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MEAN GROWTH OF H^p FUNCTIONS

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Abstract

A classical result of Hardy and Littlewood asserts that if $0 < p < q < \infty$ and f is a function which is analytic in the unit disc and belongs to the Hardy space H^p, then, if $\lambda \geq p$ and $\alpha = \frac{1}{p} - \frac{1}{q}$, we have

$$\int_0^1 (1-r)^{\lambda\alpha - 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q \, d\theta \right)^{\lambda/q} dr < \infty.$$

We prove that this result is sharp in a very strong sense. Indeed, we prove that if p, q, λ and α are as above and φ is a positive, continuous and increasing function defined in $[0,\infty)$ with $\frac{\varphi(x)}{x^q} \to \infty$, as $x \to \infty$, then there exists a function $f \in H^p$ such that

$$\int_0^1 (1-r)^{\lambda\alpha - 1} \left(\int_I \varphi\left(|f(re^{i\theta})| \right) d\theta \right)^{\lambda/q} dr = \infty,$$

for every non-degenerate interval $I \subset [0, 2\pi]$. We also prove a result of the same kind concerning functions f such that $f' \in H^p$, $0 < p < 1$.

1. Introduction and statement of results

Let Δ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T}$ the unit circle $\{\xi \in \mathbb{C} : |\xi| = 1\}$. For $0 < r < 1$ and g analytic in Δ we set

$$M_p(r,g) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| g(re^{i\theta}) \right|^p d\theta \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,g) = \max_{|z|=r} |g(z)|.$$

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For $0 < p \leq \infty$ the Hardy space H^p consists of those functions g, analytic in Δ, for which

$$\|g\|_{H^p} = \sup_{0<r<1} M_p(r,g) < \infty.$$

Hardy and Littlewood proved in [**6**] (see also [**2**, Th. 5.9]) the following.

Theorem A. *If $0 < p < q \leq \infty$ and $f \in H^p$, then*

$$M_q(r,f) = \mathrm{o}\left(\frac{1}{(1-r)^{\frac{1}{p}-\frac{1}{q}}}\right), \quad \text{as } r \to 1. \tag{1.1}$$

Considering the function $f(z) = \frac{1}{(1-z)^{\frac{1}{p}-\varepsilon}}$ for small $\varepsilon > 0$, we easily see that the exponent $\frac{1}{p} - \frac{1}{q}$ is best possible. Duren and Taylor proved in [**3**] (see also [**8**]) that the Hardy-Littlewood estimate (1.1) is sharp in a stronger sense. Namely, they proved the following result.

Theorem B. *Let $0 < p < q \leq \infty$, and let $\phi(r)$ be a positive and non-increasing function on $0 \leq r < 1$, with $\phi(r) \to 0$, as $r \to 1$. Then there exists a function $f \in H^p$ such that*

$$M_q(r,f) \neq \mathrm{O}\left(\frac{\phi(r)}{(1-r)^{\frac{1}{p}-\frac{1}{q}}}\right), \quad \text{as } r \to 1.$$

Although, as we have said, Theorem A is best possible in a strong sense, Hardy and Littlewood were able to sharpen it in one direction proving the following useful result (see [**2**, Th. 5.11]).

Theorem C. *If $0 < p < q \leq \infty$, $f \in H^p$, $\lambda \geq p$, and $\alpha = \frac{1}{p} - \frac{1}{q}$, then*

$$\int_0^1 (1-r)^{\lambda\alpha - 1} M_q(r,f)^\lambda \, dr < \infty. \tag{1.2}$$

The fact that (1.2) implies (1.1) is clear having in mind that $M_q(r,f)$ is an increasing function of r. Let us remark that Flett gave in [**4**] a proof of Theorem C based on the Marcinkiewicz interpolation theorem. Also, it is worth noticing that if we take $q < \infty$ and $\lambda = q$ then we obtain the following:

If $0 < p < q < \infty$ and $f \in H^p$,

$$\text{then } \int_0^{2\pi} \int_0^1 (1-r)^{\frac{q}{p}-2} |f(re^{i\theta})|^q \, dr \, d\theta < \infty.$$

Our first result in this paper shows that Theorem C is sharp in a very strong sense.

Theorem 1. *Let* $0 < p < q < \infty$, $\lambda \geq p$, *and* $\alpha = \frac{1}{p} - \frac{1}{q}$. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be a continuous and increasing function with*

$$\frac{\varphi(x)}{x^q} \to \infty, \quad \text{as } x \to \infty. \tag{1.3}$$

Then, there exists a function $f \in H^p$ *such that*

$$\int_0^1 (1-r)^{\lambda\alpha - 1} \left(\int_I \varphi\left(\left|f(re^{i\theta})\right|\right) d\theta \right)^{\lambda/q} dr = \infty, \tag{1.4}$$

for every non-degenerate interval $I \subset [0, 2\pi]$.

