta}', d\zeta'\rangle}{\langle b', \zeta' - z'\rangle}\right)^{\alpha-1}\right)\right|.$$

As the coefficient of the right hand side of the above inequality are bounded by the sum of the integrands of (3.4)-(3.6), Lemma 2 shows that the integral of $|d\rho\wedge d\bar{\zeta}^J\wedge \mathbf{T}^{\alpha}_{\epsilon,\lambda}(\mathfrak{s}_-,b')|$ is bounded by a constant of order $\epsilon^{\tau}|\log\epsilon|$ for some appropriately chosen $\tau<1$ (τ independent of z, λ). Therefore, we have

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0}\int |d\rho\wedge d\bar{\zeta}^J\wedge \mathbf{T}^{\alpha}_{\epsilon,\lambda}(\mathfrak{s}_-,b')|=0.$$

It remains to estimate $d\bar{\zeta}^J\wedge\mathbf{T}_{\epsilon,\lambda}$. As $\mathbf{T}_{\epsilon,\lambda}=\epsilon^{-2m}\phi(\frac{|\zeta'-z'|}{\epsilon})\theta^m_\lambda(\mathfrak{r},\mathfrak{r}^*)\wedge\langle d\bar{\zeta}',d\zeta'\rangle^m$, the length of the tangential part of $d\bar{\zeta}^J\wedge\mathbf{T}_{\epsilon,\lambda}$ is bounded by the following

$$\epsilon^{-1}\frac{|O(|\zeta-z|)d\rho\wedge\langle\mathfrak{r},d\zeta\rangle\wedge d\zeta^{\hat{i}}\wedge d\bar{\zeta}^{\hat{j}}|}{(|r|+|\zeta-z|^2)|^{(n-q-m)}|\zeta-z|^{2q}|\zeta'-z'|^{(2m-1)}}$$

where by type consideration both i, j are $>m$.

Then (3.8) of Lemma 3 gives that the integral of $|d\rho\wedge d\bar{\zeta}^J\wedge\mathbf{T}_{\epsilon,\lambda}|$ over M is bounded by a constant independent of λ, z (and even independent of ϵ). This holds if we drop the subscript λ in $\mathbf{T}_{\epsilon,\lambda}$, we denote such form by $\mathbf{T}_\epsilon$. Thus we have that the coefficient of $d\rho\wedge d\bar{\zeta}^J\wedge\mathbf{T}_{\epsilon,\lambda}$ as well as that of $d\rho\wedge d\bar{\zeta}^J\wedge\mathbf{T}_\epsilon$ are in $L^1(M)$. Lebesgue convergence theorem implies that the coefficient of $d\rho\wedge d\bar{\zeta}^J\wedge\mathbf{T}_{\epsilon,\lambda}$ converges in L^1 to that of $d\rho\wedge d\bar{\zeta}^J\wedge\mathbf{T}_\epsilon$. On the other hand, we observe that for any $\bar{\partial}_b$-closed $(0,q)$-form f and $0<\epsilon_1<\epsilon_2<d$,

$$\begin{aligned}&\int_{M\cap\{|\zeta'-z'|=\epsilon_2\}} f(\zeta)\wedge\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b')\\ &-\int_{M\cap\{|\zeta'-z'|=\epsilon_1\}} f(\zeta)\wedge\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b')\\ &=\int_{\{\epsilon_1<|\zeta'-z'|<\epsilon_2\}\cap M} f(\zeta)\wedge\bar{\partial}_\zeta\theta^m(\mathfrak{r},\mathfrak{r}^*)\wedge\Omega^0(b').\end{aligned}$$

In particular, if $f=d\bar{\zeta}^J$ (3.7) of Lemma 3 (or the estimate of $K_2(\zeta,z)$) asserts that the right hand side of the above formula is bounded by $|\epsilon_2^{\frac{1}{8}}\log\epsilon_2-\epsilon_1^{\frac{1}{8}}\log\epsilon_1|$, this and (3.22) imply that the coefficient of the (n,n) form $d\rho\wedge d\bar{\zeta}^J\wedge\mathbf{T}_\epsilon$ forms a Cauchy sequence in $L^1(M)$, hence it converges in L^1. *We denote by* $\mathbf{T}$ *the operator which assigns to all possible monomial* $d\bar{\zeta}^J$ *the limit of* $d\rho\wedge d\bar{\zeta}^J\wedge\mathbf{T}_\epsilon$ *as* $\epsilon\to 0$.

Let $\mathcal{T} = \sum_\alpha \mathbf{S}^\alpha + \mathbf{S}^m(b') + \mathbf{S}^m(\mathfrak{s}_-, b') + \mathbf{T}$. We have shown that if we denote by $K(\zeta, z)$ the tangential part of $d\bar{\zeta}^J \wedge \mathcal{T}$, then for each fixed $z \in R_d$, the integral of $|K(\zeta, z)|$ over R_{2d} is bounded by a constant independent of z. In view of the definition of $\mathfrak{r}^*$, (2.9)-(2.11) and (3.9), etc., the whole process of estimation is valid with minor modifications if we switch roles of ζ and z. Classical results of singular integrals (see e.g. [**Ra**, Appendix]) then implies $K(\zeta, z)$ defines an operator which maps $L^p(M)$ to itself, $1 \leq p \leq \infty$.

We have proved the assertion of Theorem 1 for $\bar{\partial}_b$-closed $(0, q)$ forms with coefficients in $C^1(\bar{M})$. To complete the proof of Theorem 1, we proceed as in [**Sh1**], for $1 \leq p < \infty$, let $f \in L^p_{(0,q)}(M)$ be $\bar{\partial}_b$-closed in distribution sense, we approximate f by a mollified sequence of $(0, q)$ forms f_k given by the mollification method of Friedrichs [**Fr**] whose coefficients are C^1 in a neighborhood of $\bar{M}_k$, where $M_k = \{\rho = 0\} \cap \{r < \frac{1}{k}\}$, and $f_k \to f$, $\bar{\partial}_b f_k \to 0$ in $L^p_{(0,q)}(M)$, $L^p_{(0,q+1)}(M)$ respectively. If $1 \leq q \leq n - m - 2$, Remark 2 and previous arguments applied to M_k implies that there exists $g_k \in C^1(M_k)$ such that $\bar{\partial}_b g_k = \bar{\partial}_b f_k$ and $||g_k||_p \to 0$. Since both f_k and g_k are C^1 in a neighborhood of $\bar{M}_{k+1}$, the above assertion implies there exists $u_k \in C^1(M_{k+1})$ such that $\bar{\partial}_b u_k = f_k - g_k$ and $||u_k||_p \leq C||f_k - g_k||_p$ in M_{k+1}. We remark that from the above proof for $\bar{\partial}_b$-closed $C^1_{(0,q)}(\bar{M})$ forms it is clear that the constant can be chosen independent of small perturbations of the domain M, so it depends neither on p nor on k. As the solution operator established via (2.14) is linear, we see that $u_{k+l} - u_k$ solves $\bar{\partial}_b v = f_{k+l} - g_{k+l} - f_k + g_k$ on M_{k+l+1} and satisfies $||u_{k+l} - u_k||_{L^p(M_{k+l+1})} \leq C||f_{k+l} - f_k - (g_{k+l} - g_k)||_{L^p(M_{k+l+1})} \to 0$, when $k \to \infty$. Therefore, $\{u_k\}$ forms a Cauchy sequence in $L^p(M)$, its limit u must satisfy $\bar{\partial}_b u = f$, and $||u||_p \leq C||f||_p$. For the case $p = \infty$, let $\{f_k\}$ be a fixed mollified sequence of f obtained by Friedrichs' method. Then, as $f \in L^\infty_{(0,q)}(M)$ is in $L^p_{(0,q)}(M)$ for all $1 \leq p < \infty$, we see that $f_k \in L^p(M_k)$, $f_k \to f$, $\bar{\partial}_b f_k \to 0$ in $L^p(M)$ norms for all $1 \leq p < \infty$. We argue as in the proof for $p < \infty$ to obtain g_k's, and u_k's by the very solution operator established via (2.14). It is clear that they are independent of p. Now the same argument also gives that $\{u_k\}$ forms a Cauchy sequence in $L^p(M)$ for all $1 \leq p < \infty$. So, for each $1 \leq p < \infty$ there exists a $(0, q - 1)$ form $u_p \in L^p(M)$ such that $\bar{\partial}_b u_p = f$ and $||u_p||_p \leq C||f||_p$, where C is independent of p, $1 \leq p < \infty$. However, it is easy to check that $u_p = u_{p'}$, $\forall\, p, p' \in [1, \infty)$, as M is a bounded domain. We denote it by u. Well-known results in Function Spaces then imply that u is in $L^\infty_{(0,q-1)}(M)$ and $||u||_\infty \leq C||f||_\infty$. We have completed the proof of the L^p-boundedness of the solution operator.

