

SOME REMARKS ON INDICES OF HOLOMORPHIC VECTOR FIELDS

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Abstract

One can associate several residue-type indices to a singular point of a two-dimensional holomorphic vector field. Some of these indices depend also on the choice of a separatrix at the singular point. We establish some relations between them, especially when the singular point is a generalized curve and the separatrix is the maximal one. These local results have global consequences, for example concerning the construction of logarithmic forms defining a given holomorphic foliation.

Let v be a holomorphic vector field defined on a neighborhood of $0 \in \mathbb{C}^2$ and having there an isolated singularity. Baum and Bott have associated to such a singular point two types of indices **[BB]**, defined through residues of suitable meromorphic 2-forms. One of these indices is simply the Poincaré-Hopf index of v at 0, and so it is an integer positive number which gives the multiplicity of the singular point. The other one has a more subtle meaning, and we shall call it the *Baum-Bott index*, $BB(v, 0)$.

Let S be a separatrix of v at 0, i.e. a holomorphic curve invariant by v and containing 0 (there are always separatrices, see **[CS]**). Camacho and Sad, and later Lins Neto and Suwa, have defined an index $CS(v, S, 0)$ which, roughly speaking, represents the intersection index of the trajectories of v with the separatrix S **[Cs]**, **[LN]**, **[Su]**. Gomez-Mont, Seade and Verjovsky have defined another index $GSV(v, S, 0)$ which is a sort of Poincaré-Hopf index of the restriction of v to S **[GSV]**.

Aim of this paper is to underline some simple properties of these indices, and in particular to prove a relation between them for a rather general class of isolated singularities: the class of *generalized curves* **[CLS]**, that is nondicritical singularities whose resolution does not contain saddle-nodes. In **[CLS]** it is proven that the Poincaré-Hopf index of such a singularity coincides with the Milnor number of the union of all the separatrices S (which is still a separatrix, by nondicriticalness). Here we prove that the Baum-Bott index also can be “localized” near S .

Theorem. *Let v be a holomorphic vector field near $0 \in \mathbf{C}^2$, $v(0) = 0$, and suppose that the singularity at 0 is a generalized curve. Let S be the union of all the separatrices of v at 0. Then*

$$\begin{aligned} BB(v, 0) &= CS(v, S, 0) \\ GSV(v, S, 0) &= 0. \end{aligned}$$

This result is not so strange. The Baum-Bott index is something like a “mean intersection index at 0” of the leaves of the foliation $\mathcal{F}$ generated by v , whereas the Camacho-Sad index is related to the “intersection index at 0” of the leaves of $\mathcal{F}$ with S . The leaves composing $S \setminus \{0\}$ are the only leaves which go “directly” to 0, hence a relation between $BB(v, 0)$ and $CS(v, S, 0)$ is not surprising: two leaves both different from separatrices do not “intersect at 0”. The actual proof is obtained by desingularization (the theorem is elementary for simple singularities, different from saddle-nodes; and it is false for saddle-nodes), and by the analysis of the variation of the indices under blow-ups. During the proof, the notion of *nondicritical separatrix* will naturally emerge.

In the last section of the paper we will see some global consequences of these types of results. We also include several remarks about BB , CS , and GSV which are not strictly necessary to the proof of the theorem above, but which may be useful for a geometric understanding of these indices.

1. The Baum-Bott index

Let $\mathcal{F}$ be a holomorphic foliation with isolated singularities on a complex surface X . Let $p \in X$ be a singular point of $\mathcal{F}$; near p the foliation is given either by a holomorphic vector field $v = F(z, w) \frac{\partial}{\partial z} + G(z, w) \frac{\partial}{\partial w}$ or by a holomorphic 1-form $\omega = F(z, w) dw - G(z, w) dz$. Here (z, w) are local coordinates centered at p and F, G , are holomorphic functions with $F^{-1}(0) \cap G^{-1}(0) = \{(0, 0)\}$.

Let $J(z, w)$ be the Jacobian matrix of (F, G) at (z, w) , then following [BB] we can define two indices:

$$\begin{aligned} PH(\mathcal{F}, p) &= \text{Res}_{(0,0)} \left\{ \frac{\det J}{F \cdot G} dz \wedge dw \right\} \\ BB(\mathcal{F}, p) &= \text{Res}_{(0,0)} \left\{ \frac{(\text{tr } J)^2}{F \cdot G} dz \wedge dw \right\} \end{aligned}$$

(see [GH] for the background concerning residues). These indices are well defined (i.e. they depend only on the conjugacy class of the germ of

$\mathcal{F}$ at p), and they are easily computed when p is non degenerate, that is when $J(0, 0)$ is invertible: if λ, μ are the two eigenvalues of $J(0, 0)$ then

$$PH(\mathcal{F}, p) = \frac{\det J(0, 0)}{\det J(0, 0)} = 1$$

$$BB(\mathcal{F}, p) = \frac{(\operatorname{tr} J(0, 0))^2}{\det J(0, 0)} = 2 + \frac{\lambda}{\mu} + \frac{\mu}{\lambda}.$$

The index $PH(\mathcal{F}, p)$ is nothing else than the Poincaré-Hopf index of v at p , hence it coincides with the multiplicity of the singular point and it can be also computed as a degree of a suitable map [GH]. It follows from this its topological invariance [CLS]. The index $BB(\mathcal{F}, p)$ has no meaning at the algebraic topology level, but it can be computed in the following way, à la Godbillon-Vey. On a pointed neighborhood $U^* = U \setminus \{p\}$ of p we may find a complex valued smooth 1-form β , of type $(1, 0)$, such that

$$d\omega = \beta \wedge \omega.$$

For example, we may set $\beta = \frac{F_z + G_w}{|F|^2 + |G|^2}(\bar{F} dz + \bar{G} dw)$. Using the cohomological interpretation of residues [GH] it is easy to verify that

$$BB(\mathcal{F}, p) = \frac{1}{(2\pi i)^2} \int_{\mathbf{S}^3} \beta \wedge d\beta$$

where $\mathbf{S}^3$ is a small sphere around p , oriented as a boundary of a small ball containing p . It is in fact sufficient that β is defined on a neighborhood of such a sphere.