In particular, if $0 < p < q < \infty$ *and* $\varphi : [0,\infty) \to [0,\infty)$ *is as above, then there exists a function* $f \in H^p$ *such that*

$$\int_I \int_0^1 (1-r)^{\frac{q}{p}-2} \varphi\left(\left|f(re^{i\theta})\right|\right) dr\, d\theta = \infty,$$

for every non-degenerate interval $I \subset [0, 2\pi]$.

According to a classical result of Privalov [**2**, Th. 3.11], a function f analytic in Δ has a continuous extension to the closed unit disc $\overline{\Delta}$ whose boundary values are absolutely continuous on $\partial\Delta$ if and only if $f' \in H^1$. In particular,

$$f' \in H^1 \Rightarrow f \in H^\infty. \tag{1.5}$$

This result has been shown to be sharp. Indeed, Yamashita proved in [**9**] that there exists a function f analytic in Δ with $f' \in H^p$ for all $p \in (0,1)$ but such that f is not even a normal function, and the first author has recently proved in [**5**] that no restriction on the growth of $M_1(r, f')$ other than its boundedness is enough to conclude that f is a normal function. We refer to [**1**] and [**7**] for the theory of normal functions. On the other hand, Hardy and Littlewood obtained the following generalization of (1.5) (see [**2**, Th. 5.12]).

Theorem D. *Let* f *be a function which is analytic in* Δ. *If* $0 < p < 1$ *and* $f' \in H^p$ *then* $f \in H^q$, *where* $q = p/(1-p)$.

Taking $f'(z) = (1-z)^{\varepsilon - \frac{1}{p}}$ for small $\varepsilon > 0$ shows that for each value of $p \in (0,1)$ the index q is best possible. Our next result proves the sharpness of Theorem D in a much stronger sense.

Theorem 2. *Let* $0 < p < 1$ *and* $q = p/(1-p)$. *Let* $\varphi : [0, \infty) \to [0, \infty)$ *be a continuous and increasing function satisfying (1.3). Then, there exists a function* f *analytic in* Δ *with* $f' \in H^p$ *such that*

$$\int_I \varphi\left(\left|f(e^{i\theta})\right|\right) d\theta = \infty, \tag{1.6}$$

for every non-degenerate interval $I \subset [0, 2\pi]$.

Let us remark that if p and q are as in Theorem 2 and $f' \in H^p$, then, by Theorem D, $f \in H^q$ and, hence, f has a finite non-tangential limit $f(e^{i\theta})$ for almost every θ. Hence, the left hand side of (1.6) makes sense.

2. Proof of the results

The proofs of our results will be constructive. Let α and β be two positive real numbers, and let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of real numbers with

$$0 < \delta_k < 2^{-k}, \quad \text{for all } k. \tag{2.1}$$

For $k = 1, 2, \dots,$ and $j = 1, 2, \dots, 2^k$, define

$$\theta_j^k = \frac{2\pi(2j-1)}{2^{k+1}}, \tag{2.2}$$

$$I_j^k = (\theta_j^k - \delta_k, \theta_j^k + \delta_k). \tag{2.3}$$

Notice that, for each k, the intervals I_j^k, $j = 1, 2, \dots, 2^k$, are pairwise disjoint. Set

$$r_k = 1 - \delta_k, \quad k = 1, 2, \dots . \tag{2.4}$$

For $k = 1, 2, \dots,$ define

$$f_k(z) = \sum_{j=1}^{2^k} \frac{\delta_k^{\alpha}}{(1 - r_k e^{-i\theta_j^k} z)^{\beta}}, \quad z \in \Delta. \tag{2.5}$$

Let us remark that the functions f_k are in fact analytic in the closed unit disc $\overline{\Delta}$. Actually, the functions f_k depend on α, β and the sequence $\{\delta_k\}$, however, we shall not indicate this dependence explicitely. We believe that this will not cause any confusion.

We shall make use of some lemmas to deal with the functions f_k. The proofs are elementary and some of them will be omitted. First of all, let us recall that

$$|1 - re^{i\theta}| \leq 2|\theta|, \quad 0 < r \leq 1,\ 1 - r \leq |\theta| \leq \pi, \tag{2.6}$$

$$|1 - re^{i\theta}| \geq \frac{|\theta|}{\pi}, \quad 0 < r \leq 1,\ |\theta| \leq \pi, \tag{2.7}$$

$$|1 - e^{i\theta}| \geq 2\frac{|\theta|}{\pi}, \quad |\theta| \leq \pi. \tag{2.8}$$

Lemma 1. *If $l \neq m$, then*

$$|\theta_l^k - \theta_m^k| \geq \frac{\pi}{2^{k-1}}.$$

Lemma 2. *If $\theta \in I_j^k$, then*

$$|e^{i\theta_j^k} - r_k e^{i\theta}| \leq 2\delta_k \tag{2.9}$$

and

$$|e^{i\theta_l^k} - r_k e^{i\theta}| \geq \frac{1}{2^{k-1}}, \quad \textit{for all } l \neq j. \tag{2.10}$$