Corollaries 1 and 2 are derived easily from Theorem 1 and its proof.

4. Proofs of Lemmas

We note that R_{2d} can be written as finite union of subsets $R_{i,j}$, such that in $R_{i,j}$, $|\frac{\partial\rho}{\partial x_i}\frac{\partial\sigma}{\partial x_j} - \frac{\partial\rho}{\partial x_j}\frac{\partial\sigma}{\partial x_i}| \geq \kappa$, where $2n^2\kappa$ is the lower bound of $|d\rho \wedge d\sigma|$ on $\bar{R}_{2d}$. And it can also be written as finite union of $R^{k,l}$ where in $R^{k,l}$, $|\frac{\partial\rho}{\partial\zeta_k}\frac{\partial\sigma}{\partial\zeta_l} - \frac{\partial\rho}{\partial\zeta_l}\frac{\partial\sigma}{\partial\zeta_k}| \geq |\frac{\partial\rho}{\partial\zeta_p}\frac{\partial\sigma}{\partial\zeta_q} - \frac{\partial\rho}{\partial\zeta_q}\frac{\partial\sigma}{\partial\zeta_p}|$, $\forall\, p,\, q$. Clearly, estimation over each $V \cap R_{i,j} \cap R^{k,l}$ suffices to give proofs of Lemmas 1-3. Fix an arbitrary $R_{i,j} \cap R^{k,l}$, it will be apparent from the proof that we may assume without loss of generality, say $i,\ k = 1, j,\ l = 2$, and for simplicity denote again by V the intersection of it with V.

We remark that if $|\partial''\rho| > c$ for some positive constant c in V, using standard coordinate transformations for strongly pseudoconvex domains and strongly pseudoconvex hypersurfaces, it is not hard to see Lemmas 1-3 hold. Difficulties arise because of the presence of points where $\partial''\rho = 0$. Therefore, we have to divide regions of integration into subregions where one could perform "coordinate transformations" with controllable jacobians. And the use of (3.15), (3.15') in proofs demonstrates the strong pseudoconvexity of $\{\rho < 0\}$ also plays an important role.

Proof of Lemma 1:

(A) Proof of (3.1):

Let

$$V_1 = \{\zeta \in V : |\partial''\rho(\zeta)| \leq |\zeta' - z'|^{\frac{1}{8}},\, |\partial\rho \wedge \partial\sigma| \leq |\zeta - z|^{\tau}\}$$

$$V_2 = \{\zeta \in V : |\partial''\rho(\zeta)| \leq |\zeta' - z'|^{\frac{1}{8}},\, |\partial\rho \wedge \partial\sigma| > \frac{1}{2}|\zeta - z|^{\tau}\}$$

$$V_3 = \{\zeta \in V : |\partial''\rho(\zeta)| > \frac{1}{2}|\zeta' - z'|^{\frac{1}{8}},\, |\zeta' - z'|^{\frac{1}{4}} \geq |\zeta'' - z''|\}$$

$$V_4 = \{\zeta \in V : |\partial''\rho(\zeta)| > \frac{1}{2}|\zeta' - z'|^{\frac{1}{8}},\, |\zeta' - z'|^{\frac{1}{4}} \leq |\zeta'' - z''|\}.$$

Denote by J_l, $l = 1, \dots, 4$ respectively integrals over V_l.

Denote by $\mathcal{D}$ the denominator of (3.1), then (3.15') with $t = \frac{1}{8}$ gives

$$\mathcal{D}^{-1} \lesssim |\zeta - z|^{-(2n-2p+2)}(|\sigma| + |\zeta' - z'|^2)^{-(p-\frac{1}{32})}. \tag{4.1}$$

In $R_{i,j} = R_{1,2}$, we define a map $\Theta^1 = \Theta^1_{1,2} : R_{1,2} \to \mathbf{R}^{2n}$ as follows: we write by $y = (y_0, \dots, y_{2n-1})$ the coordinate in the target space, let

$$(4.2) \qquad y_0 = \rho, \quad y_1 = \sigma, \quad y_2 = x_3, \dots, y_{2n-1} = x_{2n},$$

then the generalized jacobian $J\Theta^1 = \{\det(d\Theta^1_x)^* \circ (d\Theta^1_x)\}^{\frac{1}{2}}$ is given by $|\frac{\partial\rho}{\partial x_1}\frac{\partial\sigma}{\partial x_2} - \frac{\partial\rho}{\partial x_2}\frac{\partial\sigma}{\partial x_1}| \ge \kappa > 0$. Simple topological argument implies that $(\Theta^1)^{-1}(y)$ consists of finitely many points with its number bounded by a constant independent of y.

We now apply the general area formula to the map Θ^1 and the integral J_1 and get that

$$J_1 \lesssim \int\limits_{\substack{|y_1| \le C_1 \\ |\tilde{y}'| \le b \\ |y''| \le C_2}} \frac{dy_1 \cdots dy_{2n-1}}{|\tilde{y}|^{(2n-2p+2-2\tau)}(|y_1| + |\tilde{y}'|^2)^{(p-\frac{1}{32})}} \lesssim b^{2(\tau - \frac{31}{32})},$$

if we choose $1 > \tau > \frac{31}{32}$, where $\tilde{y}' = (y_2, \dots, y_{2m-1})$, $y'' = (y_{2m}, \dots, y_{2n-1})$, and $\tilde{y} = (\tilde{y}', y'')$.