If $\mathcal{F}$ is a foliation on a compact surface X then the sum of Baum-Bott indices at singular points is equal to $c_1^2(N_{\mathcal{F}})$, where $N_{\mathcal{F}} \in H^1(X, \mathcal{O}^*)$ is the normal bundle of $\mathcal{F}$ (which can be defined even in presence of singularities, [GM], [Br]). This is the Baum-Bott formula [BB], which can be straightforwardly generalized to the following situation. Without assuming the compactness of X , we consider a relatively compact domain $Y \subset X$ with ∂Y smooth and disjoint from the singular set of $\mathcal{F}$, $\operatorname{Sing}(\mathcal{F})$. We assume that $N_{\mathcal{F}}$ is holomorphically trivial near ∂Y (even if this is not completely necessary, the topological triviality would be sufficient), hence $\mathcal{F}$ near ∂Y is given by a nonsingular holomorphic 1-form ω and we still can construct a smooth $(1, 0)$ -form β near ∂Y such that $d\omega = \beta \wedge \omega$. Then we define

$$BB(\mathcal{F}, \partial Y) = \frac{1}{(2\pi i)^2} \int_{\partial Y} \beta \wedge d\beta$$

with ∂Y oriented as boundary of Y , and of course this number does not depend on the involved choices. The triviality of $N_{\mathcal{F}}$ near the boundary allows also to define $c_1^2(N_{\mathcal{F}}) \in \mathbf{Z}$ (here $\mathcal{F} = \mathcal{F}|_Y$).

Proposition 1 [BB].

$$BB(\mathcal{F}, \partial Y) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap Y} BB(\mathcal{F}, p) - c_1^2(N_{\mathcal{F}}).$$

We shall use this formula in the process of desingularization.

Let us return to the local situation, and let S be a separatrix of $\mathcal{F}$ at p , that is a holomorphic curve on a neighborhood of p , containing p and invariant by $\mathcal{F}$. It is not assumed that S is smooth nor irreducible at p . Using the 1-form β we may define the following index [KS]:

$$\text{Var}(\mathcal{F}, S, p) = \frac{1}{2\pi i} \int_{\partial S} \beta$$

where $\partial S = S \cap \mathbf{S}^3$ and $\mathbf{S}^3$ is again a small sphere around p ; ∂S is oriented as a boundary of $S \cap \mathbf{B}^4$, with $\mathbf{B}^4$ a small ball containing p . To give a consistent definition it is in fact sufficient that β is defined only on a neighborhood of ∂S , and that $d\omega = \beta \wedge \omega$ holds only at points of ∂S .

Suppose now that $S \subset X$ is a compact holomorphic curve invariant by the foliation $\mathcal{F}$.

Proposition 2 [KS].

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap S} \text{Var}(\mathcal{F}, S, p) = c_1(N_{\mathcal{F}}) \cdot S.$$

Propositions 1 and 2 are manifestations of the same principle. We may find a covering $\mathcal{U} = \{U_j\}$ of X , holomorphic 1-forms with isolated singularities $\omega_j \in \Omega^1(U_j)$, smooth $(1, 0)$ -forms $\beta_j \in A^{(1,0)}(U_j)$, such that:

- i) $F|_{U_j}$ is defined by ω_j (hence on $U_j \cap U_i$ one has $\omega_i = g_{ij}\omega_j$ and $\{g_{ij}\}$ is a $\mathcal{O}^*$ -cocycle defining $N_{\mathcal{F}}$).
- ii) $d\omega_j = \beta_j \wedge \omega_j$ on $U_j \setminus V_j$, where V_j is a small neighborhood of $\text{Sing}(\mathcal{F}) \cap U_j$.
- iii) $\beta_i - \beta_j = \frac{dg_{ij}}{g_{ij}}$ on $U_i \cap U_j$.

Then the 2-form Θ locally given by $\Theta = \frac{1}{2\pi i} d\beta_j$ represents (in the De Rham sense) the first Chern class of $N_{\mathcal{F}}$, hence $c_1^2(N_{\mathcal{F}}) = \int_X \Theta \wedge \Theta$, $c_1(N_{\mathcal{F}}) \cdot S = \int_S \Theta$, and Propositions 1 and 2 follow.

2. The Camacho-Sad index

Let us consider again a separatrix S at $p \in X$. Let f be a holomorphic function on a neighborhood of p and defining $S : S = \{f = 0\}$. We may assume that f is reduced, i.e. $df \neq 0$ outside p . Then [LN], [Su] there are functions g, k and a 1-form η on a neighborhood of p such that

$$g\omega = kdf + f\eta$$

and moreover k and f are prime, i.e. $k \neq 0$ on $S^* = S \setminus \{p\}$. Remark that on S we have $g\omega = kdf$, and the nonvanishing of k and df on S^* guarantees that also $g \neq 0$ on S^* .

The Camacho-Sad index [CS], [LN], [Su] is defined as

$$CS(\mathcal{F}, S, p) = -\frac{1}{2\pi i} \int_{\partial S} \frac{1}{k} \eta.$$

For example, let $\mathcal{F}$ be generated by $v(z, w) = z(\lambda + \dots) \frac{\partial}{\partial z} + w(\mu + \dots) \frac{\partial}{\partial w}$, where the dots denote terms vanishing at $(0, 0) = p$ and $\lambda, \mu \neq 0$. Let $S_1 = \{z = 0\}$, $S_2 = \{w = 0\}$. Then [Su]

$$CS(\mathcal{F}, S_1, p) = \frac{\lambda}{\mu}$$

$$CS(\mathcal{F}, S_2, p) = \frac{\mu}{\lambda}$$

$$CS(\mathcal{F}, S_1 \cup S_2, p) = 2 + \frac{\lambda}{\mu} + \frac{\mu}{\lambda}.$$

This example shows that $CS(\mathcal{F}, \cdot, p)$ is not additive on the set of separatrices (whereas $\text{Var}(\mathcal{F}, \cdot, p)$ is). More precisely [Su], if $S = S_1 \cup S_2$ then

$$CS(\mathcal{F}, S, p) = CS(\mathcal{F}, S_1, p) + CS(\mathcal{F}, S_2, p) + 2(S_1 \cdot S_2)_p$$

where $(S_1 \cdot S_2)_p$ is the local intersection number of S_1 and S_2 at p .

