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THE MULTIPLICATIVE STRUCTURE OF $K(n)^*(BA_4)$

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Abstract

Let $K(n)^*(-)$ be a Morava K-theory at the prime 2. Invariant theory is used to identify $K(n)^*(BA_4)$ as a summand of $K(n)^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2)$. Similarities with $H^*(BA_4; \mathbb{Z}/2)$ are also discussed.

Introduction

Let G be a finite group, and let N and C denote respectively the normalizer and the centralizer of a p-Sylow subgroup H of G.

For a large family of cohomology theories including the Brown-Peterson cohomology $BP^*(-)$ and Morava K-theories $K(n)^*(-)$, the author described $h^*(BG)$ when H is cyclic [**3**], and discussed the case "p-rank $(H) < 3$" in [**5**], proving in particular that $h^*(BG)$ is generated as h^*-module by at most two elements if $|N : C|$ divides $p-1$.

Results in this paper show that the condition above is really necessary, in fact we have

Theorem 0.1. *Let $K(n)^*(-)$ be a Morava K-theory at the prime* 2. *$K(n)^*(BA_4)$ restricts to those elements in*

$$K(n)^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

which algebraically depend on

$$\begin{aligned}\bar{\sigma} &= x^2 + y^2 + xy + \nu_n(x^{2^{n-1}+1}y^{2^{n-1}} + x^{2^{n-1}}y^{2^{n-1}+1}),\\ \bar{\tau}_1 &= x^3 + y^3 + x^2y + \nu_n(x^{2^{n-1}}y^{2^{n-1}+2}),\\ \bar{\tau}_2 &= x^3 + y^3 + xy^2 + \nu_n(x^{2^{n-1}+2}y^{2^{n-1}}).\end{aligned}$$

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This paper has several motivations. The knowledge of $K(n)^*(BA_4)$ could help to have explicit formulæ for the $K(n)^*$-Dickson classes. Furthermore, similarities among $H^*(BA_4;\mathbb{Z}/2)$ and $K(n)^*(BA_4)$ suggest to study $K(n)^*(BA_m)$ —whose rank as $K(n)^*$-module can be calculated [**6**]— to get information on $H^*(BA_m;\mathbb{Z}/2)$ which is not entirely known for $m \geq 16$ (see [**1**] for the cohomology of several alternating groups).

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1. Preliminaries. $H^*(BA_4)$

From now on V will denote the group $\mathbb{Z}/2\times\mathbb{Z}/2$, and $H^*(-)$ ordinary cohomology with coefficients in $\mathbb{Z}/2$.

In [**7**], the authors describe $H^*(PSL_2\mathbb{F}_q)$ for any odd q: they first calculate the cohomology of the generalized quaternion group $Q_{2^{n+1}}$ of order 2^{n+1}, and then use the diagram

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] & SL_2\mathbb{F}_q \arrow[r] & PSL_2\mathbb{F}_q \\
\mathbb{Z}/2 \arrow[r] \arrow[u, equal] & Q_{2^{n+1}} \arrow[r] \arrow[u, "i"'] & D_n \arrow[u, "j"']
\end{tikzcd}$$

where D_n is the dihedral group of order 2^n, rows are fibrations, and i and j are inclusions of 2-Sylow subgroups. Nevertheless, we show in this section that the special case

$$PSL_2\mathbb{F}_3 \cong A_4$$

can be approached in a more direct way.

The alternating group A_4 is the central term of the short exact sequence of groups

$$0 \longrightarrow V \longrightarrow A_4 \longrightarrow \mathbb{Z}/3 \longrightarrow 0,$$

therefore for any mod 2 cohomology theory $h^*(-)$, $h^*(BA_4)$ is isomorphic to the ring of invariants $[h^*(BV)]^{\mathbb{Z}/3}$ under the action determined by the map $h^*(B\phi)$ induced by an automorphism ϕ of order 3 in $\mathrm{Aut}(V)$. On $H^*(BV) \cong \mathbb{F}_2[x,y]$ the action of a generator of $\mathbb{Z}/3 \leq GL(V)$ is

$$x \xrightarrow{\alpha_H} y \quad \text{and} \quad y \xrightarrow{\alpha_H} x+y.$$

Consider now the map Φ from $F_2[x,y]$ to itself which maps any element c to the sum

$$\Phi(c) = c + \alpha_H(c) + \alpha_H^2(c);$$

Φ is commonly known as norm map. It is easy to see that

$$\operatorname{Im}\Phi = [H^*(BV)]^{\mathbb{Z}/3};$$

furthermore the image of Φ restricted to the set of monomials generates $[H^*(BV)]^{\mathbb{Z}/3}$ regarded as graded $\mathbb{F}_2$-vector space.

