

ON THE HARDY-TYPE INTEGRAL OPERATORS IN BANACH FUNCTION SPACES

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Abstract

Characterization of the mapping properties such as boundedness, compactness, measure of non-compactness and estimates of the approximation numbers of Hardy-type integral operators in Banach function spaces are given.

1. Introduction

Let X and Y be two Banach spaces of measurable functions defined on \mathbf{R}^+ . We consider the Hardy-type integral operator $K : X \rightarrow Y$ given by

$$(1.1) \quad Kf(x) = \varphi(x) \int_0^x k(x, y)\psi(y)f(y) dy, \quad x > 0,$$

where the real functions $\varphi(x)$ and $\psi(x)$ (weights) are measurable and finite almost everywhere on \mathbf{R}^+ , and the kernel $k(x, y) \geq 0$, satisfies

$$(1.2) \quad D^{-1}(k(x, z) + k(z, y)) \leq k(x, y) \\ \leq D(k(x, z) + k(z, y)), \quad x \geq z \geq y \geq 0,$$

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where the constant $D \geq 1$ does not depend on x, y, z . Typical examples of such a kernel are $(x-y)^\alpha$, $\alpha \geq 0$; $\log^\beta \left(\frac{x}{y}\right)$, $\beta \geq 0$; or $\left(\int_y^x h(s) ds\right)^\gamma$, $\gamma \geq 0$ with nonnegative $h(s)$, and their various combinations. Introduced by R. Oinarov [13], [14] the condition (1.2) extends in some sense the well-known Δ_2 -condition for convex functions [10] used in [12], [18] for convolution operators and thus, (1.2) seems to be a balance point between generality of conditions imposed on a kernel and implicitness of a criterion for the boundedness of the Hardy-type operators. A survey of the mapping properties of operators (1.1) with Oinarov's kernel in Lebesgue and Lorentz spaces can be found in [19].

The paper is devoted to operators of the form (1.1) acting in Banach spaces of Lebesgue-measurable functions on \mathbf{R}^+ (see Definition 1 below). Investigation in this area was recently initiated by E. Berezhnoi [2], [3], who, in particular, characterized weak-type estimates for the operator (1.1) with the kernel $k(x, y) \geq 0$ increasing with respect to the first variable and also strong estimates, when $k(x, y) = 1$ and the spaces X and Y satisfy an ℓ -condition (see Definition 3 below). E. Berezhnoi [3] has also obtained some necessary and/or sufficient conditions for the boundedness of operators (1.1) with restrictions on $k(x, y) \geq 0$, stronger than (1.2).

Sections 2 and 3 contain definitions and the statement of the main results and further comments, respectively. Our first result characterizes the boundedness of the operator (1.1) with kernel satisfying (1.2) in the spaces X and Y satisfying an ℓ -condition (Theorem 1). This leads to a characterization of the compactness (Theorem 2) and measure of non-compactness (Theorem 3) of the operator. Upper and lower estimates for the behaviour of the approximation numbers of operators (1.1), when $k(x, y) = 1$, are given in Theorems 6 and 7. Sections 4 and 5 provide the proofs.

2. Definitions

Definition 1 [1]. A real normed linear space $X = \{f : \|f\|_X < \infty\}$ is called a *Banach function space* (BFP) if in addition to the usual norm axioms $\|f\|_X$ satisfies the following conditions:

- (1) $\|f\|_X$ is defined for every Lebesgue-measurable function f on \mathbf{R}^+ , and $f \in X$ if, and only if, $\|f\|_X < \infty$; $\|f\|_X = 0$ if, and only if, $f = 0$ almost everywhere (a.e.);
- (2) $\|f\|_X = \||f|\|_X$ for all $f \in X$;
- (3) if $0 \leq f \leq g$ a.e., then $\|f\|_X \leq \|g\|_X$;

- (4) if $0 \leq f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$;
 (5) if $\text{mes } E < \infty$, then $\|\chi_E\|_X < \infty$;
 (6) if $\text{mes } E < \infty$, then $\int_E f(x) dx \leq C_E \|f\|_X$.

Given a BFS X , its *associate space* X' is defined by

$$X' = \left\{ g : \int_0^\infty |fg| < \infty \text{ for all } f \in X \right\},$$

and endowed with the associate norm

$$(2.1) \quad \|g\|_{X'} = \sup \left\{ \int_0^\infty |fg| : \|f\|_X \leq 1 \right\}.$$

X' is also a Banach function space satisfying axioms (1)-(6) and, moreover, X' is a norm fundamental subspace of the dual space X^* , that is the equality

$$(2.2) \quad \|f\|_X = \sup \left\{ \int_0^\infty |fg| : \|g\|_{X'} \leq 1 \right\}$$

holds for all $f \in X$ [1].

The spaces X, X' are complete normed linear spaces and $X'' = X$ [1].

The Hölder inequality

$$\left| \int_0^\infty fg \right| \leq \|f\|_X \|g\|_{X'}$$

holds for all $f \in X$ and $g \in X'$ and is sharp in both directions on the strength of (2.1) and (2.2). The relationships (2.1) and (2.2) give rise to the following

Principle of duality. $T : X \rightarrow Y$ is a bounded linear operator, that is $\|Tf\|_Y \leq C\|f\|_X$ for all $f \in X$ with a finite positive constant C , if and only if

- (i) $\|T'g\|_{X'} \leq C\|g\|_{Y'}$ for all $g \in Y'$, where the *conjugate operator* $T' : Y' \rightarrow X'$ is defined by the formulae

$$\int_0^\infty (Tf)g = \int_0^\infty f(T'g),$$

or

- (ii) $\left| \int_0^\infty (Tf)g \right| \leq C\|f\|_X \|g\|_{Y'}$ for all $f \in X$ and $g \in Y'$, with the same constant C . The least possible constant C defines the norm $\|T\|$ and, thus $\|T\|_{X \rightarrow Y} = \|T'\|_{Y' \rightarrow X'}$.

X has *absolutely continuous norm* (AC-norm), if for all $f \in X$, $\|f\chi_{E_n}\|_X \rightarrow 0$ for every sequence of sets $\{E_n\} \subset \mathbf{R}^+$ such, that $\chi_{E_n}(x) \rightarrow 0$ a.e. We assume throughout the paper that X' and Y have AC-norms.

Let ℓ be a *Banach sequence space* (BSS), what means that axioms (1)-(6) are satisfied with respect to the count measure and let $\{e_k\}$ denote the standard basis in ℓ .

Definition 2 [3]. Given a BFS X and a BSS ℓ , X is said to be ℓ -*concave*, if for any sequence of disjoint intervals $\{J_k\}$ such that $\bigcup J_k = \mathbf{R}^+$, and for all $f \in X$

$$(2.3) \quad \left\| \sum_k e_k \|\chi_{J_k} f\|_X \right\|_{\ell} \leq d_1 \|f\|_X,$$

where d_1 is a finite positive constant independent of $f \in X$ and $\{J_k\}$. Analogously, BFS Y is said to be ℓ -*convex*, if for any sequence of disjoint intervals $\{I_k\}$, such that $\bigcup I_k = \mathbf{R}^+$ and for all $g \in Y$

$$(2.4) \quad \|g\|_Y \leq d_2 \left\| \sum_k e_k \|\chi_{I_k} g\|_Y \right\|_{\ell}$$

for a finite positive constant $d_2 > 0$, independent of $g \in Y$ and $\{I_k\}$.

Definition 3 [3]. (*The Bereznoi ℓ -condition*). We say, that Banach function spaces X and Y satisfy an ℓ -*condition*, if there exists a Banach sequence space ℓ such that X is ℓ -concave and Y is ℓ -convex simultaneously.

Let ℓ' denote the associate space. We need the following

Lemma 1 [3]. *Let Y be an ℓ -convex BFS and suppose (2.4) holds. Then Y' is an ℓ' -concave BFS and*

$$(2.5) \quad \left\| \sum_k e_k \|\chi_{I_k} f\|_{Y'} \right\|_{\ell'} \leq d_2 \|f\|_{Y'},$$

for all $f \in Y'$ and $\{I_k\}$, such that $\bigcup I_k = \mathbf{R}^+$.

Throughout the paper the expressions of the form $0 \cdot \infty$, $0/0$, ∞/∞ are taken equal to zero, the inequality $A \ll B$ means $A \leq cB$, where c depends only on D , and possibly on the constants d_1 and d_2 of Definition 2; however the relationship $A \approx B$ is interpreted as $A \ll B \ll A$ or $A = cB$. χ_E denotes the characteristic function (indicator) of a set $E \subset \mathbf{R}^+$.

3. Statement of the main results

Put for all $t \geq 0$

$$(3.1) \quad A_0 = \sup_{t>0} A_0(t) = \sup_{t>0} \|\chi_{[t,\infty)}\varphi\|_Y \|\chi_{[0,t]}(\cdot)k(t, \cdot)\psi(\cdot)\|_{X'},$$

$$(3.2) \quad A_1 = \sup_{t>0} A_1(t) = \sup_{t>0} \|\chi_{[t,\infty)}(\cdot)k(\cdot, t)\varphi(\cdot)\|_Y \|\chi_{[0,t]}\psi\|_{X'}$$

and let $A = \max(A_0, A_1)$. Note, that $A_0 = A_1$, if $k(x, y) = 1$.