If $S \subset X$ is a compact holomorphic curve invariant by $\mathcal{F}$, one obtains the following formula (see [KS] for a direct proof, without desingularization).

Proposition 3 [CS], [LN], [Su].

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap S} CS(\mathcal{F}, S, p) = S \cdot S.$$

Remark the consistency of this formula with the above discussion around the non additivity of $CS(\mathcal{F}, \cdot, p)$.

3. The Gomez-Mont-Seade-Verjovsky index

We continue with hypothesis and notations of the previous section. Define

$$GSV(\mathcal{F}, S, p) = \frac{1}{2\pi i} \int_{\partial S} \frac{g}{k} d\left(\frac{k}{g}\right).$$

Let us verify that this index coincides with the one introduced in [GSV]. Let us denote by J the isomorphism from 1-forms to vector fields induced by a nonsingular holomorphic 2-form near p . The holomorphic vector field $J(\omega)$ defines $\mathcal{F}$ and it can be decomposed as a sum of two meromorphic vector fields:

$$J(\omega) = \frac{k}{g} J(df) + \frac{f}{g} J(\eta) = v_1 + v_2.$$

Observe that: i) on S we have $J(\omega) = v_1$; ii) v_1 is tangent to $S_\epsilon = \{f = \epsilon\}$, $\epsilon \neq 0$ small; iii) $v_1|_{S_\epsilon}$ has poles in correspondence of $S_\epsilon \cap \{g = 0\}$ and zeroes in correspondence of $S_\epsilon \cap \{k = 0\}$. It is then clear that the index defined in [GSV] coincides with the difference between the number of zeroes and the number of poles of $v_1|_{S_\epsilon}$, i.e. with $\frac{1}{2\pi i} \int_{\partial S_\epsilon} \frac{dk}{k} - \frac{dg}{g} = \frac{1}{2\pi i} \int_{\partial S} \frac{g}{k} d\left(\frac{k}{g}\right) = GSV(\mathcal{F}, S, p)$.

Taking the example of the previous section, one finds

$$\begin{aligned} GSV(\mathcal{F}, S_1, p) &= GSV(\mathcal{F}, S_2, p) = 1, \\ GSV(\mathcal{F}, S_1 \cup S_2, p) &= 0 \end{aligned}$$

and, as CS , also $GSV(\mathcal{F}, \cdot, p)$ is not additive on the set of separatrices. From Proposition 5 below, or by a direct computation as in [Su] for CS , one finds that if $S = S_1 \cup S_2$ then

$$GSV(\mathcal{F}, S, p) = GSV(\mathcal{F}, S_1, p) + GSV(\mathcal{F}, S_2, p) - 2(S_1 \cdot S_2)_p.$$

If $S \subset X$ is a compact curve invariant by $\mathcal{F}$ then one has the following formula (in [KS] it is formulated, in an equivalent way, using the Schwarz index, whose difference with the GSV index is the Milnor number of S at p).

Proposition 4 [Br], [KS].

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap S} GSV(\mathcal{F}, S, p) = c_1(N_{\mathcal{F}}) \cdot S - S \cdot S.$$

In fact, Proposition 4 follows also from Propositions 2 and 3 and the following relation between the three indices Var , CS , GSV .

Proposition 5. *If S is any separatrix at p then*

$$\mathrm{Var}(\mathcal{F}, S, p) = \mathrm{GSV}(\mathcal{F}, S, p) + \mathrm{CS}(\mathcal{F}, S, p).$$

Proof: Observe that, at points of ∂S :

$$\begin{aligned}\omega &= \frac{k}{g} df \\ d\omega &= \left(d\left(\frac{k}{g}\right) - \frac{1}{g}\eta \right) \wedge df\end{aligned}$$

hence $d\omega = \left(\frac{g}{k} d\left(\frac{k}{g}\right) - \frac{1}{k}\eta \right) \wedge \omega$ and consequently

$$\begin{aligned}\mathrm{Var}(\mathcal{F}, S, p) &= \frac{1}{2\pi i} \int_{\partial S} \left(\frac{g}{k} d\left(\frac{k}{g}\right) - \frac{1}{k}\eta \right) \\ &= \mathrm{GSV}(\mathcal{F}, S, p) + \mathrm{CS}(\mathcal{F}, S, p). \blacksquare\end{aligned}$$

The GSV index is, of course, an integer number, but it may be negative; this is the obstruction to the positive solution to “Poincaré problem” [Car], [Br], because in $\mathbf{CP}^2$ the inequality $c_1(N_{\mathcal{F}}) \cdot S \geq S \cdot S$ means $\deg(\mathcal{F}) + 2 \geq \deg(S)$. Here, after some definitions, we give a simple condition (related to [Car]) that implies the nonnegativity of GSV .

We recall that the singular point p is *nondicritical* if $\mathcal{F}$ has only a finite number of separatrices at p . This means that if $\pi : \tilde{X} \rightarrow X$ is the desingularization of $\mathcal{F}$ at p then $\pi^{-1}(p)$ is entirely $\tilde{\mathcal{F}}$ -invariant [CLS]. More generally, we shall say that a *separatrix* S of $\mathcal{F}$ at p is *nondicritical* if there is a sequence of blow-ups $\pi : \hat{X} \rightarrow X$, based at p , such that:

- i) π is a resolution of S , i.e. $\hat{S} = \pi^{-1}(S)$ is a curve with only normal crossing singularities;
- ii) $\pi^{-1}(p)$ is $\hat{\mathcal{F}}$ -invariant.