Proof: Let $\theta \in I_j^k$, then $|\theta - \theta_j^k| < \delta_k$, which, with (2.6), implies

$$|e^{i\theta_j^k} - r_k e^{i\theta}| = |1 - r_k e^{i(\theta - \theta_j^k)}| \leq |1 - r_k e^{i\delta_k}| \leq 2\delta_k.$$

This is (2.9). Now, let $l \neq j$ and let φ_l^k be an angle such that $e^{i\varphi_l^k} = e^{i\theta_l^k}$ and $|\theta - \varphi_l^k| \leq \pi$. Then, using (2.4), (2.8), Lemma 1 and (2.1), we obtain

$$\begin{aligned}
|e^{i\theta_l^k} - r_k e^{i\theta}| &= |e^{i\varphi_l^k} - r_k e^{i\theta}| \geq |e^{i\varphi_l^k} - e^{i\theta}| - |e^{i\theta} - r_k e^{i\theta}| \\
&= |e^{i\varphi_l^k} - e^{i\theta}| - \delta_k \geq 2\frac{|\varphi_l^k - \theta|}{\pi} - \delta_k \\
&\geq \frac{2}{\pi}\big(|\varphi_l^k - \theta_j^k| - |\theta_j^k - \theta|\big) - \delta_k \geq \frac{2}{\pi}\left(\frac{\pi}{2^{k-1}} - \delta_k\right) - \delta_k \\
&\geq \frac{1}{2^{k-2}} - 2\delta_k \geq \frac{1}{2^{k-1}}.
\end{aligned}$$

Hence, (2.10) holds. ∎

Lemma 3. *If $n < k$, then*

$$|\theta_l^n - \theta_j^k| \geq \frac{\pi}{2^k}, \quad \textit{for all } l, j.$$

Lemma 4. *If $\theta \in I_j^k$, $n < k$ and $0 < r \leq 1$, then*

$$|e^{i\theta_l^n} - r_n r e^{i\theta}| \geq \frac{1}{2^{k+1}}, \quad \textit{for all } l \in \{1, 2, \dots, 2^n\}.$$

Proof: Let $\theta \in I_j^k$ and let φ_l^n be defined as in the proof of Lemma 2. Then, using (2.7), Lemma 3, (2.3) and (2.1), we see that

$$\begin{aligned} |e^{i\theta_l^n} - r_n r e^{i\theta}| = |e^{i\varphi_l^n} - r_n r e^{i\theta}| &\geq \frac{|\varphi_l^n - \theta|}{\pi} \\ &\geq \frac{1}{\pi}\big(|\varphi_l^n - \theta_j^k| - |\theta_j^k - \theta|\big) \\ &\geq \frac{1}{\pi}\left(\frac{\pi}{2^k} - \delta_k\right) > \frac{1}{2^{k+1}}. \ \blacksquare \end{aligned}$$

We shall see that a suitable choice of the numbers α, β and the sequence $\{\delta_k\}$ will allow us to construct functions f analytic in Δ having the properties asserted in Theorems 1 and 2. Precisely, we can prove the following results.

Theorem 3. *Let $0 < p < q < \infty$, $\lambda \geq p$, and $\alpha = \frac{1}{p} - \frac{1}{q}$. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function satisfying (1.3). Then, there exist two positive numbers α and β, a sequence of real numbers $\{\delta_k\}_{k=1}^{\infty}$ which satisfies (2.1), and a sequence of positive numbers $\{c_k\}_{k=1}^{\infty}$, such that, if f is the function defined by*

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z), \quad z \in \Delta, \tag{2.11}$$

then $f \in H^p$ and (1.4) holds for every non-degenerate interval $I \subset [0, 2\pi]$.

Theorem 4. *Let $0 < p < 1$ and $q = p/(1-p)$. Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and increasing function satisfying (1.3). Then, there exist two positive numbers α and β, a sequence of real numbers $\{\delta_k\}_{k=1}^{\infty}$ which satisfies (2.1), and a sequence of positive numbers $\{c_k\}_{k=1}^{\infty}$, such that, if f is the function defined by*

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z), \quad z \in \Delta, \tag{2.12}$$

then f is analytic in Δ, $f' \in H^p$ and (1.6) holds for every non-degenerate interval $I \subset [0, 2\pi]$.

Clearly, Theorem 1 and Theorem 2 follow from Theorem 3 and Theorem 4 respectively.