The invariant of lowest positive degree in $\mathbb{F}_2[x, y]$ is

$$\sigma = \Phi(xy) = x^2 + xy + y^2.$$

This element is actually the Dickson class known in literature as $Q_{2,1}$ (see [**9**]).

The reader will find the relevant invariant theoretic computation in [**2**] to prove the algebraic dependence of every invariant on $\Phi(xy)$, $\Phi(x^2y)$, $\Phi(xy^2)$. In fact we have the following proposition.

Proposition 1.1. *As a graded ring, $H^*(BA_4)$ is isomorphic to*

$$\mathbb{F}_2[\sigma, \tau_1, \tau_2]/R,$$

where $\deg\sigma = 2$, $\deg\tau_1 = \deg\tau_2 = 3$, *and R is the ideal generated by*

$$\sigma^3 + \tau_1^2 + \tau_1\tau_2 + \tau_2^2.$$

The proposition above can be restated in terms of pure invariant theory.

Corollary 1.2. *Suppose that a $\mathbb{Z}/3$-action on $\mathbb{F}_2[x, y]$ is given by*

$$x \longrightarrow y \quad \textit{and} \quad y \longrightarrow x + y.$$

The ring of the invariants is a polynomial ring generated by

$$\sigma = x^2 + y^2 + xy, \quad \tau_1 = x^3 + y^3 + x^2y, \quad \tau_2 = x^3 + y^3 + xy^2,$$

quotiented by

$$R = (\sigma^3 + \tau_1^2 + \tau_1\tau_2 + \tau_2^2).$$

2. The Morava K-theory of BA_4

We recall that Morava K-theory at the prime 2 is a complex oriented cohomology theory with coefficients

$$K(n)^*(\{pt\}) = \mathbb{F}_2[\nu_n, \nu_n^{-1}]$$

where $\deg \nu_n = -2(2^n - 1)$, and we have

$$K(n)^*(BV) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

where $\deg x = \deg y = 2$. As noticed in section 1, $K(n)^*(BA_4)$ is isomorphic to

$$[K(n)^*(BV)]^{\mathbb{Z}/3}$$

where the $\mathbb{Z}/3$-module structure is defined by the map $K(n)^*(B\phi)$, being ϕ a generator of $\mathbb{Z}/3 \leq \mathrm{Aut}(V)$. The following lemma helps to give a concrete description of the $K(n)$-invariants.

Lemma 2.1. *One of the two generators ϕ of $\mathbb{Z}/3 \leq \mathrm{Aut}(V)$ acts as follows on $K(n)^*(BV)$:*

$$\alpha_K \stackrel{\text{def}}{=} K(n)^*(B\phi) : x \longrightarrow y \quad \textit{and} \quad \alpha_K : y \longrightarrow x + y + \nu_n x^{2^{n-1}} y^{2^{n-1}}.$$

Proof: See [**4**]. ■

The element $\alpha_K(y)$ is actually the formal sum of x and y with respect to the formal group law of mod 2 Morava K-theory

$$F_{K(n)}(x, y) \mod (x^{2^n}, y^{2^n}).$$

Consider now the norm map Ψ defined as follows:

$$\Psi : c \in K(n)^*(BV) \longmapsto c + \alpha_K(c) + \alpha_K^2(c) \in [K(n)^*(BV)]^{\mathbb{Z}/3}.$$

The map Ψ is obviously the analogue of Φ defined in section 1: it is surjective, and the invariants regarded as $\mathbb{F}_2$-vector space are spanned by the image of Ψ restricted to monomials.

Notice also that we can equip

$$K(n)^*(BV) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

with a different $\mathbb{Z}/3$-module structure just by posing

$$\alpha_H(x) = y \quad \text{and} \quad \alpha_H(y) = x + y.$$

Abusing notation, we shall use again Φ to denote the endomorphism defined on the generic element of $K(n)^*(BV)$ as follows:

$$c \longmapsto c + \alpha_H(c) + \alpha_H^2(c).$$

We are ready now to prove our main result.