For instance, if p is a nondicritical singular point of $\mathcal{F}$ then any separatrix of $\mathcal{F}$ at p is nondicritical (take $\pi =$ resolution of $\mathcal{F}$ in the above definition). At the opposite case, if either S is smooth at p or S has a normal crossing singular point at p then S is certainly nondicritical, independently on $\mathcal{F}$ (take $\pi =$ identity in the above definition).

Proposition 6. *If S is a nondicritical separatrix at p then*

$$\mathrm{GSV}(\mathcal{F}, S, p) \geq 0.$$

Proof: We shall prove the more general inequality

$$GSV(\mathcal{F}, S', p) \geq (S' \cdot S'')_p$$

where $S' \subset S$ is a union of irreducible components of S and $S'' \subset S$ is the union of those irreducible components which are not in S' . Let us consider a function $f : (X, p) \rightarrow (\mathbf{C}, 0)$ such that $S = f^{-1}(0)$, but only in the set theoretic sense: if $S_1, \dots, S_N$ are the irreducible components of S and $f_1, \dots, f_N$ their reduced equations, then $f = f_1^{p_1} \cdot \dots \cdot f_N^{p_N}$, with p_j positive integer numbers. As in [CLS], the idea is to compare $\mathcal{F}$ with the foliation $\mathcal{G}_f$ given by the level curves of f , i.e. by the 1-form $f \cdot \sum_{j=1}^N p_j \frac{df_j}{f_j}$ (remark that if S is smooth then $\mathcal{G}_f$ is nonsingular, and if S has a normal crossing singular point then $\mathcal{G}_f$ is linearizable). We denote by $\text{ord}(\cdot, p)$ the order of a foliation at p , i.e. the vanishing order at p of a 1-form generating the foliation near p .

Claim.

- i) $\text{ord}(\mathcal{F}, p) \geq \text{ord}(\mathcal{G}_f, p)$
- ii) $GSV(\mathcal{F}, S', p) \geq GSV(\mathcal{G}_f, S', p)$.

This claim is proven by induction on the (minimal) number n of blow-ups appearing in the definition of nondicritical separatrix.

$n = 0$: easy verification, left to the reader.

$n - 1 \mapsto n$: let $\pi : \tilde{X} \rightarrow X$ be a blow-up at p , $D = \pi^{-1}(p)$ its exceptional divisor, $\tilde{S}_j$ the strict transform of S_j by π , $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$, $\tilde{\mathcal{G}}_f = \pi^*(\mathcal{G}_f)$. The curve D is $\tilde{\mathcal{F}}$ -invariant, by nondicriticalness of S , and clearly also $\tilde{\mathcal{G}}_f$ -invariant. Each singular point q at $\tilde{\mathcal{G}}_f$ on D is also a singular point of $\tilde{\mathcal{F}}$; near such a point $\tilde{\mathcal{G}}_f$ is given by the levels of $\tilde{f} = f \circ \pi$, and $\tilde{f}^{-1}(0)$ is a nondicritical separatrix of $\tilde{\mathcal{F}}$ at q . The divisor D is an irreducible component of this separatrix, hence, by induction hypothesis,

$$GSV(\tilde{\mathcal{F}}, D, q) \geq GSV(\tilde{\mathcal{G}}_f, D, q).$$

The same inequality holds if $q \in \text{Sing}(\tilde{\mathcal{F}}) \setminus \text{Sing}(\tilde{\mathcal{G}}_f)$, because in that case the left hand side index is at least 1 and the right hand side index is 0. Now, using Proposition 4 and the fact that $\text{ord}(\mathcal{F}, p) = c_1(N_{\tilde{\mathcal{F}}}) \cdot D$ we obtain

$$\text{ord}(\mathcal{F}, p) = -1 + \sum_{q \in D \cap \text{Sing}(\tilde{\mathcal{F}})} GSV(\tilde{\mathcal{F}}, D, q)$$

and similarly

$$\text{ord}(\mathcal{G}_f, p) = -1 + \sum_{q \in D \cap \text{Sing}(\tilde{\mathcal{G}}_f)} GSV(\tilde{\mathcal{G}}_f, D, q).$$

Hence the first desired inequality $\text{ord}(\mathcal{F}, p) \geq \text{ord}(\mathcal{G}_f, p)$ follows from $GSV(\tilde{\mathcal{F}}, D, q) \geq GSV(\tilde{\mathcal{G}}_f, D, q)$.

Let now m_j be the order of S_j at p , so that $\tilde{f}_j = f_j \circ \pi$ vanishes on D with order m_j . Set $M = \text{ord}(\mathcal{F}, p)$, so that $\tilde{\omega} = \pi^* \omega$ vanishes on D with order M . If $q_j \in D$ is the unique intersection point between $\tilde{S}_j$ and D , then near q_j we may write

$$\begin{aligned}\tilde{\omega} &= t^M \omega_0 \\ \tilde{f}_j &= t^{m_j} f_0\end{aligned}$$

where t is a local equation of D , ω_0 has an isolated zero at q_j , f_0 is a reduced equation of $\tilde{S}_j$. The usual decomposition of ω ,

$$g\omega = kdf_j + f_j\eta,$$

implies

$$\tilde{g}t^M \omega_0 = \tilde{k}t^{m_j} df_0 + f_0(m_j t^{m_j-1} \tilde{k}dt + t^{m_j} \tilde{\eta}).$$