Theorem 1. *Let X and Y be BFS satisfying the Bereznoi ℓ -condition and let K be an integral operator of the form (1.1) with the kernel $k(x, y) \geq 0$ satisfying (1.2). Then $K : X \rightarrow Y$ is bounded, if and only if, $A < \infty$. Moreover,*

$$(3.3) \quad D^{-1}A \leq \|K\|_{X \rightarrow Y} \leq d_1 d_2 \gamma(D)A,$$

where $\gamma(D)$ depends only on D .

Remark 1. (i) The boundedness of K was characterized in [14], [19] (for Lebesgue spaces) and in [11] (for Lorentz spaces.) The case $k(x, y) = 1$ has been intensively studied for the last few decades by many authors and has led to further developments (sf. [15], [19]).

(ii) The Bereznoi ℓ -condition corresponds to the case $p \leq q$ in the $L^p - L^q$ setting and to the case $\max(r, s) \leq \min(p, q)$ in the Lorentz $L^{rs} - L^{pq}$ setting, see [6], [11]. If no ℓ -condition holds, then the lower bound in (3.3) is nevertheless valid. Moreover, there exists an operator, for which (3.3) is valid for spaces with no ℓ -condition [17].

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled and suppose the spaces X' and Y have AC-norms. Then $K : X \rightarrow Y$ is compact, if and only if $A < \infty$ and*

$$(3.4) \quad \lim_{t \rightarrow a_i} A_i(t) = \lim_{t \rightarrow b_i} A_i(t) = 0; \quad i = 0, 1,$$

where

$$(3.5) \quad a_i = \inf\{t > 0 : A_i(t) > 0\}, \quad b_i = \sup\{t > 0 : A_i(t) > 0\}; \quad i = 0, 1.$$

Remark 2. In fact, it follows from the proof of Theorem 2 below, that $a_0 = a_1, b_0 = b_1$.

The condition (3.4) has been formulated by many authors only for $a_0 = a_1 = 0, b_0 = b_1 = \infty$. However, it is easy to find a formal counterexample, for which $A < \infty$ and (3.4) is valid with $a_0 = a_1 = 0, b_0 = b_1 = \infty$, but K is non-compact. The matter is, that the condition (3.4) has to be formulated for the end-points of the “real” interval of action of operator K (see Remark 4 below for further details).

In the non-compact case we estimate the measure of non-compactness of the operator K (or, equivalently, the distance of K from the set of finite rank operators) defined by

$$\alpha(K) = \inf\{\|K - P\|; \text{rank } P < \infty\}.$$

To this end we need additional notation; put for all $0 < a < z < b < \infty$:

$$\begin{aligned} J_L^0(a) &= \sup_{0 < t < a} \|\chi_{[t,a]} \varphi\|_Y \|\chi_{[0,t]}(\cdot)k(t, \cdot)\psi(\cdot)\|_{X'}, \\ J_L^1(a) &= \sup_{0 < t < a} \|\chi_{[t,a]}(\cdot)k(\cdot, t)\varphi(\cdot)\|_Y \|\chi_{[0,t]}\psi\|_{X'}, \\ J_L(z) &= \max(J_L^0(z), J_L^1(z)), \quad J_L = \lim_{z \rightarrow a_0} J_L(z); \\ (3.6) \quad J_R^0(b) &= \sup_{b < t < \infty} \|\chi_{[t,\infty]} \varphi\|_Y \|\chi_{[b,t]}(\cdot)k(t, \cdot)\psi(\cdot)\|_{X'}, \\ J_R^1(b) &= \sup_{b < t < \infty} \|\chi_{[t,\infty]}(\cdot)k(\cdot, t)\varphi(\cdot)\|_Y \|\chi_{[b,t]}\psi\|_{X'}, \\ J_R(z) &= \max(J_R^0(z), J_R^1(z)), \quad J_R = \lim_{z \rightarrow b_0} J_R(z); \\ J &= \max(J_L, J_R). \end{aligned}$$

Theorem 3. *Let the assumptions of Theorem 2 be valid and $K : X \rightarrow Y$ be bounded. Then*

$$(3.7) \quad D^{-1}J \leq \alpha(K) \leq d_1^2 d_2^2 \gamma(D)J.$$

Utilizing the scheme from [11] we estimate from above and below the approximation numbers of the Hardy operator of the form

$$(3.8) \quad Hf(x) = \varphi(x) \int_0^x \psi(y)f(y) dy.$$

This part of the paper has been initiated by D. E. Edmunds, W. D. Evans and D. J. Harris in the work [5]. Afterwards the extension for convolution operators with the polynomial kernel was given in [7] and for the Hardy operator in Lorentz spaces in [11]. The statement of the results and proofs of this part are given in Section 5.

4. Boundedness, compactness and measure of non-compactness

We begin with an alternative proof of the criterion for the boundedness of the Hardy operator due to E. Bereznoi. Then we establish the proof for case in which the kernel satisfies Oinarov's condition. The basic idea is to apply the principle of duality to obtain the upper bound instead of using direct estimates.

Theorem 4 [3]. *Let X and Y be BFS satisfying the ℓ -condition, and let operator H be defined by (3.8). Then $H : X \rightarrow Y$ is bounded if, and only if*

$$(4.1) \quad A = \sup_{t>0} A(t) = \sup_{t>0} \|\chi_{[t,\infty)}\varphi\|_Y \|\chi_{[0,t]}\psi\|_{X'} < \infty.$$

Moreover,

$$A \leq \|H\|_{X \rightarrow Y} \leq 4d_1 d_2 A.$$

Proof: Necessity: For the lower bound we repeat the Bereznoi argument [3]. If $H : X \rightarrow Y$ is bounded, then using axioms (2) and (3) of BFS we find for arbitrary $t > 0$ and for all $f \in X$ such that $f(y)\psi(y) \geq 0$

$$\begin{aligned} \|H\|_{X \rightarrow Y} \|f\|_X &\geq \|Hf\|_Y = \left\| \varphi(x) \int_0^x f(y)\psi(y) dy \right\|_Y \\ &\geq \left\| \chi_{[t,\infty)}(x)\varphi(x) \int_0^x f(y)\psi(y) dy \right\|_Y \\ &\geq \|\chi_{[t,\infty)}(x)\varphi(x)\|_Y \int_0^t f(y)\psi(y) dy \\ &= \|\chi_{[t,\infty)}\varphi\|_Y \int_0^\infty \chi_{[0,t]}(y) f(y)\psi(y) dy. \end{aligned}$$

Consequently, applying (2.1), we have $\|H\|_{X \rightarrow Y} \geq A(t)$ for all $t > 0$ and it follows that $\|H\|_{X \rightarrow Y} \geq A$.

Sufficiency: It follows from the principle of duality that for the upper bound it is sufficient to prove the estimate

$$J \equiv \left| \int_0^\infty \varphi F g \right| \ll A \|f\|_X \|g\|_{Y'}$$

for all $f \in X$ and $g \in Y'$, where

$$F(x) = \int_0^x f(y)\psi(y) dy.$$

Suppose, that $f(y)\psi(y) \neq 0$ on a set of positive measure, then we can choose a sequence $\{x_k\} \subset \mathbf{R}^+$ such, that

$$\int_0^{x_k} |f(y)\psi(y)| dy = 2^k, \quad -\infty < k \leq N \leq \infty,$$

where $N = \sup\{k : I_k = [x_{k-1}, x_k) \neq \emptyset\}$. Then, applying Hölder's inequality, (2.3), (2.5) and (4.1), we get

$$\begin{aligned} J &\leq \int_0^\infty |\varphi F g| \leq \sum_{k \leq N} 2^{k+1} \int_{I_{k+1}} |\varphi g| \\ &= 4 \sum_{k \leq N} \int_{I_k} |f\psi| \int_{I_{k+1}} |\varphi g| \\ &\leq 4 \sum_{k \leq N} \|\chi_{I_k} f\|_X \|\chi_{I_k} \psi\|_{X'} \|\chi_{I_{k+1}} \varphi\|_Y \|\chi_{I_{k+1}} g\|_{Y'} \\ &\leq 4A \sum_{k \leq N} \|\chi_{I_k} f\|_X \|\chi_{I_{k+1}} g\|_{Y'} \\ &\leq 4A \left\| \sum_k e_k \|\chi_{I_k} f\|_X \right\|_\ell \left\| \sum_k e_k \|\chi_{I_{k+1}} g\|_{Y'} \right\|_{\ell'} \\ &\leq 4Ad_1 d_2 \|f\|_X \|g\|_{Y'}. \end{aligned}$$

Consequently, $\|H\|_{X \rightarrow Y} \leq 4d_1 d_2 A$. ■

We shall need the following modification of Theorem 4.