Proof of Theorem 3: Let α be any positive number, and let

$$\beta = \alpha + \frac{1}{p}. \tag{2.13}$$

Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of real numbers which satisfies (2.1) to be specified later. Set

$$c_k = 2^{-k\left(\frac{1}{p}+2\right)}, \quad k = 1, 2, \dots .$$

Define the functions f_k, $k = 1, 2, \dots$, as in (2.5), let $g_k = c_k f_k$ for all k and let f be defined as in (2.11). Hence,

$$f(z) = \sum_{k=1}^{\infty} g_k(z), \quad z \in \Delta.$$

Notice that

$$|g_k(z)| = |2^{-k\left(\frac{1}{p}+2\right)} f_k(z)| \le \frac{2^{-k}\delta_k^{\alpha}}{(1-|z|)^{\beta}} \le \frac{2^{-k}}{(1-|z|)^{\beta}}$$

for all $z \in \Delta$ and, hence, the series $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on every compact subset of Δ and then it defines a function f which is analytic in Δ. Now, having in mind the elementary inequality

$$(a_1 + a_2 + \cdots + a_n)^p \le n^p(a_1^p + a_2^p + \cdots + a_n^p),$$
$$p > 0,\ a_i \ge 0 \text{ for } i = 1, 2, \dots, n,$$

and the fact that for each $\gamma > 1$ there exists a constant $c = c_\gamma > 0$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^\gamma}\,d\theta \le \frac{c}{(1-r)^{\gamma-1}}, \quad 0<r<1, \tag{2.14}$$

and using (2.4) and (2.13), we obtain that

$$\begin{aligned}
\|g_k\|_{H^p}^p = \|g_k(e^{i\theta})\|_{L^p}^p &= \frac{1}{2\pi}\int_0^{2\pi} |g_k(e^{i\theta})|^p\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi} c_k^p\,2^{kp}\,\delta_k^{\alpha p}\sum_{j=1}^{2^k} \frac{1}{|1-r_k e^{-i\theta_j^k} e^{i\theta}|^{\beta p}}\,d\theta \\
&= (2^k c_k)^p\,\delta_k^{\alpha p}\,2^k \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-r_k e^{i\theta}|^{\beta p}}\,d\theta \\
&\le 2^{-kp}\,\delta_k^{\alpha p}\frac{c}{(1-r_k)^{\beta p-1}} = 2^{-kp}c,
\end{aligned}$$

where c is the positive constant which appears in (2.14) with $\gamma = \beta p > 1$.

Thus, we have proved that

$$\|g_k\|_{H^p} \le 2^{-k}c^{1/p}, \quad k = 1, 2, \dots, \tag{2.15}$$

which, clearly, implies that $f \in H^p$.

Next we turn to estimate the value of $|f(re^{i\theta})|$ when θ belongs to one of the intervals I_j^k given in (2.3), and $0<r<1$, or at least when θ is in a suitable subset of I_j^k and r is close to 1, say $r_k < r < 1$.

Suppose that $\theta \in I_j^k$ and $0<r<1$. Then

$$\begin{aligned}
|f(re^{i\theta})| &= \left|\sum_{n=1}^\infty g_n(re^{i\theta})\right| \\
&\ge |g_k(re^{i\theta})| - \sum_{n=1}^{k-1} |g_n(re^{i\theta})| - \sum_{n=k+1}^{\infty} |g_n(re^{i\theta})|.
\end{aligned} \tag{2.16}$$

We shall estimate each of these three terms separetely.

First, for $\theta \in I_j^k$ and $0 < r < 1$,

$$
\begin{aligned}
|g_k(re^{i\theta})| &= c_k \left| \sum_{l=1}^{2^k} \frac{\delta_k^\alpha}{(1 - r_k e^{-i\theta_l^k} re^{i\theta})^\beta} \right| \\
&\geq c_k \left(\frac{\delta_k^\alpha}{|1 - r_k e^{-i\theta_j^k} re^{i\theta}|^\beta} - \sum_{\substack{l=1 \\ l\neq j}}^{2^k} \frac{\delta_k^\alpha}{|1 - r_k e^{-i\theta_l^k} re^{i\theta}|^\beta} \right).
\end{aligned} \tag{2.17}
$$

If $r > r_k$, using (2.9) and (2.4), we see that

$$
\begin{aligned}
|1 - r_k e^{-i\theta_j^k} re^{i\theta}| &= |e^{i\theta_j^k} - r_k re^{i\theta}| \leq |e^{i\theta_j^k} - r_k e^{i\theta}| + |r_k e^{i\theta} - r_k re^{i\theta}| \\
&\leq 2\delta_k + r_k(1 - r) \leq 2\delta_k + (1 - r_k) = 3\delta_k.
\end{aligned}
$$

If $l \neq j$ and $r > r_k$, (2.10), (2.4) and (2.1) give

$$
\begin{aligned}
|1 - r_k e^{-i\theta_l^k} re^{i\theta}| &= |e^{i\theta_l^k} - r_k re^{i\theta}| \geq |e^{i\theta_l^k} - r_k e^{i\theta}| - |r_k e^{i\theta} - r_k re^{i\theta}| \\
&\geq \frac{1}{2^{k-1}} - r_k(1 - r) \geq \frac{1}{2^{k-1}} - (1 - r_k) = \frac{1}{2^{k-1}} - \delta_k \geq \frac{1}{2^k}.
\end{aligned}
$$