Hence $GSV(\tilde{\mathcal{F}}, \tilde{S}_j, q_j)$ is given by

$$\begin{aligned}GSV(\tilde{\mathcal{F}}, \tilde{S}_j, q_j) &= \frac{1}{2\pi i} \int_{\partial \tilde{S}_j} \frac{\tilde{g}t^M}{\tilde{k}t^{m_j}} d \left(\frac{\tilde{k}t^{m_j}}{\tilde{g}t^M} \right) \\ &= \frac{1}{2\pi i} \int_{\partial \tilde{S}_j} \frac{\tilde{g}}{\tilde{k}} d \left(\frac{\tilde{k}}{\tilde{g}} \right) + (m_j - M) \frac{dt}{t} \\ &= \frac{1}{2\pi i} \int_{\partial S_j} \frac{g}{k} d \left(\frac{k}{g} \right) + m_j(m_j - M) \\ &= GSV(\mathcal{F}, S_j, p) + m_j(m_j - M).\end{aligned}$$

Similarly, if $M_f = \text{ord}(\mathcal{G}_f, p)$:

$$GSV(\tilde{\mathcal{G}}_f, \tilde{S}_j, q_j) = GSV(\mathcal{G}_f, S_j, p) + m_j(m_j - M_f).$$

But $GSV(\tilde{\mathcal{F}}, \tilde{S}_j, q_j) \geq GSV(\tilde{\mathcal{G}}_f, \tilde{S}_j, q_j)$ by induction hypothesis, and we have proven before that $M \geq M_f$, so that

$$GSV(\mathcal{F}, S_j, p) \geq GSV(\mathcal{G}_f, S_j, p)$$

and using the formula before Proposition 4 we finally obtain the second inequality of the claim.

Let us return to the proof of Proposition 6. We take as f a reduced equation of S , then the 1-form generating $\mathcal{G}_f$ is simply df , so that $GSV(\mathcal{G}_f, S', p) = (S' \cdot S'')_p$ as a simple computation shows. By the previous claim we obtain the desired result:

$$GSV(\mathcal{F}, S', p) \geq (S' \cdot S'')_p. \quad \blacksquare$$

Recall that the singularity of $\mathcal{F}$ at p is said to be a *generalized curve* if it is nondicritical and there are no saddle-nodes in its resolution [CLS]. By nondicriticalness, the union of all separatrices of a generalized curve is still a separatrix, the “maximal” one. In this case Proposition 6 can be improved

Proposition 7. *If the singularity of $\mathcal{F}$ at p is a generalized curve and if S is the union of all separatrices then*

$$GSV(\mathcal{F}, S, p) = 0.$$

Proof: Using the same notations as in the previous proof, we have now that the inequalities of the claim become equalities [CLS]. In fact, looking at that proof, one sees that the only possibility to have a strict inequality is that $\tilde{\mathcal{G}}_f$ has on D less singularities than $\tilde{\mathcal{F}}$, and this cannot happen in our situation [CLS], so we can prove the claim with equalities instead of inequalities. Hence we obtain $GSV(\mathcal{F}, S', p) = (S' \cdot S'')_p$ and in particular $GSV(\mathcal{F}, S, p) = 0$. $\blacksquare$

Remark. Suppose that S is a nondicritical separatrix satisfying $GSV(\mathcal{F}, S, p) = 0$, then, again, all inequalities appearing in Proposition 6 and its proof must become equalities, because we are in an extremal case. This means that the resolution of $\mathcal{G}_f$ is almost a resolution of $\mathcal{F}$. More precisely, if $\pi : \tilde{X} \rightarrow X$ is the desingularization of S which appears in the definition of nondicritical separatrix, then $\pi^*(\mathcal{F})$ has only nondegenerate singularities, and all these singularities are in correspondence with the normal crossing points of $\pi^{-1}(S)$. The only difference with the generalized curve case is that some of these singularities can be dicritical (like $nzdw - mwdz = 0$, with n and m positive integers).

4. The case of almost Liouvillean singularities

The singularity of $\mathcal{F}$ at p is called *almost Liouvillean* if there exist a closed meromorphic 1-form γ_0 and a holomorphic 1-form γ_1 near p such that

$$d\omega = (\gamma_0 + \gamma_1) \wedge \omega.$$

If one can choose $\gamma_1 = 0$ then p is called *Liouvillean*. The polar divisor of $\gamma = \gamma_1 + \gamma_0$ is invariant by $\mathcal{F}$: near a point $q \in (\gamma)_\infty \setminus \{p\}$ we may choose coordinates (z, w) such that $\omega = adz$, $a \in \mathcal{O}^*$; if $\gamma_0 = b dz + c dw$, b, c meromorphic, then $d\gamma_0 = 0$ means $b_w = c_z$ and $d\omega = \gamma \wedge \omega$ implies that c is holomorphic; hence c_z also is holomorphic and consequently the poles of b (i.e. of γ) are invariant by $\mathcal{F}$.

Let $\cup_{j=1}^N S_j$ be the decomposition of $S = (\gamma)_\infty$ into irreducible components. Each S_j is a separatrix of $\mathcal{F}$ at p ; we denote by $\text{Res}(\gamma_0, S_j)$ the residue of γ_0 around S_j . We will say that the almost Liouvillean singularity is *simple* if one can choose γ_0 having only first order poles.

Proposition 8. *If the singularity of $\mathcal{F}$ at p is a simple almost Liouvillean singularity then*

$$BB(\mathcal{F}, p) = \sum_{j=1}^N \text{Res}(\gamma_0, S_j) \cdot \text{Var}(\mathcal{F}, S_j, p).$$

Proof: Let γ' be a holomorphic 1-form on a tubular neighborhood V of ∂S such that $d\omega = \gamma' \wedge \omega$. Let $\phi \in C_c^\infty(V)$ be equal to 1 on a smaller neighborhood of ∂S . Then $\beta = \phi\gamma' + (1 - \phi)\gamma$ is a smooth $(1, 0)$ -form on a neighborhood of $\mathbf{S}^3$, and $d\omega = \beta \wedge \omega$. A simple computation gives $\beta \wedge d\beta = d\phi \wedge \gamma' \wedge \gamma$ and so

$$\begin{aligned} \int_{\mathbf{S}^3} \beta \wedge d\beta &= \int_{\mathbf{S}^3 \cap V} \beta \wedge d\beta = \int_{\mathbf{S}^3 \cap V} d((1 - \phi)\gamma \wedge \gamma') \\ &= \int_{\partial(\mathbf{S}^3 \cap V)} \gamma \wedge \gamma' = \int_{\partial(\mathbf{S}^3 \cap V)} \gamma_0 \wedge \gamma'. \end{aligned}$$