Theorem 5. *Let X and Y be BFS satisfying the ℓ -condition and*

$$H_\omega f(x) = \varphi(x) \int_0^{\omega(x)} \psi(y) f(y) dy,$$

where $y = \omega(x)$ is a differentiable increasing function on \mathbf{R}^+ such that $\omega(0) = 0$, $\omega(\infty) = \infty$ and, thus, the inverse function $x = \omega^{-1}(y)$ exists. Then

$$(4.2) \quad A_\omega \leq \|H_\omega\|_{X \rightarrow Y} \leq 4d_1 d_2 A_\omega,$$

where

$$A_\omega = \sup_{t>0} \|\chi_{[0,t]}\psi\|_{X'} \|\chi_{[\omega^{-1}(t),\infty)}\varphi\|_Y = \sup_{t>0} \|\chi_{[0,\omega(t)]}\psi\|_{X'} \|\chi_{[t,\infty)}\varphi\|_Y.$$

Proof of Theorem 5: Is similar to the proof of Theorem 4. We omit details. ■

Proof of Theorem 1: Necessity: Note that the Oinarov condition (1.2) implies

$$(4.3) \quad k(x, y) \geq D^{-1}k(t, y) \text{ for all } x \geq t \geq y \geq 0.$$

Consequently, applying (4.3), we obtain for all $f \in X$ such that $f(y)\psi(y) \geq 0$

$$\begin{aligned} \|K\|_{X \rightarrow Y} \|f\|_X &\geq \left\| \varphi(x) \int_0^x k(x, y) f(y) \psi(y) dy \right\|_Y \\ &\geq D^{-1} \|\chi_{[t, \infty)} \varphi\|_Y \int_0^t k(t, y) f(y) \psi(y) dy \end{aligned}$$

and arguing as in the necessity part of Theorem 4 we find, that $\|K\|_{X \rightarrow Y} \geq D^{-1}A_0$. By the principle of duality $\|K\|_{X \rightarrow Y} = \|K'\|_{Y' \rightarrow X'}$, where

$$K'g(y) = \psi(y) \int_y^\infty k(x, y) \varphi(x) g(x) dx.$$

Applying the above argument to the operator K' , we find $\|K'\|_{X \rightarrow Y} \geq D^{-1}A_1$ and, thus,

$$\|K\|_{X \rightarrow Y} \geq D^{-1}A.$$

For *sufficiency* we need the following two lemmas.

Lemma 2. *Let $k_0(x, y) \geq 0$, $x \geq y \geq 0$ be nondecreasing and continuous with respect to x . Assume that*

$$k_0(x, y) \leq D_0(k_0(x, z) + k_0(z, y)), \quad x \geq z \geq y \geq 0$$

with $D_0 \geq 1$ independent of x, z, y . Let $f(y)$ be locally integrable, $\psi(y)$ be bounded and compactly supported and $f(y)\psi(y) \geq 0$. Let $G_0(x) = \int_0^x k_0(x, y) \psi(y) f(y) dy$ be such, that $0 < G_0(x) < \infty$ for some $x > 0$.

For a fixed number $\delta > 0$ we define $\Delta_k = \{x > 0 : G_0(x) \geq (\delta + 1)^k\}$, $k \in \mathbf{Z}$, $N = \max_{\Delta_k \neq \emptyset} k$; $x_k = \inf \Delta_k$, $k \leq N$, $x_{N+1} = \infty$ if $N < \infty$. If $\delta \geq D_0$, then $0 < \dots < x_{k-1} < x_k < \dots < x_N < \infty$ and the inequality

$$(4.4) \quad (\delta + 1)^{k-1} \leq \int_{x_{k-1}}^{x_k} k_0(x_k, y) f(y) \psi(y) dy + D_0 k_0(x_k, x_{k-1}) \int_0^{x_{k-1}} f(y) \psi(y) dy$$

holds for all $k \leq N$.

Proof of Lemma 2: Using the definition of $\{x_k\}$ and exploiting the property of $k_0(x, y)$, we find

$$\begin{aligned}
(\delta + 1)^{k-1} &= (\delta + 1)^k - \delta(\delta + 1)^{k-1} \leq G_0(x_k) - \delta(\delta + 1)^{k-1} \\
&= \int_0^{x_k} k_0(x_k, y)f(y)\psi(y) dy - \delta(\delta + 1)^{k-1} \\
&= \int_0^{x_{k-1}} k_0(x_k, y)f(y)\psi(y) dy \\
&\quad + \int_{x_{k-1}}^{x_k} k_0(x_k, y)f(y)\psi(y) dy - \delta(\delta + 1)^{k-1} \\
&\leq \int_{x_{k-1}}^{x_k} k_0(x_k, y)f(y)\psi(y) dy \\
&\quad + D_0 k_0(x_k, x_{k-1}) \int_0^{x_{k-1}} f(y)\psi(y) dy \\
&\quad + D_0 \int_0^{x_{k-1}} k_0(x_{k-1}, y)f(y)\psi(y) dy - \delta(\delta + 1)^{k-1}.
\end{aligned}$$

Now,

$$\begin{aligned}
D_0 \int_0^{x_{k-1}} k_0(x_{k-1}, y)f(y)\psi(y) dy - \delta(\delta + 1)^{k-1} \\
= D_0 G_0(x_{k-1}) - \delta(\delta + 1)^{k-1} \leq 0
\end{aligned}$$

provided $\delta \geq D_0$, and lemma is proved. ■

Remark 3. It follows from Lemma 2, that

$$G_0(x) \leq (\delta + 1)^k, \quad x \in [x_{k-1}, x_k], \quad k \leq N.$$

Lemma 3. Let $k(x, y) \geq 0$ satisfy (1.2) with $D \geq 1$ and let $k(x, y)$, $\varphi(x)$ and $\psi(y)$ be bounded functions such that $\text{supp } \varphi = \text{supp } \psi \subset (0, b)$, $b < \infty$. Then there exists $k_h(x, y) \geq 0$, $0 < h < 1$, satisfying Lemma 2 with $D_h = \max(2, D^2)$ such that $k(x, y) \leq k_h(x, y)$, $0 < h < 1$ and, moreover, if $A_{0,h}$, $A_{1,h}$ be defined by (3.1), (3.2) with $k_h(x, y)$ instead of $k(x, y)$, then

$$A_i \leq \lim_{h \rightarrow +0} A_{i,h} \leq D A_i, \quad i = 0, 1.$$

Proof of Lemma 3: Put

$$(4.5) \quad \bar{k}(x, y) = \sup_{y \leq t \leq x} k(t, y).$$

Obviously, $\bar{k}(x, y)$ is nondecreasing with respect to x and nonincreasing with respect to y and $k(x, y) \leq \bar{k}(x, y)$. Moreover, (4.3) implies, that $\bar{k}(x, y) \leq Dk(x, y)$. Hence,

$$(4.6) \quad k(x, y) \leq \bar{k}(x, y) \leq Dk(x, y)$$

and

$$(4.7) \quad \frac{1}{2} (\bar{k}(x, z) + \bar{k}(z, y)) \leq \bar{k}(x, y), \quad x \geq z \geq y \geq 0.$$

From the right hand side of (1.2) and (4.6) we have

$$(4.8) \quad \begin{aligned} \bar{k}(x, y) &\leq D^2 (k(x, z) + k(z, y)) \\ &\leq D^2 (\bar{k}(x, z) + \bar{k}(z, y)), \quad x \geq z \geq y \geq 0. \end{aligned}$$

Consequently, $\bar{k}(x, y)$ satisfies Oinarov's condition with the constant $\bar{D} = \max(2, D^2)$. Define

$$(4.9) \quad k_h(x, y) = \frac{1}{h} \int_x^{x+h} \bar{k}(t, y) dt, \quad 0 < h < 1.$$

Obviously, $k_h(x, y)$ is continuous and nondecreasing with respect to x and nonincreasing in y and

$$(4.10) \quad k(x, y) \leq \bar{k}(x, y) \leq k_h(x, y).$$

Hence, applying (4.8) we find

$$(4.11) \quad \begin{aligned} k_h(x, y) &\leq \frac{\bar{D}}{h} \int_x^{x+h} (\bar{k}(t, z) + \bar{k}(z, y)) dt = \bar{D}^2 (k_h(x, z) + \bar{k}(z, y)) \\ &\leq \bar{D}^2 (k_h(x, z) + k_h(z, y)), \quad x \geq z \geq y \geq 0. \end{aligned}$$

Consequently, $k_h(x, y)$ satisfies Oinarov's condition with the constant $D_h = \bar{D}$.