Then, (2.17) implies that

$$
\begin{aligned}
|g_k(re^{i\theta})| &\geq c_k \left(\frac{\delta_k^\alpha}{(3\delta_k)^\beta} - 2^k \delta_k^\alpha \left(2^k\right)^\beta \right) \\
&= c_k\, \delta_k^\alpha \left(\frac{1}{(3\delta_k)^\beta} - 2^{k(1+\beta)} \right), \quad \theta \in I_j^k,\ r_k < r < 1.
\end{aligned} \tag{2.18}
$$

Let us take the numbers δ_k so small that

$$
2^{k(1+\beta)} < \frac{1}{2} \frac{1}{(3\delta_k)^\beta}, \quad k = 1, 2, \dots, \tag{2.19}
$$

then, (2.18) and (2.13) give

$$
|g_k(re^{i\theta})| \geq \frac{1}{2 \cdot 3^\beta} c_k\, \delta_k^{-1/p}, \quad \theta \in I_j^k,\ r_k < r < 1. \tag{2.20}
$$

Now we look at the second term of (2.16). Again, let $\theta \in I_j^k$ and $0 < r < 1$. For all $n < k$, we have, using Lemma 4, that

$$|g_n(re^{i\theta})| \le c_n \sum_{l=1}^{2^n} \frac{\delta_n^\alpha}{\left|1 - r_n e^{-i\theta_l^n} re^{i\theta}\right|^\beta}$$

$$= c_n \delta_n^\alpha \sum_{l=1}^{2^n} \frac{1}{\left|e^{i\theta_l^n} - r_n re^{i\theta}\right|^\beta}$$

$$\le c_n \delta_n^\alpha \, 2^n \left(2^{k+1}\right)^\beta = 2^{-n\left(\frac{1}{p}+1\right)} \delta_n^\alpha \, 2^{(k+1)\beta} \le 2^{-n} 2^{(k+1)\beta},$$

which shows that

$$\sum_{n=1}^{k-1} |g_n(re^{i\theta})| \le 2^{(k+1)\beta} \sum_{n=1}^{k-1} 2^{-n} \le 2^{(k+1)\beta} \sum_{n=1}^{\infty} 2^{-n} = 2^{(k+1)\beta}.$$

So we have found that

$$\sum_{n=1}^{k-1} |g_n(re^{i\theta})| \le 2^{(k+1)\beta}, \quad \theta \in I_j^k, \; 0 < r < 1. \tag{2.21}$$

Let us take the δ_k's such that

$$\delta_k^{\alpha/\beta} < \frac{\pi}{2^k}, \quad \text{for all } k. \tag{2.22}$$

For $n = 1, 2, \dots$, define

$$J_l^n = (\theta_l^n - \delta_n^{\alpha/\beta}, \theta_l^n + \delta_n^{\alpha/\beta}), \quad l = 1, 2, \dots, 2^n. \tag{2.23}$$

Notice that (2.22) implies that, for each n, the intervals J_l^n ($l = 1, 2, \dots, 2^n$) are pairwise disjoint. Then, using (2.7), we easily obtain the following.

Lemma 5. *Let $n > k$. If $\theta \in I_j^k \setminus \bigcup_{l=1}^{2^n} J_l^n$ and $0 < r < 1$, then*

$$|e^{i\theta_l^n} - r_n re^{i\theta}| \ge \frac{1}{\pi} \delta_n^{\alpha/\beta}, \quad \textit{for all } l \in \{1, 2, \dots, 2^n\}.$$

Now we are able to estimate the third term of (2.16). Take θ and r as in Lemma 5. We have

$$|g_n(re^{i\theta})| \le c_n \sum_{l=1}^{2^n} \frac{\delta_n^\alpha}{\left|1 - r_n e^{-i\theta_l^n} re^{i\theta}\right|^\beta}$$

$$= c_n \delta_n^\alpha \sum_{l=1}^{2^n} \frac{1}{\left|e^{i\theta_l^n} - r_n re^{i\theta}\right|^\beta}$$

$$\le c_n \delta_n^\alpha\, 2^n \left(\frac{\pi}{\delta_n^{\alpha/\beta}}\right)^\beta = \pi^\beta 2^{-n\left(\frac{1}{p}+1\right)} \le \pi^\beta 2^{-n}.$$

Thus for $\theta \in I_j^k \setminus \bigcup_{n=k+1}^{\infty} \bigcup_{l=1}^{2^n} J_l^n$ and $0 < r < 1$,

$$\sum_{n=k+1}^{\infty} |g_n(re^{i\theta})| \le \sum_{n=k+1}^{\infty} \pi^\beta 2^{-n} \le \pi^\beta \sum_{n=1}^{\infty} 2^{-n} = \pi^\beta.$$

For $k = 1, 2, \dots,$ let

$$E_j^k = I_j^k \setminus \bigcup_{n=k+1}^{\infty} \bigcup_{l=1}^{2^n} J_l^n, \quad j = 1, 2, \dots, 2^k. \tag{2.24}$$

So we have proved

$$\sum_{n=k+1}^{\infty} |g_n(re^{i\theta})| \le \pi^\beta, \quad \theta \in E_j^k,\ 0 < r < 1. \tag{2.25}$$