We may choose holomorphic coordinates (z, w) near each ∂S_j , with z varying on a neighborhood of the unit circle and w on a neighborhood of zero, such that $S_j = \{w = 0\}$ and $\partial S_j = \{w = 0, |z| = 1\}$. Then, on the domain of these coordinates, $\gamma_0 = \lambda_j \frac{dw}{w} + \gamma_{0j}$, where $\lambda_j = \text{Res}(\gamma_0, S_j)$ and γ_{0j} is holomorphic, and $\gamma' = adz + b dw$ with a, b holomorphic functions. Moreover, we may assume that the small sphere $\mathbf{S}^3$ (which is not necessarily a round sphere) and the neighborhood V are chosen so

that $V = \cup_{j=1}^N V_j$, $V_j =$ neighborhood of ∂S_j , $\mathbf{S}^3 \cap V_j = \{|w| < \epsilon, |z| = 1\}$. Hence the previous integral becomes a sum of terms of the form

$$\begin{aligned} \int_{|w|=\epsilon, |z|=1} \lambda_j \frac{dw}{w} \wedge adz &= 2\pi i \lambda_j \int_{|z|=1} a(z, 0) dz \\ &= 2\pi i \lambda_j \int_{\partial S_j} \gamma' = (2\pi i)^2 \lambda_j \text{Var}(\mathcal{F}, S_j, p) \end{aligned}$$

and finally

$$BB(\mathcal{F}, p) = \sum_{j=1}^N \lambda_j \text{Var}(\mathcal{F}, S_j, p). \quad \blacksquare$$

Example. Let $\mathcal{F}$ be generated by

$$\omega = z(\lambda + \dots) dw - w(\mu + \dots) dz \quad (\lambda, \mu \neq 0)$$

and let α, β be any two complex numbers such that $\alpha\lambda + \beta\mu = \lambda + \mu$. Then if we set $\gamma_0 = \alpha \frac{dz}{z} + \beta \frac{dw}{w}$, we may find a suitable holomorphic 1-form γ_1 such that $d\omega = (\gamma_0 + \gamma_1) \wedge \omega$. Let $S_1 = \{z = 0\}$, $S_2 = \{w = 0\}$, then

$$\text{Var}(\mathcal{F}, S_1, p) = \frac{\lambda}{\mu} + 1$$

$$\text{Var}(\mathcal{F}, S_2, p) = \frac{\mu}{\lambda} + 1$$

and, using Proposition 8,

$$BB(\mathcal{F}, p) = \alpha \left(\frac{\lambda}{\mu} + 1 \right) + \beta \left(\frac{\mu}{\lambda} + 1 \right) = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2.$$

Example. Let $\mathcal{F}$ be generated by

$$\omega = z^{p+1} dw - w(1 + \lambda z^p) dz, \quad p \geq 1, \lambda \in \mathbf{C}$$

(the formal normal form of a saddle-node), then $d\omega = \gamma \wedge \omega$ with $\gamma = (p+1) \frac{dz}{z} + \frac{dw}{w}$. If $S_1 = \{z = 0\}$, $S_2 = \{w = 0\}$, one finds

$$CS(\mathcal{F}, S_1, p) = 0, \quad CS(\mathcal{F}, S_2, p) = \lambda$$

$$GSV(\mathcal{F}, S_1, p) = 1, \quad GSV(\mathcal{F}, S_2, p) = p + 1$$

$$BB(\mathcal{F}, p) = (p+1)(1+0) + 1(p+1+\lambda) = 2p+2+\lambda.$$

5. The case of generalized curves

The second statement of the theorem mentioned in the introduction has been proved in Proposition 7. We now prove the first statement, in a form which is a little more general (see the remark after Proposition 7).

Proposition 9. *If S is a nondicritical separatrix with the property $GSV(\mathcal{F}, S, p) = 0$ then*

$$BB(\mathcal{F}, p) = CS(\mathcal{F}, S, p).$$

Proof: Let $\pi : \tilde{X} \rightarrow X$ be the desingularization of S at p appearing in the definition of nondicritical separatrix, $D = \cup_{j=1}^N D_j = \pi^{-1}(p)$ the exceptional divisor. Let $U \subset X$ be a (small) spherical neighborhood of p and $\tilde{U} = \pi^{-1}(U)$, $\tilde{\mathcal{F}} = \pi^*(\mathcal{F}|_U)$. Let $\omega \in \Omega^1(U)$ be a 1-form generating $\mathcal{F}$; the 1-form $\tilde{\omega} = \pi^*\omega$ vanishes on every D_j , with vanishing order m_j , so that

$$c_1(N_{\tilde{\mathcal{F}}}^*) = \sum_{j=1}^N m_j [D_j].$$

Let $p_1, \dots, p_M$ be the singularities of $\tilde{\mathcal{F}}$ which are not corners of D , and let $p_{M+1}, \dots, p_L$ be the ones which coincide with corners. We clearly have (see section 1) $BB(\mathcal{F}, p) = BB(\tilde{\mathcal{F}}, \partial\tilde{U})$ and hence, by Proposition 1:

$$BB(\mathcal{F}, p) = \sum_{j=1}^L BB(\tilde{\mathcal{F}}, p_j) - c_1^2(N_{\tilde{\mathcal{F}}}).$$