Let $A_{0,h}(t)$, $A_{1,h}(t)$ be determined by (3.1) and (3.2) with k_h instead of k , respectively. Then applying (4.6) and (4.10) we find

$$(4.12) \quad \begin{aligned} A_0(t) \leq A_{0,h}(t) &= \|(\chi_{[t+h,\infty)} - \chi_{[t+h,\infty)} + \chi_{[t,\infty)})\varphi\|_Y \|\chi_{[0,t]}k_h(t, \cdot)\psi(\cdot)\|_{X'} \\ &\leq D \|\chi_{[t+h,\infty)}\varphi\|_Y \|\chi_{[0,t+h]}(\cdot)k(t+h, \cdot)\psi(\cdot)\|_{X'} \\ &\quad + \|\varphi\|_\infty \|k\|_\infty \|\psi\|_\infty \|\chi_{[0,b]}\|_{X'} \|\chi_{[t,t+h]}\chi_{[0,b]}\|_Y. \end{aligned}$$

It implies

$$(4.13) \quad A_0 \leq A_{0,h} \leq DA_0 + C \sup_{t>0} \|\chi_{[t,t+h]}\chi_{[0,b]}\|_Y$$

with a finite constant C . Since $A_{0,h}$ decreases, when $h \rightarrow +0$, and $\|\chi_{[t,t+h]}\chi_{[0,b]}\|_Y$ is continuous in t and compactly supported the result for $A_{0,h}$ follows from (4.13) by letting $h \rightarrow +0$. The argument for $A_{1,h}$ is analogous. Lemma 3 is proved. ■

Now we continue with the sufficient part of Theorem 1. By the principle of duality it is sufficient to show, that

$$J \equiv \left| \int_0^\infty \varphi G g \right| \ll A \|f\|_X \|g\|_{Y'}$$

for all compactly supported $f \in X$ and $g \in Y'$, where

$$G(x) = \int_0^x k(x, y) f(y) \psi(y) dy.$$

Assume that $\varphi(x)$ and $\psi(y)$ are bounded compactly supported functions. Because of (4.10) we have

$$J \leq \int_0^\infty |\varphi G g| \leq \int_0^\infty |\varphi G_h g| := J_h,$$

where

$$G_h(x) = \int_0^x k_h(x, y) f(y) \psi(y) dy$$

and without loss of generality we take $f(y)\psi(y) \geq 0$. Hence, we may and shall apply Lemma 2 with $D_0 = \max(2, D^2)$, $\delta = D_0$ and the sequence

of intervals $I_k = [x_{k-1}, x_k)$ with (4.4) holding. By Remark 2 we obtain

$$\begin{aligned}
J_h &\leq \sum_{k \leq N} \int_{x_k}^{x_{k+1}} |\varphi G_h g| \leq \sum_{k \leq N} (\delta + 1)^{k+1} \int_{x_k}^{x_{k+1}} |\varphi g| \\
&\leq (\delta + 1)^2 \sum_{k \leq N} (\delta + 1)^{k-1} \int_{I_{k+1}} |\varphi g| \\
&\leq (\delta + 1)^2 \left[\sum_{k \leq N} \int_{I_k} k_h(x_k, y) \psi(y) f(y) dy \int_{I_{k+1}} |\varphi g| \right. \\
&\quad \left. + D_0 \sum_{k \leq N} k_h(x_k, x_{k-1}) \int_0^{x_{k-1}} \psi(y) f(y) dy \int_{I_{k+1}} |\varphi g| \right] \\
&:= (\delta + 1)^2 [J_{1,h} + D_0 J_{2,h}].
\end{aligned}$$

Using the Hölder inequality, the ℓ -condition with Lemma 1 and (4.13), we find

$$\begin{aligned}
(4.14) \quad J_{1,h} &= \sum_{k \leq N} \int_{I_k} k_h(x_k, y) \psi(y) f(y) dy \int_{I_{k+1}} |\varphi g| \\
&\leq \sum_{k \leq N} \|\chi_{I_k} f\|_X \|\chi_{I_k}(\cdot) k_h(x_k, \cdot) \psi(\cdot)\|_{X'} \|\chi_{I_{k+1}} \varphi\|_Y \|\chi_{I_{k+1}} g\|_{Y'} \\
&\leq \sum_{k \leq N} \|\chi_{I_k} f\|_X \|\chi_{[0, x_k]}(\cdot) k_h(x_k, \cdot) \psi(\cdot)\|_{X'} \|\chi_{[x_k, \infty)} \varphi\|_Y \|\chi_{I_{k+1}} g\|_{Y'} \\
&\leq A_{0,h} \sum_{k \leq N_0} \|\chi_{I_k} f\|_X \|\chi_{I_{k+1}} g\|_{Y'} \leq d_1 d_2 A_{0,h} \|f\|_X \|g\|_{Y'}.
\end{aligned}$$

For the term $J_{2,h}$ we write

$$\begin{aligned}
(4.15) \quad J_{2,h} &= \sum_{k \leq N} k_h(x_k, x_{k-1}) \left[\int_0^{x_{k-2}} \psi(y) f(y) dy \right. \\
&\quad \left. + \int_{x_{k-2}}^{x_{k-1}} \psi(y) f(y) dy \right] \int_{I_{k+1}} |\varphi g| := J_{2,h}^{(1)} + J_{2,h}^{(2)}.
\end{aligned}$$

The estimate for $J_{2,h}^{(2)}$ is analogous to that for $J_{1,h}$. Applying the above scheme for estimating $J_{1,h}$ and using that $k_h(x, y)$ is non-increasing with

respect to y , we get,

$$\begin{aligned}
J_{2,h}^{(2)} &= \sum_{k \leq N} k_h(x_k, x_{k-1}) \int_{x_{k-2}}^{x_{k-1}} \psi(y) f(y) dy \int_{I_{k+1}} |\varphi g| \\
&\leq \sum_{k \leq N} \int_{I_{k-1}} k_h(x_k, y) \psi(y) f(y) dy \int_{I_{k+1}} |\varphi g| \\
(4.16) \quad &\leq \sum_{k \leq N} \|\chi_{I_{k-1}} f\|_X \|\chi_{I_{k-1}}(\cdot) k_h(x_k, \cdot) \psi(\cdot)\|_{X'} \|\chi_{I_{k+1}} \varphi\|_Y \|\chi_{I_{k+1}} g\|_{Y'} \\
&\leq \sum_{k \leq N} \|\chi_{I_{k-1}} f\|_X \|\chi_{[0, x_k]}(\cdot) k_h(x_k, \cdot) \psi(\cdot)\|_{X'} \|\chi_{[x_k, \infty)} \varphi\|_Y \|\chi_{I_{k+1}} g\|_{Y'} \\
&\leq d_1 d_2 A_{0,h} \|f\|_X \|g\|_{Y'}.
\end{aligned}$$

For the term $J_{2,h}^{(1)}$ we obtain

$$\begin{aligned}
J_{2,h}^{(1)} &= \sum_{k \leq N} k_h(x_k, x_{k-1}) \int_0^{x_{k-2}} \psi(y) f(y) dy \int_{I_{k+1}} |\varphi g| \\
&= \int_0^\infty \left(\int_0^{\Omega(x)} f(y) \psi(y) dy \right) \sum_{k \leq N} k_h(x_k, x_{k-1}) \chi_{I_{k+1}}(x) |\varphi(x) g(x)| dx,
\end{aligned}$$

where

$$\Omega(x) = \sum_{k \leq N} x_{k-2} \chi_{I_{k+1}}(x).$$

Let $y = \omega(x)$ be a function that satisfies the hypothesis of Theorem 5 and such that $\omega(x_k) = x_{k-2}$, $k \leq N$; $\Omega(x) \leq \omega(x) \leq x$, $x > 0$. Then

$$J_{2,h}^{(1)} \leq \int_0^\infty \left(\int_0^{\omega(x)} f(y) \psi(y) dy \right) Q(x) g(x) dx,$$

where

$$Q(x) = \left(\sum_{k \leq N} k_0(x_k, x_{k-1}) \chi_{I_{k+1}}(x) \right) \varphi(x).$$

Applying Hölder's inequality and the upper bound in (4.2) of Theorem 5 we obtain

$$\int_0^\infty \left(\int_0^{\omega(x)} f(y) \psi(y) dy \right) Q(x) g(x) dx \leq 4d_1 d_2 \mathcal{A}_\omega \|f\|_X \|g\|_{Y'},$$

where

$$(4.17) \quad \mathcal{A}_\omega = \sup_{t>0} \|\chi_{[0,t]}\psi\|_{X'} \|\chi_{[\omega^{-1}(t),\infty)}Q\|_Y.$$

For a fixed number $t \in (0, \infty)$ there exists a union of disjoint intervals such that

$$[\omega^{-1}(t), \infty) \subseteq \bigcup_{k_0 \leq k \leq N} I_{k+1}.$$

It is easy to see from the definition of the function $\omega(x)$, that $\omega^{-1}(t) \in I_{k+1}$ if and only if $t \in I_{k-1}$ and for any $x \in [\omega^{-1}(t), \infty)$ we have two choices

$$(i) \quad t < x_{k_0-1} < x_{k_0} \leq \omega^{-1}(t) < x < x_{k_0+1},$$

when

$$x \in [\omega^{-1}(t), \infty) \cap [x_{k_0}, x_{k_0+1});$$

or

$$(ii) \quad t < x_{k-1} < x_k \leq x < x_{k+1},$$

when

$$x \in I_{k+1}, \quad k > k_0.$$

Since $k_h(x, y)$ is nondecreasing with respect to x and nonincreasing with respect to y we have in both cases

$$k_h(x_k, x_{k-1})\chi_{I_{k+1}}(x) \leq k_h(x, t), \quad k \geq k_0.$$

Consequently,

$$\|\chi_{[\omega^{-1}(t), \infty)}Q\|_Y \leq \|\chi_{[t, \infty)}(\cdot)k_h(\cdot, t)\varphi(\cdot)\|_Y.$$

Using this and (4.17) we obtain that

$$\mathcal{A}_\omega \leq A_{1,h}.$$

Thus,

$$(4.18) \quad J_{2,h}^{(1)} \leq d_1 d_2 A_{1,h} \|f\|_X \|g\|_{Y'}.$$

Combining the estimates (4.12)-(4.18) and using Lebesgue's dominated convergence theorem we get the upper bound

$$(4.19) \quad J \leq d_1 d_2 \gamma(D) A \|f\|_X \|g\|_{Y'}$$

where

$$(4.20) \quad \gamma(D) = D (1 + \max(2, D^2))^2 (1 + 2 \max(2, D^2)).$$

By Fatou's theorem we obtain (4.19) for f, g, φ, ψ with no restriction. Theorem 1 is proved. ■

Remark 4. (i) There are three natural analogues of Theorem 1. The first is a restriction to an interval of real axis, the second deals with the associate operator and the third is concerned the non-Volterra case if the kernel is symmetric with respect to x and y . We omit details.