For the integration over V_2, as $|\partial\rho \wedge \partial\sigma| > 0$, we would like to define a map from $R^{k,l} = R^{1,2}$ to $\mathbf{R}^{2n}$ so that $y_0 = \rho$, $y_1 = \sigma$, $y_2 = s$, $y_3 = x_5, \dots, y_{2n-2} = x_{2n}$, and $y_{2n-1} = r$ with $|\frac{\partial\rho}{\partial\zeta_1}\frac{\partial\sigma}{\partial\zeta_2} - \frac{\partial\rho}{\partial\zeta_2}\frac{\partial\sigma}{\partial\zeta_1}|^2$ as its generalized jacobian. Yet this map may fail to have finite multiplicity over R_{2d}, so we modify the map using the technique first introduced by Range-Siu [**Ra-Siu**] (see also Shaw [**Sh1**]), namely, let P_ρ, P_σ, P_s, P_r, be respectively the second order Taylor polynomials of ρ, σ, s, and r expanded at z. They have the following properties:

(4.3) $\rho(\zeta) = P_\rho(\zeta, z) + o(|\zeta - z|^2)$, where o is uniform in z,

(4.4) $d_\zeta\rho = d_\zeta P_\rho(\zeta, z) + o(|\zeta - z|)$, where o is uniform in z,

(4.5) $\sigma(\zeta') = P_\sigma(\zeta', z') + o(|\zeta' - z'|^2)$, where o is uniform in z',

(4.6) $d'_\zeta\sigma = d'_\zeta P_\sigma(\zeta', z') + o(|\zeta' - z'|)$, where o is uniform in z',

(4.7) $s = \operatorname{Im}\langle \mathfrak{s}, \zeta' - z' \rangle = P_s(\zeta', z') + o(|\zeta' - z'|^2)$, where o is uniform in z',

(4.8) $d'_\zeta \operatorname{Im}\langle \mathfrak{s}, \zeta' - z' \rangle = d'_\zeta P_s(\zeta', z') + o(|\zeta' - z'|)$, where o is uniform in z',

(4.9) $r = \operatorname{Im}\langle \mathfrak{r}, \zeta - z \rangle = P_r(\zeta, z) + o(|\zeta - z|^2)$, where o is uniform in z,

(4.10) $d_\zeta \operatorname{Im}\langle \mathfrak{r}, \zeta - z \rangle = d_\zeta P_r(\zeta, z) + o(|\zeta - z|)$, where o is uniform in z.

We may shrink δ so that for $|\zeta' - z'| < \delta$ and $|\zeta - z| < \delta$ there exists a constants $c > 0$ independent of z such that

(4.11) $|s| + |\zeta' - z'|^2 \geq c(|P_s(\zeta', z')| + |\zeta' - z'|^2)$,

(4.12) $|\sigma| + |s| + |\zeta' - z'|^2 \geq c(|P_\sigma(\zeta', z')| + |P_s(\zeta', z')| + |\zeta' - z'|^2)$,

(4.13) $|r| + |\zeta - z|^2 \geq c(|P_r(\zeta, z)| + |\zeta - z|^2)$.

These follow directly from Taylor expansions for C^2 functions and (2.9)-(2.11).

For each fixed $z \in R_d$, we now define the map $\Theta^2 : R^{1,2} \to \mathbf{R}^{2n}$ by $y_0 = P_\rho(\zeta, z)$, $y_1 = P_\sigma(\zeta, z)$, $y_2 = P_s(\zeta, z)$, $y_k = x_{k+2}$, $3 \leq k \leq 2n-2$, $y_{2n-1} = P_r(\zeta, z)$. In view of (4.4), (4.6), (4.8), (4.10), we have

$$
\begin{aligned}
J\Theta^2 &= |d\rho \wedge dr \wedge d\sigma \wedge ds \wedge_5^{2n} dx_k| + o(|\zeta - z|)|\partial\rho \wedge \partial\sigma| \\
&\quad + o(|\zeta' - z'|)|\partial\rho \wedge \partial\sigma| + O(|\zeta - z|^2) \\
&= |\partial\rho/\partial\zeta_1 \partial\sigma/\partial\zeta_2 - \partial\rho/\partial\zeta_2 \partial\sigma/\partial\zeta_1|^2 + o(|\zeta - z|)|\partial\rho \wedge \partial\sigma| \\
&\quad + o(|\zeta' - z'|)|\partial\rho \wedge \partial\sigma| + O(|\zeta - z|^2).
\end{aligned} \tag{4.14}
$$

In particular, if $|\partial\rho \wedge \partial\sigma| > |\zeta - z|^\tau$, we then have $|\partial\rho/\partial\zeta_1 \partial\sigma/\partial\zeta_2 - \partial\rho/\partial\zeta_2 \partial\sigma/\partial\zeta_1| > \frac{1}{2n^2}|\zeta - z|^\tau$ and (4.14) implies that $J\Theta^2 > c|\zeta - z|^{2\tau}$ for some constant c independent of ζ, z. Moreover, we have the following estimates:

$$
c \leq |\partial\rho/\partial\zeta_1 \partial\sigma/\partial\zeta_2 - \partial\rho/\partial\zeta_2 \partial\sigma/\partial\zeta_1|^2 / J\Theta^2 \leq C \tag{4.15}
$$

$$
|\partial\rho \wedge \partial\sigma| / J\Theta^2 \leq C|\zeta - z|^{-\tau}. \tag{4.16}
$$

Clearly, (4.15) is equivalent to

$$
c \leq |\partial\rho \wedge \partial\sigma|^2 / J\Theta^2 \leq C. \tag{4.15'}
$$

From (4.12), (4.13) and (3.15') we have that

$$
\begin{aligned}
\mathcal{D}^{-1} \lesssim (|P_r| + |\zeta - z|^2)^{-(n-q-p+\frac{1}{2})}|\zeta - z|^{-2q-\frac{3}{4}} \\
(|P_\sigma| + |P_s| + |\zeta' - z'|^2)^{-(p-\frac{3}{128})}.
\end{aligned} \tag{4.17}
$$

We now apply the general area formula to the map Θ^2 and J_2, then (4.15') and (4.17) give

$$J_2 \lesssim \int_{\substack{|y_j|\le C_1, j=1,2,2n-1\\ |\tilde y'|\le b\\ |y''|\le C_2}} \frac{dy_1\cdots dy_{2n-1}}{(|y_{2n-1}|+|\tilde y|^2)^{(n-q-p+\frac12)}|\tilde y|^{2q+\frac34}(|y_1|+|y_2|+|\tilde y'|^2)^{(p-\frac{3}{128})}}$$

where $\tilde y' = (y_3, \dots, y_{2m-2})$, $y'' = (y_{2m-1}, \dots, y_{2n-2})$, and $\tilde y = (\tilde y', y'')$, if $m > 2$; and when $m = 2$, $\tilde y'$ is void. Direct computation gives the integral is bounded by $Cb^{\frac14}$.

In case $\{|\partial''\rho| > \frac12|\zeta'-z'|^{\frac18}\}$, we use general coarea formula applied to π the projection map from $\mathbf{C}^n$ to $\mathbf{C}^m$. Note that $J^*\pi = |d''\rho|$. And for almost all $\zeta' \in \pi(\bar M)$ the set $V_3 \cap \Gamma_{\zeta'}$ is contained in a $2n-2m-1$ submanifold by the Sard-type theorem [**Si**, p. 56]. For such ζ', we use the coordinate transformation $h''_{\zeta'}$ defined as follows:

In the subset of M where $\partial''\rho \neq 0$, for $j > m$, let $\mathcal{V}^j = \{\zeta \in M, \partial''\rho \neq 0, |\partial\rho/\partial\zeta_j| \ge |\partial\rho/\partial\zeta_l|\, \forall\, l > m\}$. Fix a $\mathcal{V}^j$, say $j = n$, the map $h''_{\zeta'}$ from $\mathcal{V}^n \cap \Gamma_{\zeta'} \cap B_\delta(z)$ to $\mathbf{R}^{2n-2m}$ is defined by $t''_0 = P_\rho(\zeta, z)$, $t''_1 = x_{2m+1}$, $t''_{2n-2m-2} = x_{2n-2}$, $t''_{2n-2m-1} = P_r(\zeta, z)$. Then previous argument for $J\Theta^2$ also applies here, and we have $Jh''_{\zeta'} \simeq |\partial''\rho|^{-1}(|\partial\rho/\partial\zeta_n|^2 + O(|\zeta - z|)|\partial''\rho| + O(|\zeta - z|^2))$. In particular, if $|\partial''\rho| > |\zeta - z|^\tau$, reasoning as before, $Jh''_{\zeta'} \ge c|\zeta - z|^\tau$ and $c \le |\partial''\rho|/Jh''_{\zeta'} \le C$. Moreover, the map has the property that the inverse image $|{h''_{\zeta'}}^{-1}(t'')|$ is a finite set with uniformly bounded cardinal number independent of ζ' and t''.