Every p_j is a nondegenerate singularity with exactly two separatrices contained in $\pi^{-1}(S)$. If $j = 1, \dots, M$ we denote by S_j^0 the separatrix transverse to D , which projects by π to an irreducible component $\hat{S}_j$ of S , and by S_j^1 the separatrix contained in D . If $j = M+1, \dots, L$ we denote by S_j^1 the union of the two separatrices, both contained in D . By the previous computations:

$$BB(\tilde{\mathcal{F}}, p_j) = \text{Var}(\tilde{\mathcal{F}}, S_j^0, p_j) + \text{Var}(\tilde{\mathcal{F}}, S_j^1, p_j) \quad \text{if } 1 \leq j \leq M$$

$$BB(\tilde{\mathcal{F}}, p_j) = \text{Var}(\tilde{\mathcal{F}}, S_j^1, p_j) \quad \text{if } M+1 \leq j \leq L.$$

By Proposition 2 the sum of terms $\text{Var}(\tilde{\mathcal{F}}, S_j^1, p_j)$ is equal to $c_1(N_{\tilde{\mathcal{F}}}) \cdot D$, and so we find

$$BB(\mathcal{F}, p) = \sum_{j=1}^M \text{Var}(\tilde{\mathcal{F}}, S_j^0, p_j) + c_1(N_{\tilde{\mathcal{F}}}) \cdot D - c_1^2(N_{\tilde{\mathcal{F}}}).$$

We now study the relation between $\text{Var}(\tilde{\mathcal{F}}, S_j^0, p_j)$ and $\text{Var}(\mathcal{F}, \hat{S}_j, p)$, by an argument similar to that used in Propositions 6 and 7. Let ω_0 be a 1-form generating $\tilde{\mathcal{F}}$ near p_j and vanishing only at p_j ; hence $\tilde{\omega} = h\omega_0$ where h vanishes on D at order m_k , k being the index such that $p_j \in D_k$. Let β be a smooth $(1, 0)$ -form near $\partial\hat{S}_j$ such that $d\omega = \beta \wedge \omega$, and let $\tilde{\beta} = \pi^*\beta$, so that near ∂S_j^0 we have $d\tilde{\omega} = \tilde{\beta} \wedge \tilde{\omega}$. We obtain, again near ∂S_j^0 , $d\omega_0 = \left(\tilde{\beta} - \frac{dh}{h}\right) \wedge \omega_0$, hence

$$\begin{aligned} \text{Var}(\tilde{\mathcal{F}}, S_j^0, p_j) &= \frac{1}{2\pi i} \int_{\partial S_j^0} \tilde{\beta} - \frac{dh}{h} \\ &= \frac{1}{2\pi i} \int_{\partial \hat{S}_j} \beta - \frac{1}{2\pi i} \int_{\partial S_j^0} \frac{dh}{h} = \text{Var}(\mathcal{F}, \hat{S}_j, p) - m_k. \end{aligned}$$

If l_k , $k = 1, \dots, N$, is the number of singularities of $\tilde{\mathcal{F}}$ on D_k different from corners, we obtain

$$\begin{aligned} \sum_{j=1}^M \text{Var}(\tilde{\mathcal{F}}, S_j^0, p_j) &= \sum_{j=1}^M \text{Var}(\mathcal{F}, \hat{S}_j, p) - \sum_{k=1}^N l_k m_k \\ &= \text{Var}(\mathcal{F}, S, p) - \sum_{k=1}^N l_k m_k \end{aligned}$$

and hence

$$BB(\mathcal{F}, p) = \text{Var}(\mathcal{F}, S, p) - \sum_{k=1}^N l_k m_k + c_1(N_{\tilde{\mathcal{F}}}) \cdot D - c_1^2(N_{\tilde{\mathcal{F}}}).$$

On the other hand, the numbers l_k can be computed from Proposition 4: if c_k is the numbers of corners on D_k then

$$l_k + c_k = c_1(N_{\tilde{\mathcal{F}}}) \cdot D_k - D_k^2$$

but $c_k = D_k \cdot D - D_k^2$, so that

$$l_k = c_1(N_{\tilde{\mathcal{F}}}) \cdot D_k - D \cdot D_k = (c_1(N_{\tilde{\mathcal{F}}}) - D) \cdot D_k.$$

As a consequence, we have

$$\begin{aligned} \sum_{k=1}^N l_k m_k &= (c_1(N_{\tilde{\mathcal{F}}}) - D) \cdot \sum_{k=1}^N m_k D_k \\ &= (c_1(N_{\tilde{\mathcal{F}}}) - D) \cdot c_1(N_{\tilde{\mathcal{F}}}^*) = c_1(N_{\tilde{\mathcal{F}}}) \cdot D - c_1^2(N_{\tilde{\mathcal{F}}}) \end{aligned}$$

and finally

$$BB(\mathcal{F}, p) = \text{Var}(\mathcal{F}, S, p).$$

The proof is then achieved by Propositions 5 and 7. ■

Remark that by Proposition 5 and the additivity of Var , this result can be reformulated as follows:

$$BB(\mathcal{F}, p) = \sum_{j=1}^M [CS(\mathcal{F}, S_j, p) + GSV(\mathcal{F}, S_j, p)]$$

where $\cup_{j=1}^M S_j$ is any decomposition of S (for example, the decomposition into irreducible components).

6. Some global remarks

Let X be a compact surface and let $\mathcal{F}$ be a holomorphic foliation on X leaving invariant a compact curve $S \subset X$. If S is nondicritical (i.e., it is a nondicritical separatrix at every singular point of $\mathcal{F}$ on S) the Propositions 4 and 6 imply the inequality

$$S \cdot S \leq c_1(N_{\mathcal{F}}) \cdot S.$$

We study here the limit case $S \cdot S = c_1(N_{\mathcal{F}}) \cdot S$, as it was done (among other things) in [CL] for the case of a curve in $\mathbf{CP}^2$ with only normal crossing singularities.