(ii) Note, that if $k(x_0, y_0) = \infty$ for some $\infty > x_0 \geq y_0 > 0$, then Oinarov's condition implies

$$k(x, y) = \infty, \quad x \geq x_0 \geq y_0 \geq y > 0.$$

Consequently, the convention $0 \cdot \infty = 0$ yields, that $A < \infty$ is possible, only if

$$\|\chi_{[x_0, \infty)} \varphi\|_Y + \|\chi_{[0, y_0]} \psi\|_{X'} = 0.$$

Thus, such an operator K is actually reduced to the interval $[y_0, x_0]$, where it coincides with

$$K_0 f(x) = \varphi(x) \int_{y_0}^x k(x, y) \psi(y) f(y) dy, \quad y_0 \leq x \leq x_0,$$

being the null-operator outside of the interval. Thus we may and shall assume the kernel to be bounded $k(x, y) \leq c_\tau < \infty$ on every domain of the form

$$\Omega_\tau = \{(x, y) : \infty > \tau \geq x \geq y \geq 0\}.$$

Proof of Theorem 2: Necessity: That $A = \max(A_0, A_1) < \infty$ follows from Theorem 1. If $f \in X$, $f(y)\psi(y) \geq 0$ then exploiting Oinarov's condition, we find

$$\begin{aligned} \infty > A \|f\|_X &\gg \|Kf\|_Y \geq \|\chi_{[t, \infty)} Kf\|_Y \\ &\geq D^{-1} \|\chi_{[t, \infty)} \varphi\|_Y \int_{a_0}^t k(t, y) \psi(y) f(y) dy. \end{aligned}$$

Now, by the principle of duality for an arbitrary fixed $\gamma \in (0, 1)$ we may find a function f_t , such that $\text{supp } f_t \subseteq [a_0, t]$, $f_t(y)\psi(y) \geq 0$, $\|f_t\|_X = 1$ and

$$(4.21) \quad \infty > A \gg \|Kf_t\|_Y \geq \gamma D^{-1} \|\chi_{[t, \infty)} \varphi\|_Y \|\chi_{[a_0, t]} k(t, \cdot) \psi(\cdot)\|_{X'}.$$

Given $G \in X'$ the Hölder inequality and absolute continuity of the norm in X' yield

$$(4.22) \quad \left| \int_0^\infty f_t G \right| \leq \| \chi_{[a_0, t]} G \|_{X'} \rightarrow 0, \quad t \rightarrow a_0.$$

Since $K' : Y' \rightarrow X'$ is also a compact operator, for any given $\varepsilon > 0$ there exists a finite number of functions $G_1, G_2, \dots, G_{n_\varepsilon}$ such that

$$(4.23) \quad \min_{1 \leq n \leq n_\varepsilon} \| K' g - G_n \|_{X'} \leq \varepsilon$$

for every $g \in Y'$, $\|g\|_{Y'} \leq 1$. Given $\varepsilon > 0$ and f_t with $\|f_t\|_X = 1$ we find by the principle of duality and (4.23) $g \in Y'$, $\|g\|_{Y'} \leq 1$ and G_n , such that

$$\| K f_t \|_Y \leq (1 - \varepsilon) \left| \int_0^\infty K f_t g \right|$$

and

$$\| K' g - G_n \|_{X'} \leq \varepsilon,$$

respectively, and using (4.21) we obtain

$$\begin{aligned} \| K f_t \|_Y &\leq (1 - \varepsilon) \left| \int_0^\infty f_t K' g \right| \\ &\leq (1 - \varepsilon) \varepsilon + (1 - \varepsilon) \left| \int_0^\infty f_t G_n \right| \leq \varepsilon, \quad t \rightarrow a_0. \end{aligned}$$

Consequently, $\| K f_t \|_Y \rightarrow 0$, $t \rightarrow a_0$, and a part of (3.4), namely $\lim_{t \rightarrow a_0} A_0(t) = 0$, now follows from (4.21). Analogously, beginning with the inequality

$$\begin{aligned} \infty > A &\gg \| K f \|_Y \geq \| \chi_{[t, \infty)} K f \|_Y \\ &\geq D^{-1} \| \chi_{[t, \infty)}(\cdot) \varphi(\cdot) k(\cdot, t) \|_Y \int_{a_1}^t \psi(y) f(y) dy, \end{aligned}$$

we prove, that $\lim_{t \rightarrow a_1} A_1(t) = 0$. The dual assertions on infinity follows from the similar observations for the associate operator. For proving *sufficiency* we need the following result.

Lemma 4. *Let X and Y be BFS on a separable σ -finite measure space and $T : X \rightarrow Y$ be an integral operator of the form $Tf(\alpha) = \int T(\alpha, \beta) f(\beta) d\beta$. If both X' and Y have AC-norms and*

$$(4.24) \quad A_T = \left\| \| T(\alpha, \cdot) \|_{X'} \right\|_Y < \infty,$$

then T is compact.

Proof of Lemma 4: It is sufficient to establish, that the set of the functions of the form

$$\sum_{i=1}^{i_0} \mu_i(\alpha) \eta_i(\beta)$$

is dense in the space $Y[X']$ with the norm defined by the right side of (4.24), where $\mu_i \in Y$ and $\eta_i \in X'$. On the strength of ([9, Chapter XI, Lemma 2]) it is true if the both spaces X' and Y are “order continuous”. This is fulfilled, when X' and Y have AC-norms, because of ([1, Chapter 1, Proposition 3.5]), and so the Lemma is proved. ■

We continue the proof of the sufficiency part of Theorem 2. Let us show first, that

$$(4.25) \quad a_0 = a_1, \quad b_0 = b_1.$$

To this end assume, for instance, that $0 \leq a_0 < a_1$. Then $A_0(t) = A_1(t) = 0$, $t \in [0, a_0]$ and it follows from the Landau resonance theorem ([1, Lemma 2.6]) and Theorem 1, restricted to the interval $[0, a_0]$, that for a.e. $x \in [0, a_0]$

$$(4.26) \quad \varphi(x)k(x, y)\psi(y) = 0 \text{ for a.e. } y \in [0, x].$$

From (3.5) we find, that

$$A_1(t) = \|\chi_{[t, \infty)}(\cdot)k(\cdot, t)\varphi(\cdot)\|_Y \|\chi_{[0, t]}\psi\|_{X'} = 0, \quad a_0 < t \leq a_1.$$

If

$$\|\chi_{[0, t]}\psi\|_{X'} = 0, \quad a_0 < t \leq a_1,$$

then $\psi(y) = 0$ for a.e. $y \in [0, t]$ by the first axiom of BFS and, hence, $A_0(t) = 0$, $t > a_0$, which contradicts the definition of a_0 . If

$$\|\chi_{[t, \infty)}(\cdot)k(\cdot, t)\varphi(\cdot)\|_Y = 0, \quad a_0 < t \leq a_1,$$

then for all $a_0 < t \leq a_1$

$$\varphi(x)k(x, t) = 0 \text{ for a.e. } x \in [t, \infty)$$

and for all $g \in Y'$ such, that $\text{supp } g \subseteq [a_0, a_1]$, $\varphi(x)g(x) \geq 0$ and arbitrary $f \in X$ such, that $f(t)\psi(t) \geq 0$, we find

$$\begin{aligned} \int_{a_0}^{a_1} Kf(x)g(x) dx &= \int_{a_0}^{a_1} \varphi(x)g(x) dx \int_{a_0}^x k(x, t)\psi(t)f(t) dt \\ &= \int_{a_0}^{a_1} \psi(t)f(t) dt \int_t^{a_1} \varphi(x)k(x, t)g(x) dx = 0 \end{aligned}$$

and, again by the Landau theorem, we get for a.e. $x \in [a_0, a_1]$

$$(4.27) \quad \varphi(x)k(x, y)\psi(y) = 0 \text{ for a.e. } y \in [a_0, x].$$

Now by (4.26) and (4.27)

$$\int_0^{a_1} Kf(x)g(x) dx = \int_{a_0}^{a_1} \varphi(x)g(x) dx \int_0^{a_0} k(x, t)\psi(t)f(t) dt$$

and by the Hölder inequality and Oinarov's condition we find

$$\begin{aligned} & \left| \int_0^{a_1} Kf(x)g(x) dx \right| \\ & \leq D \int_{a_0}^{a_1} k(x, a_0)|\varphi(x)g(x)| dx \int_0^{a_0} |\psi(t)f(t)| dt \\ & \quad + D \int_{a_0}^{a_1} |\varphi(x)g(x)| dx \int_0^{a_0} k(a_0, t)|\psi(t)f(t)| dt \\ & \leq D \left(\|\chi_{[a_0, a_1]}(\cdot)\varphi(\cdot)k(\cdot, a_0)\|_{Y'} \|\chi_{[0, a_0]}\psi\|_{X'} \right. \\ & \quad \left. + \|\chi_{[a_0, a_1]}\varphi\|_{Y'} \|\chi_{[0, a_0]}(\cdot)k(a_0, \cdot)\psi(\cdot)\|_{X'} \right) \|\chi_{[0, a_1]}g\|_{Y'} \|\chi_{[0, a_1]}f\|_X \\ & \leq D (A_0(a_0) + A_1(a_0)) \|\chi_{[0, a_1]}g\|_{Y'} \|\chi_{[0, a_1]}f\|_X = 0. \end{aligned}$$

Hence, by the principle of duality we obtain $\|K\|_{X_{[0, a_1]} \rightarrow Y_{[0, a_1]}} = 0$, and, in particular, Theorem 1, restricted to the interval $[0, a_1]$, implies $A_0(t) = 0, 0 \leq t \leq a_1$. Thus, $a_0 = a_1$, and by similar arguments it can be proved that $b_0 = b_1$. For this reason we may and shall assume further for simplicity, that $a_0 = a_1 = 0, b_0 = b_1 = \infty$.