We conclude from (2.16), (2.20), (2.21) and (2.25), that

$$|f(re^{i\theta})| \ge \frac{1}{2\cdot 3^\beta}\, c_k\, \delta_k^{-1/p} - 2^{(k+1)\beta} - \pi^\beta, \quad \theta \in E_j^k,\ r_k < r < 1. \tag{2.26}$$

Take the δ_k's so small that

$$2^{(k+1)\beta} + \pi^\beta < \frac{1}{4\cdot 3^\beta}\, c_k\, \delta_k^{-1/p}. \tag{2.27}$$

Then (2.26) gives

$$|f(re^{i\theta})| \ge \frac{1}{4\cdot 3^\beta}\, c_k\, \delta_k^{-1/p}, \quad \theta \in E_j^k,\ r_k < r < 1. \tag{2.28}$$

From (1.3) it is clear that

$$\frac{\varphi(\lambda_0 x)}{x^q} \to \infty, \quad \text{as } x \to \infty,$$

for every constant $\lambda_0 > 0$. Taking

$$\lambda_k = \frac{1}{4 \cdot 3^\beta} c_k, \quad k = 1, 2, \dots ,$$

for each k, we have

$$\left(\frac{\varphi(\lambda_k x)}{x^q} \right)^{\lambda/q} \to \infty, \quad \text{as } x \to \infty,$$

and hence there exists $\varepsilon_k > 0$ such that

$$\varepsilon^{\lambda/p} \varphi \left(\lambda_k \varepsilon^{-1/p} \right)^{\lambda/q} > k, \quad 0 < \varepsilon \le \varepsilon_k.$$

Let us choose the numbers δ_k satisfying

$$0 < \delta_k \le \varepsilon_k, \quad k = 1, 2, \dots . \tag{2.29}$$

Then it follows that

$$\delta_k^{\lambda/p} \varphi \left(\frac{1}{4 \cdot 3^\beta} c_k \, \delta_k^{-1/p} \right)^{\lambda/q} > k, \quad k = 1, 2, \dots . \tag{2.30}$$

Furthermore, we take the numbers δ_k so small that

$$\sum_{n=k+1}^{\infty} 2^{n+1} \delta_n^{\alpha/\beta} \le \delta_k, \quad k = 1, 2, \dots , \tag{2.31}$$

which implies

$$|E_j^k| \ge \delta_k, \quad j = 1, 2, \dots , 2^k, \tag{2.32}$$

for all $k = 1, 2, \dots$, where $|E_j^k|$ denotes the Lebesgue measure of the set E_j^k.

Now, if k is any positive integer, and $j \in \{1, 2, \dots, 2^k\}$, using (2.28), the fact that φ is increasing, (2.30) and (2.32), we obtain

$$\begin{aligned}
&\int_0^1 (1-r)^{\lambda\alpha-1} \left(\int_{I_j^k} \varphi\left(\left|f(re^{i\theta})\right|\right) d\theta \right)^{\lambda/q} dr \\
&\qquad \geq \int_{r_k}^1 (1-r)^{\lambda\alpha-1} \left(\int_{E_j^k} \varphi\left(\left|f(re^{i\theta})\right|\right) d\theta \right)^{\lambda/q} dr \\
&\qquad \geq \int_{r_k}^1 (1-r)^{\lambda\alpha-1} \left(\int_{E_j^k} \varphi\left(\frac{1}{4\cdot 3^\beta}\, c_k\, \delta_k^{-1/p}\right) d\theta \right)^{\lambda/q} dr \\
&\qquad = \varphi\left(\frac{1}{4\cdot 3^\beta}\, c_k\, \delta_k^{-1/p}\right)^{\lambda/q} |E_j^k|^{\lambda/q} \int_{r_k}^1 (1-r)^{\lambda\alpha-1}\, dr \\
&\qquad \geq k\, \delta_k^{-\lambda/p} \delta_k^{\lambda/q} \frac{(1-r_k)^{\lambda\alpha}}{\lambda\alpha} = k\delta_k^{-\lambda\alpha} \frac{\delta_k^{\lambda\alpha}}{\lambda\alpha} = \frac{1}{\lambda\alpha} k.
\end{aligned}$$

Thus, we have seen that

$$\int_0^1 (1-r)^{\lambda\alpha-1} \left(\int_{I_j^k} \varphi\left(\left|f(re^{i\theta})\right|\right) d\theta \right)^{\lambda/q} dr \geq \frac{1}{\lambda\alpha} k, \tag{2.33}$$

$$j = 1, 2, \dots, 2^k,\; k = 1, 2, \dots .$$

Now, if $I \subset [0, 2\pi]$ is a non-degenerate interval, then it is clear that there exists k_0 such that for every $k \geq k_0$ there exists $j_k \in \{1, 2, \dots, 2^k\}$ with $I_{j_k}^k \subset I$. Then, using (2.33), we see that

$$\begin{aligned}
&\int_0^1 (1-r)^{\lambda\alpha-1} \left(\int_I \varphi\left(\left|f(re^{i\theta})\right|\right) d\theta \right)^{\lambda/q} dr \\
&\qquad \geq \lim_{k\to\infty} \int_0^1 (1-r)^{\lambda\alpha-1} \left(\int_{I_{j_k}^k} \varphi\left(\left|f(re^{i\theta})\right|\right) d\theta \right)^{\lambda/q} dr = \infty.
\end{aligned}$$