The first remark is that $c_1(N_{\mathcal{F}}) \cdot S = S \cdot S$ implies that for every irreducible component S_j we also have

$$c_1(N_{\mathcal{F}}) \cdot S_j = S \cdot S_j.$$

This holds because for every $p \in \text{Sing}(\mathcal{F}) \cap S$ we have not only $GSV(\mathcal{F}, S, p) = 0$ but also $GSV(\mathcal{F}, S_j, p) = (S_j \cdot \overline{S \setminus S_j})_p$ (see section 3), and hence

$$\begin{aligned} c_1(N_{\mathcal{F}}) \cdot S_j &= S_j \cdot S_j + \sum_{p \in \text{Sing}(\mathcal{F}) \cap S_j} GSV(\mathcal{F}, S_j, p) \\ &= S_j \cdot S_j + \sum_p (S_j \cdot \overline{S \setminus S_j})_p = S_j \cdot S. \end{aligned}$$

In other words, the line bundle $N_{\mathcal{F}}^* \otimes \mathcal{O}(S)$ is topologically trivial on S :

$$c_1(N_{\mathcal{F}}^* \otimes \mathcal{O}(S)) \cdot S_j = 0 \quad \forall j = 1, \dots, N.$$

The second remark, contained in [CLS] when the singularities of $\mathcal{F}$ are generalized curves, is the following. Near $p \in \text{Sing}(\mathcal{F}) \cap S$ choose a holomorphic 1-form ω generating $\mathcal{F}$ and a reduced equation f of S , so that $\frac{\omega}{f}$ has a first order pole on S . Let $\pi : \tilde{X} \rightarrow X$ be the resolution of S at p appearing in the definition of nondicriticalness and set $\tilde{\omega} = \pi^*\omega$, $\tilde{f} = f \circ \pi$. Because $GSV(\mathcal{F}, S, p) = 0$, we have also $\text{ord}(\mathcal{F}, p) = \text{ord}(\mathcal{G}_f, p)$, that is the vanishing order of $\tilde{f}$ on every component of $\pi^{-1}(p)$ is equal to the vanishing order of $\tilde{\omega}$ plus 1 (this is checked, as usual, by induction). That is, $\frac{\tilde{\omega}}{\tilde{f}}$ has again a first order pole on $\pi^{-1}(S)$. Or, equivalently, $\tilde{S} = \pi^{-1}(S)$ is still a nondicritical $\tilde{F}$ -invariant curve satisfying the extremal equality $c_1(N_{\tilde{\mathcal{F}}}) \cdot \tilde{S} = \tilde{S} \cdot \tilde{S}$.

After these two remarks, we can state the following proposition, based on a theorem of Deligne [De], [Nog].

Proposition 10. *Let X be a compact Kähler surface having the property that the “real Chern class map” $H^1(X, \mathcal{O}^*) \xrightarrow{c} H^2(X, \mathbf{R})$ is injective. Let $\mathcal{F}$ be a holomorphic foliation on X leaving invariant a nondicritical curve S such that:*

- i) $c_1(N_{\mathcal{F}}) \cdot S = S \cdot S$
- ii) $S \cdot S \leq c_1^2(N_{\mathcal{F}})$
- iii) *the intersection form of X restricted to the subspace generated by the irreducible components $S_1, \dots, S_N$ of S has at least one positive eigenvalue, i.e. there exist $m_1, \dots, m_N \in \mathbf{Z}$ such that $(\sum m_j S_j)^2 > 0$.*

Then $\mathcal{F}$ is generated by a closed meromorphic 1-form Ω , having S as first order polar divisor.

Proof: Let us consider the line bundle $L = N_{\mathcal{F}}^* \otimes \mathcal{O}(S)$. By i) and ii):

$$c_1^2(L) = c_1^2(N_{\mathcal{F}}) + S^2 - 2c_1(N_{\mathcal{F}}) \cdot S = c_1^2(N_{\mathcal{F}}) - S^2 \geq 0.$$

On the other hand, we have seen that $c_1(L) \cdot S_j = 0$ for every j . By iii) and Hodge index theorem [BPV, ch. IV], we must have

$$c_1^2(L) \leq 0.$$

Hence the only possibility is $c_1^2(L) = 0$, and, again by Hodge index theorem, the real Chern class of L is zero. The hypothesis on X implies that L is the trivial line bundle, that is $N_{\mathcal{F}} = \mathcal{O}(S)$ and so $\mathcal{F}$ can be generated by a meromorphic 1-form Ω such that:

- a) the zero divisor $(\Omega)_0$ is empty
- b) the polar divisor $(\Omega)_{\infty}$ is S , and it has order one.

It remains to prove that Ω is closed. Let $\pi : \tilde{X} \rightarrow X$ be the composition of the resolutions of the singular points of S , so that $\tilde{S} = \pi^{-1}(S)$ is a normal crossing curve, invariant by $\tilde{\mathcal{F}} = \pi^{-1}(\mathcal{F})$. As remarked before, $\tilde{\Omega} = \pi^*(\Omega)$ has on $\tilde{S}$ a first order pole. Because $\tilde{S}$ is invariant, $d\tilde{\Omega}$ has the same property. In other words, $\tilde{\Omega}$ is a so called “logarithmic form”, and a theorem of Deligne [De], [Nog] asserts that $\tilde{\Omega}$ is closed. Of course, this means that also Ω is closed. ■

Three remarks on hypotheses:

- 1) the hypothesis on the injectivity of the real Chern class map can be omitted, but then the conclusion must be twisted with a flat line bundle: Ω will be a closed meromorphic 1-form with values in a suitable flat line bundle;
- 2) by Proposition 3 and 9, the hypothesis i) implies that $S \cdot S = \sum_{p \in \text{Sing}(\mathcal{F}) \cap S} BB(\mathcal{F}, p)$, whereas by Proposition 1 $c_1^2(N_{\mathcal{F}}) = \sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p)$. Hence i) implies ii) if $BB(\mathcal{F}, p) \geq 0$ for every singular point outside S , and in particular if S contains all the singularities of $\mathcal{F}$;
- 3) obviously i) implies ii) also if the rank of the Néron-Severi group of X [BPV] is one.

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