Let $0 < a < b < \infty$ and put

$$P_a f = \chi_{[0, a]} f, \quad Q_b f = \chi_{[b, \infty)} f, \quad P_{ab} f = \chi_{[a, b]} f.$$

Then we have

$$(4.28) \quad \begin{aligned} Kf &= (P_a + P_{ab} + Q_b)K(P_a + P_{ab} + Q_b)f \\ &= P_a K P_a f + Q_b K Q_b f + P_{ab} K P_{ab} f + Q_b K P_a f + Q_b K P_{ab} f + P_{ab} K P_a f. \end{aligned}$$

By Theorem 1 restricted to the intervals $[0, a]$ or $[b, \infty)$ and (3.4) we have

$$(4.29) \quad \begin{aligned} \|P_a K P_a\| &\leq \left\{ \sup_{0 < t < a} A_0(t) + \sup_{0 < t < a} A_1(t) \right\} \rightarrow 0, \quad a \rightarrow 0, \\ \|Q_b K Q_b\| &\leq \left\{ \sup_{t > b} A_0(t) + \sup_{t > b} A_1(t) \right\} \rightarrow 0, \quad b \rightarrow \infty. \end{aligned}$$

It follows from Lemma 4 that the operator $Q_b K P_a$ is compact. Indeed,

$$\begin{aligned}
 A_{Q_b K P_a} &\leq \left\| \left\| \chi_{[0,b]}(\cdot) k(x, \cdot) \psi(\cdot) \right\|_{X'}, \chi_{[b,\infty)}(x) \varphi(x) \right\|_Y \\
 &\leq D \left(\left\| \chi_{[b,\infty)}(x) k(x, b) \varphi(x) \right\|_Y \left\| \chi_{[0,b]} \psi \right\|_{X'} \right. \\
 (4.30) \quad &\quad \left. + \left\| \chi_{[b,\infty)} \varphi \right\|_Y \left\| \chi_{[0,b]}(\cdot) k(b, \cdot) \psi(\cdot) \right\|_{X'} \right) \\
 &\leq D (A_0(b) + A_1(b)) < \infty.
 \end{aligned}$$

Analogously, we find

$$\begin{aligned}
 A_{Q_b K P_{ab}} &\leq D (A_0(b) + A_1(b)) < \infty, \\
 (4.31) \quad A_{P_{ab} K P_a} &\leq D (A_0(a) + A_1(a)) < \infty.
 \end{aligned}$$

Note, that $0 < \left\| \chi_{[a,\infty)} \varphi \right\|_Y, \left\| \chi_{[0,b]} \psi \right\|_{X'} < \infty$, otherwise $A_0(a) = A_1(b) = 0$, and

$$k(x, y) \leq c_b < \infty, \quad b \geq x \geq y \geq a.$$

By Remark 4(ii), we may write

$$\begin{aligned}
 A_{P_{ab} K P_{ab}} &= \left\| \left\| \chi_{[a,b]}(\cdot) k(x, \cdot) \psi(\cdot) \right\|_{X'}, \chi_{[a,b]}(x) \varphi(x) \right\|_Y \\
 &\leq c_b \left\| \chi_{[a,\infty)} \varphi \right\|_Y \left\| \chi_{[0,b]} \psi \right\|_{X'} < \infty
 \end{aligned}$$

and hence, by Lemma 4, operator $P_{ab} K P_{ab}$ is compact too. Using this and (4.29)-(4.31) we see, that K is a limit of compact operators. This ends the proof of Theorem 2. ■

Proof of Theorem 3: We assume for simplicity, that $a_0 = a_1 = 0$, $b_0 = b_1 = \infty$. Let $0 < a < b < \infty$. By Theorem 1 we obtain

$$D^{-1} J_L(a) \leq \|P_a K P_a\| \leq d_1 d_2 \gamma(D) J_L(a),$$

$$D^{-1} J_R(b) \leq \|Q_b K Q_b\| \leq d_1 d_2 \gamma(D) J_R(b),$$

where the constant $\gamma(D)$ is defined by (4.20). Now, using (4.28) and taking into account the compactness of the last four components there, we see, that,

$$\alpha(K) \leq \|P_a K P_a + Q_b K Q_b\|.$$

Put $S = P_a K P_a$ and $T = Q_b K Q_b$. Then

$$\begin{aligned}
 \|S + T\| &= \sup_{f \neq 0} \frac{\|Sf + Tf\|_Y}{\|f\|_X} \\
 &\leq d_2 \sup \left\{ \frac{\left\| \|Sf\|_Y + \|Tf\|_Y \right\|_\ell}{\|f\|_X}; f \neq 0, f\psi \geq 0, P_{ab}f = 0 \right\}.
 \end{aligned}$$

If $P_{ab}f = 0$, then by the Bereznoi ℓ -condition

$$\|f\|_X \geq d_1^{-1} \left\| \|g\|_X + \|h\|_X \right\|_\ell,$$

where $g = P_a f$ and $h = Q_b f$. Hence,

$$\begin{aligned} \|S + T\| &\leq d_1 d_2 \sup_{f \neq 0, f=g+h} \frac{\left\| \|Sf\|_Y + \|Tf\|_Y \right\|_\ell}{\left\| \|g\|_X + \|h\|_X \right\|_\ell} \\ &= d_1 d_2 \sup_{f \neq 0, f=g+h} \left\| \frac{\|Sg\|_Y}{\|g\|_X} \cdot \frac{\|g\|_X}{\left\| \|g\|_X + \|h\|_X \right\|_\ell} \right. \\ &\quad \left. + \frac{\|Th\|_Y}{\|h\|_X} \cdot \frac{\|h\|_X}{\left\| \|g\|_X + \|h\|_X \right\|_\ell} \right\|_\ell \\ &\leq d_1^2 d_2^2 \gamma(D) \left\| J_L(a) \frac{\|g\|_X}{\left\| \|g\|_X + \|h\|_X \right\|_\ell} + J_R(b) \frac{\|h\|_X}{\left\| \|g\|_X + \|h\|_X \right\|_\ell} \right\|_\ell \\ &\leq d_1^2 d_2^2 \gamma(D) J. \end{aligned}$$

To obtain the lower bound, let $\theta > \alpha(K)$. If Y has AC-norm, then Y is separable [1]. Hence, there exists $T : X \rightarrow Y$ such that $\text{rank } T < \infty$ and $\|Kf - Tf\|_Y \leq \theta \|f\|_X$ for all $f \in X$. Since range of the operator T is formed by a finite number of functions from Y , we can approximate each of them by a bounded function with compact support [1] and, thus, given $\varepsilon > 0$, there exist $T_0 : X \rightarrow Y$ with $\text{rank } T_0 = \text{rank } T$ and the numbers $0 < \delta < N < \infty$, such that

$$\|T - T_0\| < \varepsilon \text{supp } T_0 f \subset [\delta, N] \text{ for all } f \in X.$$

Hence,

$$\|Kf - T_0 f\|_Y \leq (\theta + \varepsilon) \|f\|_X \text{ for all } f \in X.$$

Let f be such that $\text{supp } f \subset [0, \delta] \cup [N, \infty)$ and $f \psi \geq 0$. Then

$$(\theta + \varepsilon) \|f\|_X \geq \|Kf\|_Y = \|KP_\delta f + KQ_N f\|_Y \geq \|P_\delta KP_\delta f + Q_N KQ_N f\|_Y,$$

since all the functions involved are non-negative. Thus

$$(\theta + \varepsilon) \|f\|_X \geq \|P_\delta KP_\delta f\|_Y$$

for all $f \in X$ with $\text{supp } f \subset [0, \delta]$ and $f\psi \geq 0$ and

$$(\theta + \varepsilon)\|f\|_X \geq \|Q_N K Q_N f\|_Y$$

for all $f \in X$ with $\text{supp } f \subset [N, \infty)$ and $f\psi \geq 0$. Hence, applying the lower bound from Theorem 1, we obtain

$$(\theta + \varepsilon) \geq D^{-1} J_L(\delta) \text{ and } (\theta + \varepsilon) \geq D^{-1} J_R(N),$$

Letting $\theta \rightarrow \alpha(K)$, $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, $N \rightarrow \infty$ we establish the lower bound. Theorem 3 is proved. ■

Remark 5. Theorem 3 for Lebesgue spaces was proved in [7], the case $k(x, y) = 1$ was given in [6].