Theorem 2.2. *$K(n)^*(BA_4)$ restricts to those elements in $K(n)^*(BV)$ which algebraically depend on*

$$\Psi(xy) = \bar{\sigma}, \quad \Psi(x^2y) = \bar{\tau}_1 \quad \textit{and} \quad \Psi(xy^2) = \bar{\tau}_2.$$

Proof: Since $K(n)^*(-)$ is $2(2^n - 1)$-periodic we can look at classes in $K(n)^*(BV)$ whose degree is between 2 and $2(2^n - 1)$. In this range, elements of type

$$\nu_n^2 x^h y^k$$

are necessarily zero, since either h or k is greater than 2^n. It follows that for any monomial $x^h y^k \in K(n)^*(BV)$ we have

$$\Psi(\nu_n x^h y^k) = \nu_n \Phi(x^h y^k). \tag{1}$$

An element $c \in K(n)^*(BV)$ is invariant under α_K if and only if $\Psi(c) = c$, and supposing

$$2 \le t \le 2(2^n - 1),$$

we have

$$c = p(x, y) + \nu_n q(x, y),$$

where $p(x, y)$ and $q(x, y)$ are homogeneous polynomials of $\mathbb{F}_2[x, y]$ of degree t and $t + 2(2^n - 1)$ respectively. If $\Psi(c) = c$, it follows from the considerations above that $\Phi(p(x, y)) = p(x, y)$, and by Corollary 1.2 there exists a polynomial r_1 in three indeterminates such that

$$r_1(\sigma, \tau_1, \tau_2) = p(x, y).$$

Define now

$$\Psi(xy) = \bar{\sigma}, \quad \Psi(x^2y) = \bar{\tau}_1 \quad \text{and} \quad \Psi(xy^2) = \bar{\tau}_2.$$

The element

$$c - r_1(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2) = \nu_n s(x, y)$$

is invariant under α_K. Notice now that $s(x, y)$ can be regarded as a polynomial in $\mathbb{F}_2[x, y]$; it follows by (1) that $s(x, y)$ is invariant under α_H, and again by Corollary 1.2 there exists a polynomial r_2 in three indeterminates such that

$$r_2(\sigma, \tau_1, \tau_2) = s(x, y).$$

We finally get

$$c = r_1(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2) - \nu_n r_2(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2)$$

as we claimed. ■

Theorem 2.2 also gives some information on $K(n)^*(BA_5)$. Notice in fact that 2-Sylow subgroups in A_5 are abelian, and a 2-Sylow normalizer in A_5 is isomorphic to A_4. It follows by a theorem in [**8**] that BA_4 and BA_5 are stably 2-homotopy equivalent. Hence the map induced by inclusion

$$K(n)^*(BA_5) \longrightarrow K(n)^*(BA_4)$$

is an isomorphism.

Remark 2.3. The element

$$\bar{\sigma}^3 + \bar{\tau}_1^2 + \bar{\tau}_1\bar{\tau}_2 + \bar{\tau}_2^2$$

is zero in $K(n)^*(BA_4)$, as the analogous algebraic expression in σ, τ_1, τ_2 for ordinary cohomology. The relation above is not however of minimal positive degree: the element

$$\nu_n^2 \bar{\sigma}^{2^n}$$

is zero and has degree four.

It is known that the subring of $H^{\mathrm{even}}(BA_4)$ generated by Chern classes is proper (see, for example [**10**, p. 100]), and the reader could ask if $\bar{\sigma}$, $\bar{\tau}_1$, $\bar{\tau}_2$ are $K(n)$-Chern classes of suitable representations.

We recall that up to equivalence the group A_4 has just four distinct complex irreducible representations. Three of them are one-dimensional, and their restriction to V is trivial. The fourth one has instead non-trivial total Chern class in $K(n)^*(BA_4)$, as the next proposition shows.

Proposition 2.4. *Let ξ be a 3-dimensional irreducible representation of A_4. The restriction $\xi_{|V}$ to the 2-Sylow subgroup V has Chern classes*

$$c_1(\xi_{|V}) = \nu_n \bar{\sigma}^{2^{n-1}}, \quad c_2(\xi_{|V}) = \bar{\sigma}, \quad c_3(\xi_{|V}) = \bar{\tau}_1 + \bar{\tau}_2 + \nu_n \bar{\sigma}^{2^{n-1}+1}$$

in $K(n)^*(BV)$.

Proof: Let g_1 and g_2 be two generators in V. Consider two one-dimensional representations ρ_1 and ρ_2 defined as follows

$$\rho_i : g_i \longmapsto -1 \quad \rho_i : g_{3-i} \longmapsto 1$$

for $i = 1, 2$. The transfer ξ of ρ_1 to A_4 represents the equivalence class of the 3-dimensional irreducible representations of A_4; its restriction to V is given by

$$\rho_1 \oplus \rho_2 \oplus (\rho_1 \otimes \rho_2).$$

It follows that the total Chern class $c.(\xi_{|V})$ is equal to

$$(1 + x)(1 + y)(1 + x + y + \nu_n x^{2^{n-1}} y^{2^{n-1}}).$$

Hence the proposition follows. ■

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