Hence, Theorem 3 is proved taking α, β and the sequence $\{c_k\}$ as above, and the δ_k's satisfying (2.1), (2.19), (2.22), (2.27), (2.29) and (2.31), which is clearly possible. ∎

Proof of Theorem 4: Let α be any positive number, and let

$$\beta = \alpha + \frac{1}{p} - 1. \tag{2.34}$$

Suppose that $\{\delta_k\}_{k=1}^{\infty}$ is a sequence of real numbers which satisfies (2.1). Set

$$c_k = 2^{-2k/p}, \quad k = 1, 2, \dots ,$$

define the functions f_k, $k = 1, 2, \dots$, as in (2.5), and let $g_k = c_k f_k$ for all k. Then,

$$g_k'(z) = c_k \sum_{j=1}^{2^k} \frac{\delta_k^{\alpha}\, \beta\, r_k\, e^{-i\theta_j^k}}{(1 - r_k e^{-i\theta_j^k} z)^{\beta+1}},$$

and

$$|g_k'(z)| \leq c_k\, \beta \sum_{j=1}^{2^k} \frac{\delta_k^{\alpha}}{|1 - r_k e^{-i\theta_j^k} z|^{\beta+1}}.$$

Now, using the elementary inequality

$$(a_1 + a_2 + \dots + a_n)^p \leq a_1^p + a_2^p + \dots + a_n^p, \quad a_i \geq 0 \text{ for } i = 1, 2, \dots, n,$$

which holds since $0 < p < 1$, (2.14) with $\gamma = (\beta + 1)p > 1$, (2.4) and (2.34), we have

$$\begin{aligned}
\|g_k'\|_{H^p}^p = \|g_k'(e^{i\theta})\|_{L^p}^p &= \frac{1}{2\pi} \int_0^{2\pi} |g_k'(e^{i\theta})|^p \, d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} c_k^p\, \beta^p\, \delta_k^{\alpha p} \sum_{j=1}^{2^k} \frac{1}{|1 - r_k e^{-i\theta_j^k} e^{i\theta}|^{(\beta+1)p}} \, d\theta \\
&= c_k^p\, \beta^p\, \delta_k^{\alpha p}\, 2^k \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - r_k e^{i\theta}|^{(\beta+1)p}} \, d\theta \\
&\leq 2^{-k}\, \beta^p\, \delta_k^{\alpha p} \frac{c}{(1 - r_k)^{(\beta+1)p-1}} = 2^{-k} \beta^p\, c.
\end{aligned}$$

So we have obtained that

$$\|g_k'\|_{H^p}^p \leq 2^{-k} \beta^p\, c, \quad k = 1, 2, \dots . \tag{2.35}$$

Let us define f by (2.12). It is clear that f is analytic in Δ. Since

$$f'(z) = \sum_{k=1}^{\infty} g_k'(z),$$

using (2.35), we deduce that

$$\|f'\|_{H^p}^p \le \sum_{k=1}^{\infty} \|g_k'\|_{H^p}^p \le \sum_{k=1}^{\infty} 2^{-k} \beta^p\, c = \beta^p\, c < \infty,$$

and, hence, $f' \in H^p$.

We shall argue as in the proof of Theorem 3. If $\theta \in I_j^k$, we have

$$(2.36)\qquad \begin{aligned} |f(e^{i\theta})| &= \left| \sum_{n=1}^{\infty} g_n(e^{i\theta}) \right| \\ &\ge |g_k(e^{i\theta})| - \sum_{n=1}^{k-1} |g_n(e^{i\theta})| - \sum_{n=k+1}^{\infty} |g_n(e^{i\theta})|. \end{aligned}$$

First, we apply Lemma 2 to get

$$\begin{aligned} |g_k(e^{i\theta})| &= c_k \left| \sum_{l=1}^{2^k} \frac{\delta_k^\alpha}{(1 - r_k e^{-i\theta_l^k} e^{i\theta})^\beta} \right| \\ &\ge c_k \left(\frac{\delta_k^\alpha}{|1 - r_k e^{-i\theta_j^k} e^{i\theta}|^\beta} - \sum_{\substack{l=1 \\ l\neq j}}^{2^k} \frac{\delta_k^\alpha}{|1 - r_k e^{-i\theta_l^k} e^{i\theta}|^\beta} \right) \\ &\ge c_k\, \delta_k^\alpha \left(\frac{1}{(2\delta_k)^\beta} - 2^k 2^{(k-1)\beta} \right). \end{aligned}$$