5. Approximation numbers

We begin with the reminder, that for any positive integer m , the m -th approximation number a_m of a bounded linear map $T : X \rightarrow Y$ is defined by

$$(5.1) \quad a_m(T) = \inf\{\|T - P\|; P \text{ a bounded linear operator and rank } P < m\}.$$

For further information on the approximation numbers we refer the reader to the monographs [4], [8] and [16]. We consider the operator $H : X \rightarrow Y$ of the form (3.8) and suppose, that H is compact. We also assume for simplicity, that $a_0 = 0$, $b_0 = \infty$ for the operator H . By Theorem 2 we get

$$A = \sup_{t>0} A(t) = \sup_{t>0} \|\chi_{[t,\infty)}\varphi\|_Y \|\chi_{[0,t]}\psi\|_{X'} < \infty,$$

$$\lim_{t \rightarrow 0} A(t) = \lim_{t \rightarrow \infty} A(t) = 0.$$

Given sufficiently small ε , $0 < \varepsilon < \|H\|$, we choose the numbers $0 = c_0 < c_1 < c_2 < \dots < c_{N-1} < c_N < c_{N+1} = \infty$ and intervals $I_k = [c_k, c_{k+1})$, $k = 0, 1, \dots, N$, such that

$$(5.3) \quad A[c_1] = A[c_N] = \varepsilon,$$

where

$$A[c_1] = \sup_{0 < t < c_1} A(t) = \sup_{0 < t < c_1} \|\chi_{[t,c_1)}\varphi\|_Y \|\chi_{[0,t]}\psi\|_{X'}$$

$$A[c_N] = \sup_{c_N < t < \infty} A(t) = \sup_{c_N < t < \infty} \|\chi_{[t,\infty)}\varphi\|_Y \|\chi_{[c_N,t]}\psi\|_{X'}.$$

Lemma 5. *Let X and Y be BFS satisfying the Bereznoi ℓ -condition, and suppose Y and Y' have AC-norms. Let $0 < a < b < \infty$, $I = (a, b)$ and*

$$\begin{aligned}
 F(x) &= \int_a^x \psi(y)f(y) dy, \quad a \leq x \leq b; \\
 (5.4) \quad F_I &= \frac{1}{\mu(I)} \int_I F d\mu, \\
 \mu(I) &= \int_I d\mu,
 \end{aligned}$$

where $d\mu(x) = \varphi(x)g(x) dx$, and $g(x)$ is a function on I satisfying the inequality

$$(5.5) \quad (1 - \delta) \|\chi_{[a,b]}\varphi\|_Y \|\chi_{[a,b]}g\|_{Y'} \leq \int_I \varphi(x)g(x) dx$$

for a sufficiently small $0 < \delta \leq 0,01$. Then

$$\frac{3}{10} \max(B_0, B_1) \leq \sup_{f \neq 0} \frac{\|\chi_{[a,b]}\varphi(F - F_I)\|_Y}{\|\chi_{[a,b]}f\|_X} \leq 8 \frac{2}{25} d_1^2 d_2^2 \max(B_0, B_1),$$

where

$$\begin{aligned}
 B_0 &= \sup_{a < x < c} \|\chi_{[x,c]}\psi\|_{X'} \|\chi_{[a,x]}\varphi\|_Y, \\
 B_1 &= \sup_{c < x < b} \|\chi_{[c,x]}\psi\|_{X'} \|\chi_{[x,b]}\varphi\|_Y.
 \end{aligned}$$

Proof of Lemma 5: Given $f \in X_{[a,b]}$, $c \in (a, b)$ we put

$$\Psi_c(x) = \begin{cases} - \int_x^c \psi(y)f(y) dy, & a \leq x < c, \\ \int_c^x \psi(y)f(y) dy, & c \leq x < b \end{cases}$$

and $\Psi_{c,I} = \frac{1}{\mu(I)} \int_I \Psi_c d\mu$. Then

$$F(x) - F_I = \Psi_c(x) - \Psi_{c,I}.$$

To obtain the lower bound, we take $f \in X_{[a,b]}$ such that $\text{supp } f \subseteq [a, c]$ and suppose, that the inequality

$$\|\chi_{[a,b]}\varphi(F - F_I)\|_Y \leq C \|\chi_{[a,b]}f\|_X$$

holds for all $f \in X_{[a,b]}$ with a constant C independent of f . Then

$$\begin{aligned}
C \|\chi_{[a,c]} f\|_X &\geq \|\chi_{[a,c]} \varphi(F - F_I)\|_Y = \|\chi_{[a,c]} \varphi(\Psi_c - \Psi_{c,I})\|_Y \\
&\geq \|\chi_{[a,c]} \varphi \Psi_c\|_Y - |\Psi_{c,I}| \|\chi_{[a,c]} \varphi\|_Y \\
&= \|\chi_{[a,c]} \varphi \Psi_c\|_Y - \frac{1}{\mu(I)} \left| \int_I \Psi_c d\mu \right| \|\chi_{[a,c]} \varphi\|_Y \\
&\geq \|\chi_{[a,c]} \varphi \Psi_c\|_Y - \frac{1}{\mu(I)} \int_I |\Psi_c| d\mu \|\chi_{[a,c]} \varphi\|_Y \\
&= \|\chi_{[a,c]} \varphi \Psi_c\|_Y - \frac{1}{\mu(I)} \int_I |\Psi_c \varphi \varphi^{-1}| d\mu \|\chi_{[a,c]} \varphi\|_Y \\
&\geq \|\chi_{[a,c]} \varphi \Psi_c\|_Y - \frac{1}{\mu(I)} \|\chi_{[a,c]} \varphi \Psi_c\|_Y \|\chi_{[a,c]} g\|_{Y'} \|\chi_{[a,c]} \varphi\|_Y \\
&= \left(1 - \frac{1}{\mu(I)} \|\chi_{[a,c]} g\|_{Y'} \|\chi_{[a,c]} \varphi\|_Y\right) \|\chi_{[a,c]} \varphi \Psi_c\|_Y \\
&= \left(1 - \frac{V(a,c)}{\mu(I)}\right) \|\chi_{[a,c]} \varphi \Psi_c\|_Y,
\end{aligned}$$

where $V(a,c) = \|\chi_{[a,c]} \varphi\|_Y \|\chi_{[a,c]} g\|_{Y'}$. Because of the absolute continuity of the norms Y and Y' , we can for any fixed $\beta \in (0, 1 - \delta)$ find a point $c \in (a, b)$ such that $V(a,c) = \beta \mu(I)$, therefore by Theorem 4 restricted to the interval $[a, c]$ we have $C \geq (1 - \beta)B_0$. A similar argument applied for all f such that $\text{supp } f \subset [c, b)$ gives

$$C \|\chi_{[c,b]} f\|_X \geq \left(1 - \frac{W(c,b)}{\mu(I)}\right) \|\chi_{[c,b]} \varphi \Psi_c\|_Y,$$

where $W(c,b) = \|\chi_{[c,b]} \varphi\|_Y \|\chi_{[c,b]} g\|_{Y'}$.

$$\begin{aligned}
1 - \frac{W(c,b)}{\mu(I)} &= \frac{1}{\mu(I)} (\mu(I) - W(c,b)) \\
&= \frac{1}{\mu(I)} \left((1 - \delta) \|\chi_{[a,b]} \varphi\|_Y \|\chi_{[a,b]} g\|_{Y'} - W(c,b) \right) \\
&\geq \beta(1 - \delta) - \frac{W(c,b)}{\mu(I)}.
\end{aligned}$$

If $c \rightarrow b$, then $\beta \rightarrow (1 - \delta)$ and $W(c, b) \rightarrow 0$, therefore we can choose $c \in (a, b)$ such that

$$\frac{W(c, b)}{\mu(I)} \leq \frac{\beta(1 - \delta)}{2}.$$

By Theorem 4 we get $C \geq \frac{\beta(1 - \delta)}{2} B_1$. Now, if we take β such that $1 - \beta = \frac{\beta(1 - \delta)}{2}$, then $\beta = \frac{2}{3 - \delta}$ and the required lower bound $C \geq \frac{3}{10} \max(B_0, B_1)$.