The condition $|\zeta' - z'|^{\frac14} \ge |\zeta'' - z''|$ implies that $Jh''_{\zeta'} \gtrsim |\zeta' - z'|^{\frac18}$. As for the image $\pi(V)$ in $\mathbf{C}^m$, we use the coordinate transformation $h'_{z'}$ for $\{\sigma < 0\} \subset \mathbf{C}^m$ as defined in [**Ra**, Chapter V, Lemma 3.4] which has its jacobian bounded from below. Thus,

$$J_3 \lesssim \int_{\substack{|y_j|\le C_1, j=1,2\\ |\tilde y'|\le b}} \left(\int_{\substack{|y_{2n-1}|\le C_1\\ |y''|\le C_2}} \frac{dy_{2m+1}\cdots dy_{2n-1}}{(|y_{2n-1}| + |\tilde y|^2)^{(n-q-p+\frac12)}|\tilde y|^{2q}} \right) \frac{dy_1\cdots dy_{2m}}{(|y_1| + |y_2| + |\tilde y'|^2)^p|\tilde y'|^{\frac14}}$$

where $\tilde y' = (y_3, \dots, y_{2m})$, $y'' = (y_{2m+1}, \dots, y_{2n-2})$, and $\tilde y = (\tilde y', y'')$.

Again direct computation shows that it is bounded by $Cb^{\frac34}$ if $p > 2$, and by $Cb^{\frac34}|\log b|$ if $p = 2$.

In V_4, since $|\zeta' - z'|^{\frac{1}{4}} \leq |\zeta'' - z''|$, we have for $2 > \vartheta > 1$

$$\mathcal{D}^{-1} \lesssim |\zeta - z|^{-(2n-2m-\vartheta)}(|\sigma| + |s| + |\zeta' - z'|^2)^{-p}|\zeta' - z'|^{-\frac{(2m-2p+\vartheta+1)}{4}},$$

and

$$J_4 \lesssim \int\limits_{\substack{|y_j|\leq C_1, j=1,2\\ |\tilde{y}'|\leq b}} \left(\int\limits_{\Gamma_{\zeta'}} \frac{d\mathbf{v}_{\Gamma_{\zeta'}}}{|\tilde{y}|^{(2n-2m-\vartheta)}}\right) \frac{dy_1 \cdots dy_{2m}}{(|y_1| + |y_2| + |\tilde{y}'|^2)^p |\tilde{y}'|^{\frac{(2m-2p+\vartheta+1+\frac{1}{2})}{4}}}$$

where $\tilde{y}' = (y_3, \ldots, y_{2m})$, and $\tilde{y} = (\tilde{y}', x_{2m+1}, \ldots, x_{2n})$.

As $\Gamma_{\zeta'}$ is compact and for almost all ζ', $\Gamma_{\zeta'} \cap V_4$ is contained in a $(2n-2m-1)$ manifold, $|x|^{-(2n-2m-\vartheta)}$ is integrable over such $\Gamma_{\zeta'}$ with finite bound, for, $\vartheta > 1$ and the volume of $\Gamma_{\zeta'}$ is bounded by a constant independent of z'. Elementary integration then gives J_4 is bounded by $b^{\frac{7}{8}}|\log b|$.

(B) Proof of (3.2):

We write $d\rho\wedge\partial\rho\wedge\partial\sigma$ as $\sum_J a_J\, dx_J$, where $J = (i, j, k)$, $1 \leq i, j, k \leq 2n$ are stricly increasing triple indices and $dx_J = dx_i \wedge dx_j \wedge dx_k$. R_{2d} is then decomposed as finite union of closed sets R_J, where in R_J $|a_J| \geq |a_{J'}|$ for all J'. We consider now $V \cap R_J$ for arbitrary J, and denote it again by V. We assume the worst case (as far as integrability is concerned) where $i, j, k \leq 2m$, say $J = (1, 2, 3)$.

Let

$$V_1 = \{\zeta \in V : |d\rho \wedge \partial\rho \wedge \partial\sigma| \leq |\zeta - z|^\tau\}$$

$$V_2 = \{\zeta \in V : |\partial''\rho(\zeta)| \leq |\zeta' - z'|^{\frac{1}{8}},\ |d\rho \wedge \partial\rho \wedge \partial\sigma| > \frac{1}{2}|\zeta - z|^\tau\}$$

$$V_3 = \{\zeta \in V : |\partial''\rho(\zeta)| > \frac{1}{2}|\zeta' - z'|^{\frac{1}{8}},\ |\zeta' - z'|^{\frac{1}{4}} \geq |\zeta'' - z''|\}$$

$$V_4 = \{\zeta \in V : |\partial''\rho(\zeta)| > \frac{1}{2}|\zeta' - z'|^{\frac{1}{8}},\ |\zeta' - z'|^{\frac{1}{4}} \leq |\zeta'' - z''|\}.$$

Denote by J_l, $l = 1, 4$ respectively integrals over V_l.

Apply the general area formula to the map Θ^1 and the integral J_1, we get that

$$J_1 \lesssim \int\limits_{\substack{|y_1|\le C_1\\ |\tilde y'|\le b\\ |y''|\le C_2}} \frac{dy_1\cdots dy_{2n-1}}{|\tilde y|^{(2n-2p-\tau)}(|y_1|+|\tilde y'|^2)^p} \lesssim \begin{cases} b^\tau & \text{if } p>1,\\ b^\tau|\log b| & \text{if } p=1,\end{cases}$$

where $\tilde y' = (y_2,\dots,y_{2m-1})$, $y''=(y_{2m},\dots,y_{2n-1})$, and $\tilde y = (\tilde y', y'')$.

As for the integration over V_2, denote by $\mathcal{D}$ the denominator of (3.2), then (3.15') with $t=\frac18$ gives

$$\mathcal{D}^{-1} \lesssim (|r|+|\zeta-z|^2)^{-(n-p-q)}|\zeta-z|^{-(2q+\frac12)}(|\sigma|+|\zeta'-z'|^2)^{-(p-\frac{1}{64})}.$$

We introduce the following coordinate transformation:

In each R_J, $J=(i,j,k)$ for each fixed $z\in R_d$, we define the map Θ^J : $R_J \to \mathbf{R}^{2n}$ by letting $y_0 = P_\rho(\zeta,z)$, $y_1 = P_s(\zeta,z)$, $y_l = x_{l-1}$, $2\le l\le i$, $y_l = x_l$, $i+1\le l\le j-1$, $y_l = x_{l+1}$, $j\le l\le k-2$, $y_l = x_{l+2}$, $k-1\le l\le 2n-2$, $y_{2n-1} = P_r(\zeta,z)$. In view of (4.4), (4.8), (4.10), we have

$$J\Theta^J = |d\rho\wedge dr\wedge ds\wedge dx^{\hat J}| + o(|\zeta-z|)|\partial\rho\wedge\partial\sigma| + o(|\zeta'-z'|)|\partial\rho|^2 + R \tag{4.18}$$

where R contains functions in $O(|\zeta-z|^k|\zeta'-z'|^l)$ with $k+l\ge 2$.