Notice that the sequence $\{\delta_k\}$ may be supposed to satisfy

$$2^k 2^{(k-1)\beta} < \frac{1}{2} \frac{1}{(2\delta_k)^\beta}, \quad k = 1, 2, \dots .$$

Then,

$$|g_k(e^{i\theta})| \ge c_k\, \delta_k^\alpha \frac{1}{2} \frac{1}{(2\delta_k)^\beta} = \frac{1}{2\cdot 2^\beta} c_k\, \delta_k^{\alpha-\beta} = \frac{1}{2^{\beta+1}} c_k\, \delta_k^{-1/q}.$$

So, we have proved that

$$|g_k(e^{i\theta})| \geq \frac{1}{2^{\beta+1}} c_k \, \delta_k^{-1/q}, \quad \theta \in I_j^k. \tag{2.37}$$

Next, take $\theta \in I_j^k$ and $n < k$. Using Lemma 4, we deduce that

$$\begin{aligned} |g_n(e^{i\theta})| &\leq c_n \sum_{l=1}^{2^n} \frac{\delta_n^\alpha}{\left|1 - r_n e^{-i\theta_l^n} e^{i\theta}\right|^\beta} \\ &= c_n \delta_n^\alpha \sum_{l=1}^{2^n} \frac{1}{\left|e^{i\theta_l^n} - r_n e^{i\theta}\right|^\beta} \\ &\leq c_n \delta_n^\alpha \, 2^n \left(2^{k+1}\right)^\beta = 2^{n\left(1-\frac{2}{p}\right)} \delta_n^\alpha \, 2^{(k+1)\beta} \leq 2^{-n} 2^{(k+1)\beta}. \end{aligned}$$

Hence,

$$\sum_{n=1}^{k-1} |g_n(e^{i\theta})| \leq 2^{(k+1)\beta}, \quad \theta \in I_j^k. \tag{2.38}$$

For every positive integer n, define J_l^n, $l = 1, 2, \dots, 2^n$, by (2.23), and suppose, as in the proof of Theorem 3, that (2.22) is satisfied. Notice that Lemma 5 holds for every $r \in (0,1)$, and so it also does for $r = 1$. Finally, define the sets E_j^k, for $k = 1, 2, \dots$, by (2.24). Then, the same argument used in the proof of Theorem 3 shows that

$$\sum_{n=k+1}^{\infty} |g_n(e^{i\theta})| \leq \pi^\beta, \quad \theta \in E_j^k. \tag{2.39}$$

It follows from (2.36), (2.37), (2.38) and (2.39), that

$$|f(e^{i\theta})| \geq \frac{1}{2^{\beta+1}} \, c_k \, \delta_k^{-1/q} - 2^{(k+1)\beta} - \pi^\beta, \quad \theta \in E_j^k,$$

and, taking the numbers δ_k sufficiently small, we have

$$|f(e^{i\theta})| \geq \frac{1}{2^{\beta+2}} \, c_k \, \delta_k^{-1/q}, \quad \theta \in E_j^k. \tag{2.40}$$

For $k = 1, 2, \dots$, let

$$\lambda_k = \frac{1}{2^{\beta+2}} \, c_k,$$

and notice that (1.3) implies

$$\frac{\varphi(\lambda_k x)}{x^q} \to \infty, \quad \text{as } x \to \infty,$$

and so there exists $\varepsilon_k > 0$ such that

$$\varepsilon\,\varphi\left(\lambda_k \varepsilon^{-1/q}\right) > k, \quad 0 < \varepsilon \le \varepsilon_k.$$

We may assume that the numbers δ_k also satisfy

$$0 < \delta_k \le \varepsilon_k, \quad k = 1, 2, \dots .$$

Therefore,

$$\delta_k \varphi\left(\frac{1}{2^{\beta+2}}\, c_k\, \delta_k^{-1/q}\right) > k, \quad k = 1, 2, \dots . \tag{2.41}$$

Also, as in the proof of Theorem 3, we can take the numbers δ_k small enough so that (2.31) holds, and then

$$|E_j^k| \ge \delta_k, \quad j = 1, 2, \dots, 2^k, \tag{2.42}$$

for all $k = 1, 2, \dots$.

From (2.40), the fact that φ is increasing, (2.41) and (2.42), we conclude that, for each set E_j^k, we have

$$\begin{aligned}\int_{E_j^k} \varphi\left(\left|f(e^{i\theta})\right|\right)\, d\theta &\ge \int_{E_j^k} \varphi\left(\frac{1}{2^{\beta+2}} c_k\, \delta_k^{-1/q}\right) d\theta \\ &= \varphi\left(\frac{1}{2^{\beta+2}} c_k\, \delta_k^{-1/q}\right) |E_j^k| \\ &\ge k\, \delta_k^{-1}\, \delta_k = k.\end{aligned}$$

An argument similar to that used at the end of the proof of Theorem 3 shows that this implies that (1.6) holds for every non-degenerate interval $I \subset [0, 2\pi]$. This finishes the proof. ■

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