Sufficiency: Using Hölder's inequality and the Bereznoi ℓ -condition, we see that

$$\begin{aligned} \|\chi_{[a,b]}\varphi(F - F_I)\|_Y &= \|\chi_{[a,b]}\varphi(\Psi_c - \Psi_{c,I})\|_Y \\ &\leq \|\chi_{[a,b]}\varphi\Psi_c\|_Y + |\Psi_{c,I}| \|\chi_{[a,b]}\varphi\|_Y \\ &\leq \|\chi_{[a,b]}\varphi\Psi_c\|_Y \\ &\quad + \frac{1}{\mu(I)} \|\chi_{[a,b]}\varphi\Psi_c\|_Y \|\chi_{[a,b]}g\|_{Y'} \|\chi_{[a,b]}\varphi\|_Y \\ &\leq \frac{2}{(1 - \delta)} \|\chi_{[a,b]}\varphi\Psi_c\|_Y \\ &\leq \frac{2}{(1 - \delta)} d_2 \left\| \|\chi_{[a,c]}\varphi\Psi_c\|_Y + \|\chi_{[c,b]}\varphi\Psi_c\|_Y \right\|_\ell \\ &\leq \frac{8d_1^2 d_2^2}{1 - \delta} \max(B_0, B_1) \|\chi_{[a,b]}f\|_X, \end{aligned}$$

and the required result follows. The proof of Lemma 5 is complete. ■

By Lemma 5 the norm of the operator

$$H_I f(x) = \chi_I(x)\varphi(x)(F(x) - F_I)$$

depends continuously on the interval I . We choose the intervals $I_k = [c_k, c_{k+1}]$, $k = 1, \dots, N - 1$ so, that

$$(5.6) \quad \begin{aligned} \|H_{I_k}\| &= \varepsilon, \quad k = 1, \dots, N - 2, \\ \|H_{I_{N-1}}\| &\leq \varepsilon. \end{aligned}$$

Now we follow the construction from ([7, Section 3.1]) adjusted to the present case. Let for $k = 1, \dots, N - 1$

$$F_k(x) = \chi_{I_k}(x) \int_{c_k}^x \psi(y) f(y) dy,$$

$$P_k f(x) = \chi_{I_k}(x) \{Hf(x) - \varphi(x)(F_k(x) - F_{k, I_k})\}.$$

Observe, that the operator $P = \sum_{k=1}^{N-1} P_k$ is a bounded linear operator $P : X \rightarrow Y$ and $\text{rank } P \leq N - 1$.

Theorem 6. *Let X and Y be BFS satisfying the Bereznoi ℓ -condition, and X', Y, Y' have AC-norms. Let $H : X \rightarrow Y$ defined by (3.8) be a compact operator. Given $\varepsilon > 0$, $\|H\| > \varepsilon$, let the integer $N > 2$ and intervals $I_k = [c_k, c_{k+1}]$, $k = 0, 1, \dots, N$ be chosen so that (5.3) and (5.6) hold. Then the upper bound*

$$(5.7) \quad a_N(H) \leq d_1 d_2 \varepsilon,$$

is valid, where the constants d_1, d_2 are determined by (2.3), (2.4).

Proof of Theorem 6: Using Theorem 4 and the Bereznoi ℓ -condition we see that

$$\begin{aligned} \|Hf - Pf\|_Y &\leq d_2 \left\| \varepsilon \|\chi_{[0, c_1]} f\|_X + \varepsilon \|\chi_{[c_N, \infty)} f\|_X + \sum_{k=1}^{N-1} \|H_{I_k} f\|_Y \right\|_{\ell} \\ &\leq \varepsilon d_2 \left\| \sum_{k=0}^N \|\chi_{I_k} f\|_X \right\|_{\ell} \leq \varepsilon d_1 d_2 \|f\|_X, \end{aligned}$$

and thus, $a_N(H) \leq d_1 d_2 \varepsilon$. ■

Theorem 7. *Let the assumptions of Theorem 6 hold and, moreover, let X be ℓ_{p_1} -convex BFS and Y be ℓ_{p_2} -concave BFS for BSS ℓ_{p_1}, ℓ_{p_2} with $p_2 \geq p_1 > 1$. Then the following lower bound*

$$(5.8) \quad a_N(H) \geq \frac{1}{2,02} \varepsilon d_1^{-1} d_2^{-1} N^{\frac{1}{p_2} - \frac{1}{p_1}}$$

holds.

Proof of Theorem 7: Let $\lambda \in (0, 1)$. We take the sequence of functions $f_k \in X$, such that $\text{supp } f_k \subset I_k$, and

$$(5.9) \quad \frac{\|\chi_{I_i} \varphi F_i\|_Y}{\|f_i\|_X} \geq \lambda \varepsilon, \quad i = 0, N$$

$$(5.10) \quad \frac{\|\chi_{I_k} \varphi (F_k - F_{k, I_k})\|_Y}{\|f_k\|_X} \geq \lambda \varepsilon, \quad k = 1, 2, \dots, N - 1,$$

where

$$F_k(x) = \int_{c_k}^x \psi(y) f_k(y) dy, \quad k = 0, 1, \dots, N - 1.$$

In this construction we follow [5]. Let $\tilde{P} : X \rightarrow Y$ be a bounded linear map and $\text{rank } \tilde{P} \leq N$. Then we choose constants $\nu_0, \nu_1, \nu_2, \dots, \nu_N$ such that

$$(5.11) \quad \tilde{P} \left(\sum_{k=0}^N \nu_k f_k \right) = 0.$$

Put $f = \sum_{k=0}^N \nu_k f_k$ and as before

$$F(x) = \int_0^x \psi(y) f(y) dy, \quad x > 0.$$

For all $x \in I_k$

$$F(x) = \nu_k F_k(x) + \mu_k, \quad k = 0, 1, \dots, N - 1,$$

for some constant μ_k . For all $c \in \mathbf{R}$

$$\begin{aligned} \|\chi_I \varphi (F - F_I)\|_Y &\leq \|\chi_I \varphi (F - c - (F - c)_I)\|_Y \\ &\leq \|\chi_I \varphi (F - c)\|_Y + |(F - c)_I| \|\chi_I \varphi\|_Y \\ &\leq \frac{2}{1 - \delta} \|\chi_I \varphi (F - c)\|_Y \leq 2,02 \|\chi_I \varphi (F - c)\|_Y. \end{aligned}$$

Hence,

$$(5.12) \quad \|\chi_I \varphi (F - F_I)\|_Y \leq 2,02 \inf_{c \in \mathbf{R}} \|\chi_I \varphi (F - c)\|_Y.$$

Applying (5.9)-(5.12), ℓ_{p_2} -concavity of Y and ℓ_{p_1} -convexity of X , we find

$$\begin{aligned}
& \|Hf - \tilde{P}f\|_Y^{p_2} = \|Hf\|_Y^{p_2} \\
& \geq d_1^{-p_2} \left(\|\chi_{I_0} \varphi F_0\|_Y^{p_2} + \sum_{k=1}^{N-1} \|\chi_{I_k} \varphi F\|_Y^{p_2} + \|\chi_{I_N} \varphi F_N\|_Y^{p_2} \right) \\
& \geq d_1^{-p_2} \left((\lambda\varepsilon)^{p_2} \|\nu_0 f_0\|_X^{p_2} + \sum_{k=1}^{N-1} \|\chi_{I_k} \varphi (\nu_k F_k + \mu_k)\|_Y^{p_2} + (\lambda\varepsilon)^{p_2} \|\nu_N f_N\|_X^{p_2} \right) \\
& \geq d_1^{-p_2} \left((\lambda\varepsilon)^{p_2} \|\nu_0 f_0\|_X^{p_2} \right. \\
& \quad \left. + \left(\frac{1}{2, 02} \right)^{p_2} \sum_{k=1}^{N-1} (|\nu_k| \|\chi_{I_k} \varphi (F_k - F_{k, I_k})\|_Y)^{p_2} + (\lambda\varepsilon)^{p_2} \|\nu_N f_N\|_X^{p_2} \right) \\
& \geq d_1^{-p_2} \left((\lambda\varepsilon)^{p_2} \|\nu_0 f_0\|_X^{p_2} \right. \\
& \quad \left. + \left(\frac{\lambda\varepsilon}{2, 02} \right)^{p_2} \sum_{k=1}^{N-1} (\|\nu_k\| \|f_k\|_X)^{p_2} + (\lambda\varepsilon)^{p_2} \|\nu_N f_N\|_X^{p_2} \right) \\
& = d_1^{-p_2} \left((\lambda\varepsilon)^{p_2} \|\nu_0 f_0\|_X^{p_2} + \left(\frac{\lambda\varepsilon}{2, 02} \right)^{p_2} \sum_{k=1}^{N-1} \|\nu_k f_k\|_X^{p_2} + (\lambda\varepsilon)^{p_2} \|\nu_N f_N\|_X^{p_2} \right) \\
& \geq \left(\frac{\lambda\varepsilon}{2, 02d_1} \right)^{p_2} \sum_{k=0}^N \|\nu_k f_k\|_X^{p_2} \\
& \geq \left(\frac{\lambda\varepsilon}{2, 02d_1} \right)^{p_2} \left(\sum_{k=0}^N \|\nu_k f_k\|_X^{p_1} \right)^{p_1/p_2} (N+1)^{1-p_2/p_1} \\
& \geq \left(\frac{\lambda\varepsilon}{2, 02d_1d_2} \right)^{p_2} (N+1)^{1-p_2/p_1} \|f\|_X^{p_2}.
\end{aligned}$$

Thus,

$$a_N(H) \geq \frac{1}{2, 02} d_1^{-1} d_2^{-1} \lambda\varepsilon (N+1)^{1/p_2-1/p_1},$$

and, letting $\lambda \rightarrow 1$ the required lower bound follows. ■

Remark 6. For Lebesgue spaces Theorems 6 and 7 were proved in [5], the extension to Lorentz spaces was given in [11].

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