In particular, if $|d\rho\wedge\partial\rho\wedge\partial\sigma| > |\zeta-z|^\tau$, we then have $|a_J| \ge \frac{1}{2n^3}|\zeta-z|^\tau$ in R_J and (4.18) implies that $J\Theta^J \ge c|\zeta-z|^\tau$ for some constant c independent of ζ, z; and $c \le |\bar\partial\rho\wedge\partial\rho\wedge\partial\sigma|/J\Theta^J \le C$, thus the general area formula applied to Θ^J and J_2 gives

$$J_2 \lesssim \int\limits_{\substack{|y_j|\le C_1, j=1,2n-1\\ |\tilde y'|\le b\\ |y''|\le C_2}} \frac{dy_1\cdots dy_{2n-1}}{(|y_{2n-1}|+|\tilde y|^2)^{n-q-p}|\tilde y|^{2q+\frac12}(|y_1|+|\tilde y'|^2)^{p-\frac{1}{64}}}$$

$$\lesssim \begin{cases} b^{\frac{17}{32}} & \text{if } p>1,\\ b^{\frac12} & \text{if } p=1,\end{cases}$$

where $\tilde y' = (y_2,\dots,y_{2m-2})$, $y''=(y_{2m-1},\dots,y_{2n-2})$, and $\tilde y = (\tilde y', y'')$.

Estimations of J_3 and J_4 follow mostly arguments for the corresponding cases of (3.1) except we have to take care the case where $p=1$ which is obvious, we skip them.

The case where one of indices in J is greater than $2m$ is simpler, it takes the following decomposition:

$$V_1 = \{\zeta \in V : |d\rho \wedge \partial\rho \wedge \partial\sigma| \le |\zeta - z|^\tau\}$$

$$V_2 = \left\{\zeta \in V : |d\rho \wedge \partial\rho \wedge \partial\sigma| > \frac{1}{2}|\zeta - z|^\tau\right\},$$

and coordinate transformations Θ^1 for V_1, Θ^J for V_2.

The estimation of (3.3) is even simpler where we use the coordinate transformation Θ^1 to get the desired estimate.

Proof of Lemma 2:

(A) Proof of (3.4):

Let κ, τ be positive numbers to be chosen such that $2 > \kappa > 1$, and $\kappa\tau < 1$, let

$$V_1 = \{\zeta \in V : |\partial''\rho(\zeta)| \le |\zeta' - z'|^\tau,\ |\partial\rho \wedge \partial\sigma| \le |\zeta - z|^{\frac{\kappa\tau}{2}}\}$$

$$V_2 = \left\{\zeta \in V : |\partial''\rho(\zeta)| \le |\zeta' - z'|^\tau,\ |\partial\rho \wedge \partial\sigma| > \frac{1}{2}|\zeta - z|^{\frac{\kappa\tau}{2}}\right\}$$

$$V_3 = \left\{\zeta \in V : |\partial''\rho(\zeta)| > \frac{1}{2}|\zeta' - z'|^\tau,\ |\zeta' - z'|^{\kappa\tau} \ge |\zeta'' - z''|\right\}$$

$$V_4 = \left\{\zeta \in V : |\partial''\rho(\zeta)| > \frac{1}{2}|\zeta' - z'|^\tau,\ |\zeta' - z'|^{\kappa\tau} \le |\zeta'' - z''|\right\}.$$

Denote by J_l, $l = 1, \dots, 4$ respectively integrals over V_l.

Denote by $\mathcal{D}$ the denominator of (3.4), then for points ζ satisfying $|\partial''\rho(\zeta)| \le |\zeta' - z'|^\tau$, (3.15') with $t = \tau$ gives

$$\mathcal{D}^{-1} \lesssim |\zeta - z|^{(-2n+2m-\frac{1}{2})}(|\sigma| + |\zeta' - z'|^2)^{-1}|\zeta' - z'|^{(-2m+1+\frac{\tau}{4})}.$$

As in the proof for Lemma 1, we apply the general area formula to the map Θ^1 and the integral J_1 and get that

$$J_1 \lesssim \int\limits_{\substack{|y_1| \le C_1 \\ |\tilde{y}'| \le b \\ |y''| \le C_2}} \frac{dy_1 \cdots dy_{2n-1}}{|\tilde{y}|^{2(n-m+\frac{1}{4}-\frac{\kappa\tau}{4})}(|y_1| + |\tilde{y}'|^2)|\tilde{y}'|^{(2m-1-\frac{5\tau}{4})}}$$

$$\lesssim b^{\frac{(2\kappa+5)\tau}{4}-\frac{3}{2}}|\log b| \text{ if we choose } \frac{1}{\kappa} > \tau > \frac{6}{2\kappa+5}, \left(\text{so } \kappa < \frac{5}{4}\right)$$

where $\tilde{y}' = (y_2, \dots, y_{2m-1})$, $y'' = (y_{2m}, \dots, y_{2n-1})$, and $\tilde{y} = (\tilde{y}', y'')$.

As for the integration over V_2, we write $d\rho \wedge \partial\sigma \wedge \bar\partial\sigma$ as $\sum_J a_J\, dx_J$, where $J = (i, j, k)$, $1 \le i, j, k \le 2n$ are stricly increasing triple indices and $dx_J = dx_i \wedge dx_j \wedge dx_k$. Then, V_2 can be decomposed into finite union of Borel sets V_J, where in V_J $|a_J| \ge |a_{J'}|$ for all J'. Moreover, the hypothesis $|\partial\rho \wedge \partial\sigma| > \frac{1}{2}|\zeta - z|^{\frac{\kappa\tau}{2}}$ and $|\partial''\rho| < |\zeta' - z'|^\tau$, $\frac{\kappa\tau}{2} < \tau$ imply that $J = (i, j, k)$ with $i, j, k \le 2m$. In V_J we define the following coordinate transformation:

We assume without loss of generality, that $J = (1, 2, 2m)$ and define the map $\Theta_J : V_J \to \mathbf{R}^{2n}$ by $y_0 = P_\rho(\zeta, z)$, $y_1 = P_s(\zeta, z)$, $y_2 = P_\sigma(\zeta, z)$, $y_k = x_k$, $3 \le k \le 2m-1$, $y_k = x_{k+1}$, $2m \le k \le 2n-1$. We have $J\Theta_J = |d\rho \wedge d\sigma \wedge ds \wedge dx_3 \wedge \cdots \wedge \widehat{dx_{2m}} \wedge \cdots \wedge dx_{2n}| + o(|\zeta' - z'|)|\partial\rho \wedge \partial\sigma| + o(|\zeta - z|)|\partial\sigma|^2 + O(|\zeta - z|^2)$. Since $|d\rho \wedge \partial\sigma \wedge \bar\partial\sigma| \ge |\partial\rho \wedge \partial\sigma|^2 > \frac{1}{4}|\zeta - z|^{\kappa\tau}$, we then have $|a_J| \ge \frac{1}{4(2n)^3}|\partial\rho \wedge \partial\sigma|^2 > \frac{1}{4(2n)^3}|\zeta - z|^{\kappa\tau}$ in V_J. As $\kappa\tau < 1$ the above estimate of $J\Theta_J$ implies that $J\Theta_J > c|\zeta - z|^{\kappa\tau}$ for some constant c independent of ζ, z. Moreover, we have $|\partial\rho \wedge \partial\sigma|/J\Theta_J < C|\zeta - z|^{-\frac{1}{2}\kappa\tau}$, where $C = 2(2n)^{\frac{3}{2}}c^{\frac{1}{2}}$.

We now apply the general area formula to the map Θ_J and J_2, the hypothesis $|\partial''\rho| \le |\zeta' - z'|^\tau$ and the discussion above give

$$
\begin{aligned}
J_2 &\lesssim \int\limits_{\substack{|y_j| \le C_1, j=1,2,\\ |\tilde y'| \le b \\ |y''| \le C_2}} \frac{dy_1 \cdots dy_{2n-1}}{(|\tilde y|^2)^{n-q-m}|\tilde y|^{2q+\frac{1}{2}\kappa\tau}(|y_1| + |y_2| + |\tilde y'|^2)^2|\tilde y'|^{2m-3-\tau}} \\
&\lesssim b^{\tau - \frac{\kappa\tau}{2}}|\log b|
\end{aligned}
$$

where $\tilde y' = (y_3, \ldots, y_{2m-1})$, $y'' = (y_{2m}, \ldots, y_{2n-1})$, and $\tilde y = (\tilde y', y'')$.

In $\{|\partial''\rho| > \frac{1}{2}|\zeta' - z'|^\tau\}$, as in the proof of Lemma 1, we use general coarea formula applied to the projection map π, then use coordinate transformations $h''_{\zeta'}$ for the fibre $V_3 \cap \Gamma_{\zeta'}$ and $h'_{z'}$ for $\pi(V)$. Recall that $J^*\pi = |d''\rho|$, and $Jh'_{z'}$ is bounded from below. The condition $|\zeta' - z'|^{\kappa\tau} \ge |\zeta'' - z''|$ for points in V_3 implies that $Jh''_{\zeta'} \approx |\partial''\rho|$.

Then in V_3, as $|\zeta' - z'|^{\kappa\tau} \ge |\zeta'' - z''|$, we have

$$
\begin{aligned}
\mathcal{D}^{-1} \lesssim (|r| + |\zeta - z|^2)^{-(n-q-m)}|\zeta - z|^{-(2q+1)} \\
(|\sigma| + |s| + |\zeta' - z'|^2)^{-2}|\zeta' - z'|^{-(2m-3-\kappa\tau)},
\end{aligned}
$$

and

$$J_3 \lesssim \int\limits_{\substack{|y_j| \le C_1, j=1,2\\ |\tilde{y}'| \le b}} \left\{ \int\limits_{\substack{|y_{2n-1}| \le C_1\\ |y''| \le C_2}} \frac{dy_{2m+1} \cdots dy_{2n-1}}{(|y_{2n-1}| + |\tilde{y}|^2)^{n-q-m} |\tilde{y}|^{2q+1}} \right\}$$

$$\times \frac{dy_1 \cdots dy_{2m}}{(|y_1| + |y_2| + |\tilde{y}'|^2)^2 |\tilde{y}'|^{2m-3-(\kappa-1)\tau}}$$

$$\lesssim b^{(\kappa-1)\tau} |\log b|$$

where $\tilde{y}' = (y_3, \ldots, y_{2m})$, $y'' = (y_{2m+1}, \ldots, y_{2n-2})$, and $\tilde{y} = (\tilde{y}', y'')$.

While in V_4, as $|\zeta' - z'|^{\kappa\tau} \le |\zeta'' - z''|$, let $\vartheta > 1$ be a constant such that $\vartheta\kappa\tau < 1$, we have

$$\mathcal{D}^{-1} \lesssim (|r| + |\zeta - z|^2)^{-(n-q-m)} |\zeta - z|^{-(2q-\vartheta)}$$
$$(|\sigma| + |\zeta' - z'|^2)^{-2} |\zeta' - z'|^{-(2m-3+\vartheta\kappa\tau)}.$$

Using coordinate transformation $h'_{z'}$ for $\pi(V_4)$ and arguing as in estimating J_4 of (3.1), we have

$$J_4 \lesssim \int\limits_{\substack{|y_j| \le C_1, j=1,2\\ |\tilde{y}'| \le b}} \left\{ \int\limits_{\Gamma_{\zeta'}} \frac{d\mathbf{v}_{\Gamma_{\zeta'}}}{|\tilde{y}|^{2n-2m-\vartheta}} \right\} \frac{dy_1 \cdots dy_{2m}}{(|y_1| + |y_2| + |\tilde{y}'|^2)^2 |\tilde{y}'|^{2m-3+\vartheta\kappa\tau}}$$

$$\lesssim b^{1-\vartheta\kappa\tau} |\log b|$$

where $\tilde{y}' = (y_3, \ldots, y_{2m})$, and $\tilde{y} = (\tilde{y}', x_{2m+1}, \ldots, x_{2n})$.

(B) Proof of (3.5):

Let τ, κ be positive numbers to be chosen such that $\kappa > 1$, and $\kappa\tau < 1$, let

$$V_1 = \{\zeta \in V : |\partial'' \rho(\zeta)| \le |\zeta' - z'|^\tau\}$$

$$V_2 = \left\{\zeta \in V : |\partial'' \rho(\zeta)| > \frac{1}{2} |\zeta' - z'|^\tau,\ |\zeta' - z'|^{\kappa\tau} \ge |\zeta'' - z''|\right\}$$

$$V_3 = \left\{\zeta \in V : |\partial'' \rho(\zeta)| > \frac{1}{2} |\zeta' - z'|^\tau,\ |\zeta' - z'|^{\kappa\tau} \le |\zeta'' - z''|\right\}.$$

Denote by J_l, $l = 1, 2, 3$ respectively integrals over V_l.

Denote by $\mathcal{D}$ the denominator of (3.5), then for points ζ satisfying $|\partial''\rho(\zeta)| \le |\zeta' - z'|^\tau$

$$|\partial''\rho\wedge\partial\rho\wedge\partial\sigma|\mathcal{D}^{-1} \lesssim |\zeta-z|^{(-2n+2m+\frac{\tau}{2})}(|\sigma|+|\zeta'-z'|^2)^{-1}|\zeta'-z'|^{(-2m+2+\frac{\tau}{2})}.$$

As in the proof for (3.4), we apply the general area formula to the map Θ^1 and the integral J_1 and get that

$$J_1 \lesssim \int\limits_{\substack{|y_1|\le C_1\\ |\tilde{y}'|\le b\\ |y''|\le C_2}} \frac{dy_1 \cdots dy_{2n-1}}{|\tilde{y}|^{2(n-m-\frac{\tau}{4})}(|y_1| + |\tilde{y}'|^2)|\tilde{y}'|^{2m-2-\frac{\tau}{2}}} \lesssim b^{\frac{\tau}{2}}|\log b|$$

where $\tilde{y}' = (y_2, \dots, y_{2m-1})$, $y'' = (y_{2m}, \dots, y_{2n-1})$, and $\tilde{y} = (\tilde{y}', y'')$.

In $\{|\partial''\rho| > \frac{1}{2}|\zeta' - z'|^\tau\}$, as in the proof of (3.4), we use general coarea formula applied to π and the coordinate transformations $h''_{\zeta'}$ for the fibre $V_2 \cap \Gamma_{\zeta'}$. As for $\pi(V)$, it is easy to see that we could decompose $\pi(V)$ into finite union of open subsets, on each of them we define a map $\tilde{h}'$ to $\mathbf{C}^m$ so that $\tilde{h}'_1 = -\sigma(\zeta')$ and $J_{\mathbf{R}}\tilde{h}' > \frac{1}{2m}|d\sigma|$. Recall $J^*\pi = |d''\rho|$. The condition $|\zeta' - z'|^{\kappa\tau} \ge |\zeta'' - z''|$ for points in V_2 implies that $Jh''_{\zeta'} \gtrsim |\partial''\rho|$. Then in V_2, the hypothesis $|\zeta' - z'|^{\kappa\tau} \ge |\zeta'' - z''|$ gives

$$\mathcal{D}^{-1} \lesssim (|r| + |\zeta - z|^2)^{-(n-q-m)}|\zeta - z|^{-(2q+1)} \\ (|\sigma(\zeta')| + |\zeta' - z'|^2)^{-1}|\zeta' - z'|^{-(2m-2-\kappa\tau)},$$

and

$$J_2 \lesssim \int\limits_{\substack{|y_j|\le C_1, j=1,2\\ |\tilde{y}'|\le b}} \left\{ \int\limits_{\substack{|y_{2n-1}|\le C_1\\ |y''|\le C_2}} \frac{dy_{2m+1} \cdots dy_{2n-1}}{(|y_{2n-1}| + |\tilde{y}|^2)^{n-q-m}|\tilde{y}|^{2q+1}} \right\} \\ \times \frac{dy_1 \cdots dy_{2m}}{(|y_1| + |\tilde{y}'|^2)|\tilde{y}'|^{2m-2-(\kappa-1)\tau}} \\ \lesssim b^{(\kappa-1)\tau}|\log b| \quad \because \kappa > 1$$

where $\tilde{y}' = (y_2, \dots, y_{2m})$, $y'' = (y_{2m+1}, \dots, y_{2n-2})$, and $\tilde{y} = (\tilde{y}', y'')$.

While in V_3, as $|\zeta'-z'|^{\kappa\tau}\le|\zeta''-z''|$, let $\vartheta>1$ be a constant such that $\vartheta\kappa\tau<1$, we have

$$\mathcal{D}^{-1}\lesssim(|r|+|\zeta-z|^2)^{-(n-q-m)}|\zeta-z|^{-(2q-\vartheta)}\\ (|\sigma|+|\zeta'-z'|^2)^{-1}|\zeta'-z'|^{-(2m-2+\vartheta\kappa\tau)}.$$

Using coordinate transformation $\tilde{h}'$ for $\pi(V_3)$ and arguing as in the proof of (3.1) for J_4, give

$$J_3\lesssim\int\limits_{\substack{|y_j|\le C_1,j=1,\\ |\tilde{y}'|\le b}}\left\{\int\limits_{\Gamma_{\zeta'}}\frac{d\mathbf{v}_{\Gamma_{\zeta'}}}{|\tilde{y}|^{2n-2m-\vartheta}}\right\}\frac{dy_1\cdots dy_{2m}}{(|y_1|+|\tilde{y}'|^2)|\tilde{y}'|^{2m-2+\vartheta\kappa\tau}}$$
$$\lesssim b^{1-\vartheta\kappa\tau}|\log b|$$

where $\tilde{y}'=(y_2,\dots,y_{2m})$, and $\tilde{y}=(\tilde{y}',x_{2m+1},\dots,x_{2n})$.

The estimation of (3.6) follows the same idea as that of (3.5), and it is simpler, we skip them.

Proof of Lemma 3:

(A) Proof of (3.7):

Let

$$V_1=\{\zeta\in V:|\partial''\rho(\zeta)|\le|\zeta'-z'|^{\frac{1}{8}}\}$$
$$V_2=\left\{\zeta\in V:|\partial''\rho(\zeta)|>\frac{1}{2}|\zeta'-z'|^{\frac{1}{8}},\ |\zeta'-z'|^{\frac{1}{4}}\ge|\zeta''-z''|\right\}$$
$$V_3=\left\{\zeta\in V:|\partial''\rho(\zeta)|>\frac{1}{2}|\zeta'-z'|^{\frac{1}{8}},\ |\zeta'-z'|^{\frac{1}{4}}\le|\zeta''-z''|\right\}.$$

Denote by J_l, $l=1,2,3$ respectively integrals over V_l.

Since in V_1, $|\partial''\rho|\le|\zeta'-z'|^{\frac{1}{8}}$ is small, we must have $|\partial'\rho|$ large, say $>\frac{1}{2}$, and V_1 could be written as finite union of submanifolds each with $x^{\hat{j}}$, $j\le 2m$ as local coordinate system, such that coordinate maps defined by $y_0=\rho$, $y_1=x_1,\dots,y_j=x_{j+1},\dots,y_{2m}=x_{2m+1},\dots,y_{2n-1}=x_{2n}$, $1\le j\le 2m$ have their generalized jacobian bounded from below by $\frac{1}{4n}$. Therefore, it is obvious we have

$$J_1\lesssim\int\limits_{\substack{a\le|y'|\le b\\ |y''|\le C_2}}\frac{dy_1\cdots dy_{2n-1}}{|y|^{2(n-m)}|y'|^{(2m-1-\frac{1}{8})}}\lesssim b^{\frac{1}{8}}|\log b|-a^{\frac{1}{8}}|\log a|,$$

where $y'=(y_1,\dots,y_{2m-1})$, $y''=(y_{2m},\dots,y_{2n-1})$, and $y=(y',y'')$.

In V_2, we use general coarea formula applied to the projection map π, and we use the coordinate transformation $h''_{\zeta'}$ for the fibre $\Gamma_{\zeta'}$. As in the proof for Lemma 2, we have $Jh''_{\zeta} \gtrsim |\zeta' - z'|^{\frac{1}{8}}$ in V_2. We get

$$J_2 \lesssim \int_{a \le |y'| \le b} \left\{ \int_{\substack{|y_{2n-1}| \le C_1 \\ |y''| \le C_2}} \frac{dy_{2m+1} \cdots dy_{2n-1}}{(|y_{2n-1}| + |\tilde y|^2)^{(n-q-m)} |\tilde y|^{2q}} \right\} \frac{dy_1 \cdots dy_{2m}}{|y'|^{(2m-1+\frac{1}{8})}}$$

$$\lesssim \begin{cases} b^{\frac{7}{8}} |\log b| - a^{\frac{7}{8}} |\log a| & \text{if } a > 0, \\ b^{\frac{7}{8}} |\log b| & \text{if } a = 0, \end{cases}$$

where $y' = (y_1, \dots, y_{2m})$, $y'' = (y_{2m+1}, \dots, y_{2n-2})$, and $\tilde y = (y', y'')$.

In V_3, again, we use general coarea formula applied to π. The condition $|\zeta' - z'|^{\frac{1}{4}} \le |\zeta'' - z''|$ gives that $|\zeta - z|^{-1} \le |\zeta' - z'|^{-\frac{1}{4}}$, thus we have

$$J_3 \lesssim \int_{a \le |x'| \le b} \left(\int_{\Gamma_{\zeta'}} \frac{d\mathbf{v}_{\Gamma_{\zeta'}}}{|x|^{2n-2m-2}} \right) \frac{dx_1 \cdots dx_{2m}}{|x'|^{(2m-1+\frac{1}{2})}} \lesssim b^{\frac{1}{2}} - a^{\frac{1}{2}}$$

where $x' = (x_1, \dots, x_{2m})$, and we have used the fact that $|x|^{-(2n-2m-2)}$ is integrable over $\Gamma_{\zeta'}$ for almost all ζ' with finite bound, since the volume of $\Gamma_{\zeta'}$ is bounded by constant independent of ζ'.

Thus (3.7) holds if we let $e = \frac{1}{8}$ and C large.

(B) Proof of (3.8):

Let

$$V_1 = \{\zeta \in V : |\partial'' \rho(\zeta)| \le |\zeta' - z'|^{\frac{1}{2}}\}$$

$$V_2 = \left\{\zeta \in V : |\zeta - z|^{\frac{1}{4}} \ge |\partial'' \rho(\zeta)| > \frac{1}{2} |\zeta' - z'|^{\frac{1}{2}} \right\}$$

$$V_3 = \left\{\zeta \in V : |\partial'' \rho(\zeta)| > \frac{1}{2} |\zeta - z|^{\frac{1}{4}} \right\}.$$

Denote by J_l, $l = 1, 2, 3$ respectively integrals over V_l.

For V_1, we argue exactly as in the proof of the corresponding case of (3.7) and get

$$J_1 \lesssim \int\limits_{\substack{|y'|\le b\\ |y''|\le C_2}} \frac{dy_1\cdots dy_{2n-1}}{|y|^{(2n-2m-1)}|y'|^{(2m-2)}} \lesssim b$$

where $y' = (y_1, \dots, y_{2m-1})$, $y'' = (y_{2m}, \dots, y_{2n-1})$, and $y = (y', y'')$.

To calculate the integral over V_2, as in previous proofs, we use general coarea formula applied to π. The condition $|\partial''\rho(\zeta)| \le |\zeta - z|^{\frac14}$ implies

$$J_2 \lesssim \int\limits_{a\le|x'|\le b} \left(\int\limits_{\Gamma_{\zeta'}} \frac{d\mathbf{v}_{\Gamma_{\zeta'}}}{|x|^{2n-2m-1\frac14}} \right) \frac{dx_1\cdots dx_{2m}}{|x'|^{(2m-1)}} \lesssim b$$

where $x' = (x_1, \dots, x_{2m})$, and we have used the fact that $|x|^{-(2n-2m-1\frac14)}$ is integrable over $\Gamma_{\zeta'}$ with uniform bound for almost all ζ'.

In V_3, we use general coarea formula applied to π, and the coordinate transformation $h''_{\zeta'}$ for the fibre $\Gamma_{\zeta'}$. The condition $|\partial''\rho(\zeta)| > \frac12|\zeta - z|^{\frac14}$ implies $c \le |\partial''\rho(\zeta)|/Jh''_{\zeta} \le C$ for some constants c, C independent of ζ' and z. Moreover, $|{h''_{\zeta'}}^{-1}(y'')|$ is bounded by a non-negative integer uniformly in ζ' and z. Thus,

$$J_3 \lesssim \int\limits_{a\le|y'|\le b} \left(\int\limits_{\substack{|y_{2n-1}|\le C_1\\ |y''|\le C_2}} \frac{dy_{2m+1}\cdots dy_{2n-1}}{(|y_{2n-1}| + |\tilde y|^2)^{(n-q-m-\frac12)}|\tilde y|^{2q}} \right) \frac{dy_1\cdots dy_{2m}}{|y'|^{(2m-1)}}$$

$$\lesssim b$$

where $y' = (y_1, \dots, y_{2m})$, $y'' = (y_{2m+1}, \dots, y_{2n-2})$, and $\tilde y = (y', y'')$.

The proof is complete.

References

[B] L. Bitar, Résolution de $\bar\partial$ sur des portions de sphères ou d'anneaux, *Michigan Math. J.* **37** (1990), 347–355.

[C-L1] C. H. Chang and H. P. Lee, Semi-global solvability for tangential Cauchy-Riemann operators at critical degrees, in preparation.

[C-L2] C. H. CHANG AND H. P. LEE, Non-solvable examples for the tangential Cauchy-Riemann operators, in preparation.

[C-L3] C. H. CHANG AND H. P. LEE, L^p estimates for $\bar{\partial}$ solutions in some weakly pseudoconvex domains in $\mathbf{C}^n$, Preprint.

[F] H. FEDERER, "*Geometric Measure Theory*," Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[Fol-St] G. B. FOLLAND AND E. M. STEIN, Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27** (1974), 429–522.

[For] J. E. FORNÆSS, Embedding strictly pseudoconvex domains in convex domains, *Amer. J. Math.* **98** (1976), 529–569.

[Fr] K. O. FRIEDRICHS, The identity of weak and strong extensions of differential operators, *Trans. Amer. Math. Soc.* **55** (1944), 132–151.

[H-P] R. HARVEY AND J. POLKING, Fundamental solutions in complex analysis, I, II, *Duke Math. J.* **46** (1979), 253–340.

[H-W] R. HARVEY AND R. O. WELLS, JR., Zero sets of non-negative strictly plurisubharmonic functions, *Math. Ann.* **201** (1973), 165–170.

[He] G. M. HENKIN, The Lewy equation and analysis on pseudoconvex manifolds, *Russian Math. Surveys* **32(3)** (1977), 59–130.

[Ma-Mi] L. MA AND J. MICHEL, Local regularity for the tangential Cauchy-Riemann complex, *J. Reine Angew. Math.* **442** (1993), 63–90.

[Mi-Sh1] J. MICHEL AND M. C. SHAW, C^∞-regularity of solutions of the tangential CR-equations on weakly pseudoconvex manifolds, Preprint.

[Mi-Sh2] J. MICHEL AND M. C. SHAW, The $\bar{\partial}$ problem on domains with piecewise smooth boundaries with applications, Preprint.

[Ra] M. R. RANGE, "*Holomorphic Functions and Integral Representations in Several Complex Variables*," Springer-Verlag, New York, 1986.

[Ra-Siu] M. R. RANGE AND Y. T. SIU, Uniform estimates for the $\bar{\partial}$-equation on domains with piecewise smooth strictly pseudoconvex boundaries, *Math. Ann.* **206** (1973), 325–354.

[Ro] J. P. ROSAY, Some application of Cauchy-Fantappié forms to (local) problems in $\bar{\partial}_b$, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **13** (1986), 225–243.

[Se-He] A. G. SERGEEV AND G. M. HENKIN, Uniform estimates for solutions of the $\bar{\partial}$-equation in pseudoconvex polyhedra, *Math. USSR Sb.* **40** (1981), 469–507.

[Sh1] M. C. Shaw, L^p estimates for local solutions of $\bar{\partial}_b$ on strongly pseudoconvex CR manifolds, *Math. Ann.* **288** (1990), 36–62.

[Sh2] M. C. Shaw, Semi-global existence theorems of $\bar{\partial}_b$ for $(0, n-2)$ forms on pseudoconvex boundaries in $\mathbf{C}^n$, Colloque D'Analyse Complexe et Géométrie (Marseille, 1992), *Astérisque* **217** (1993), 227–240.

[Sh3] M. C. Shaw, Local existence theorems with estimates for $\bar{\partial}_b$ on weakly pseudoconvex boundaries, *Math. Ann.* **294** (1992), 677–700.

[Sh4] M. C. Shaw, Homotopy formula for $\bar{\partial}_b$ in CR manifolds with mixed Levi signatures, *Math. Z.* **224(1)** (1997), 113–135.

[Si] L. Simon, "*Lectures on Geometric Measure Theory*," Proceedings of the Centre for Mathematical Analysis, Australian National University **3**, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[Sk] H. Skoda, Valeurs au bord pour les solutions de l'operateur $\bar{\partial}$ et caractérisation des zéros des fonctions de la classe Nevalinna, *Bull. Soc. Math. France* **104(3)** (1976), 225–299.

[W] S. M. Webster, On the local solution of the tangential Cauchy-Riemann equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **6** (1989), 167–182.

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