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MOLECULES AND LINEARLY ORDERED IDEALS OF MV-ALGEBRAS

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Abstract

We show that an ideal I of an MV-algebra A is linearly ordered if and only if every non-zero element of I is a molecule. The set of molecules of A is contained in $\mathrm{Inf}(A) \cup B_2(A)$ where $B_2(A)$ is the set of all elements $x \in A$ such that $2x$ is idempotent. It is shown that $I \neq \{0\}$ is weakly essential if and only if $B^{\perp} \subset B(A)$. Connections are shown among the classes of ideals that have various combinations of the properties of being implicative, essential, weakly essential, maximal or prime.

1. Introduction

In [**8**] and [**15**] we deduced various properties of MV-algebras on the assumption that certain ideals were linearly ordered. It was also shown in [**8**] that atoms generate linearly ordered ideals. In this paper, we characterize completely those elements that generate linearly ordered ideals. We also characterize linearly ordered ideals in terms of a property of their elements. This property is the concept of a molecule.

Molecules were introduced by A. Abian in [**1**] as a generalization of atoms. Not much has been done using this concept although there was a paper [**20**] by Yaqub on the molecules of Post algebras. Recall that a non-zero element m of a partially ordered set P with zero is a molecule if every two non-zero elements of P which are less than or equal to m have a non-zero lower bound. It was shown in [**1**] that an element of a Boolean algebra is a molecule if and only if it is an atom. In general, molecules exist even when atoms do not; and in a chain, there is at most one atom while every non-minimum element is a molecule.

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We shall characterize molecules in MV-algebras and deduce some of their properties. We shall also show when the orthogonal complement of the ideal generated by a molecule is implicative. Other properties of ideals such as "essential" and "weakly essential" are also studied. We shall show connections among the classes of ideals that have various combinations of the properties of being implicative, essential, weakly essential, maximal or prime.

In order not to lengthen the paper, we shall not review the definitions and basic concepts of MV-algebras. Rather, we refer the reader to the references for the theory of BCK and MV-algebras, in particular to [**2**], [**4**], [**8**], [**16**], [**17**], [**18**] and [**19**]. We shall follow the notation and terminology of [**8**] and shall assume the results there without further reference, as well as the results in [**2**] and [**4**]. We shall state explicitly the results of [**14**] and [**15**] that we need, if and when we need them. We use freely the BCK-algebra operation found in [**8**] and [**18**] as we feel that computations involving the BCK-algebra operation are often more transparent than those using just the MV-algebra operations.

In this paper, A shall denote a general MV-algebra, $B(A)$ its Boolean subalgebra of idempotents, $At(A)$ its set of atoms, and $\mathrm{Mol}(A)$ its set of molecules. I shall denote an ideal of A. In order to avoid trivialities, it shall always be assumed that $A \neq \{0, 1\}$. We shall also denote $B(A)-\{1\}$ by $B_1(A)$. Then the only idempotent in $A - B_1(A)$ is 1.

Recall that I is implicative if whenever $x^n \in I$ for some integer $n \geq 1$, then $x \in I$ (see [**8**, Theorem 3.6] and [**11**, Theorem 2.4]). These are precisely the ideals which give quotients A/I which are Boolean algebras. Let $\mathrm{Inf}(A) = \{x \in A \mid x^2 = 0\} = \{x \wedge \overline{x} \mid x \in A\}$, $N(A) = \{x \in A \mid x^n = 0$ for some integer $n \geq 1\}$ and $\mathrm{Rad}(A) =$ intersection of all maximal ideals of A. Observe that I is implicative if and only if $\mathrm{Inf}(A) \subset I$ (see [**11**]). Also $\mathrm{Rad}\, A = \{x \in A | nx \leq \overline{x}$ for all integers $n \geq 0\}$ (see [**6**], [**7**] and [**11**]). Let $I\,\mathrm{Rad}(A)$ be the intersection of all implicative ideals of A. It was observed in [**11**] that we have $\mathrm{Rad}(A) \subset \mathrm{Inf}(A) \subset N(A) \subset I\,\mathrm{Rad}(A)$, and $\langle \mathrm{Inf}(A) \rangle = \langle N(A) \rangle = I\,\mathrm{Rad}(A)$.

Recall that I is essential if $I^{\perp} = \{0\}$. Also $I \neq \{0\}$ is weakly essential if for all ideals J such that $J \cap \{A - B_1(A)\} \neq \emptyset$ we have $I \cap J \neq \{0\}$ (see [**10**]). Observe that if J is a proper ideal, then $J \cap \{A - B_1(A)\}$ is idempotent free. Thus $J \cap \{A - B_1(A)\} \neq \emptyset$ means that it contains a non-zero non-idempotent. It was shown in [**10**, Theorem 3.15] that if J is a proper ideal, then $J \cap \{A - B_1(A)\} \neq \emptyset$ if and only if $J \cap \mathrm{Inf}(A) \neq \{0\}$. It is easily verified that essential ideals are weakly essential (see [**10**]). In [**15**, Theorem 2.13] we showed that if $I \neq \{0\}$ is implicative, then it is weakly essential, and in [**15**, Theorem 3.11], we showed that if I is weakly essential, then $I^{\perp} \subset B(A)$. We also showed in [**15**, Theorem 2.3] that if I

is prime then $I^\perp$ is linearly ordered. It was also shown in [**8**, Lemma 5.1] that if I is linearly ordered and contains an idempotent $x \neq 0$, then x is the largest element of I. Finally, observe that if $a \in At(A)$, then either $a^2 = 0$ or $a^2 = a$, that is, either $a \in \mathrm{Inf}(A)$ or $a \in B(A)$. Thus we have $At(A) \subset \mathrm{Inf}(A) \cup B(A)$.

2. Molecules

Definition 2.1. A non-zero element m of a poset P with 0 is a molecule if whenever $0 < x,\ y \leq m$, then $\{x, y\}$ a non-zero lower bound.

Thus $m \in A$ is a molecule if and only if whenever $x,\ y \in A$ satisfy $0 < x,\ y \leq m$, then $x \wedge y > 0$. Let $\mathrm{Mol}(A)$ denote the set of all molecules of A. Observe that $At(A) \subset \mathrm{Mol}(A)$.

Recall that if $\emptyset \neq X \subset A$, then $X^\perp = \{a \in A \mid a \wedge x = 0$ for all $x \in X\}$. Then $X^\perp$ is an ideal of A; it is proper if $X \neq \{0\}$ (see [**2**] and [**8**]). The ideal $\langle X \rangle$ generated by X is the set of all $a \in A$ such that $a \leq k_1x_1 + \cdots + k_nx_n$ for some integer $n \geq 1$, integers $k_1, \ldots, k_n \geq 0$, and $x_1, \ldots, x_n \in X$. We shall denote $\langle \{a\} \rangle$ by $\langle a \rangle$. Observe that $\{a\}^\perp = \langle a \rangle^\perp$.

Theorem 2.2. *$m \in \mathrm{Mol}(A)$ if and only if $\langle m \rangle^\perp$ is a prime ideal.*

Proof: Suppose that $m \in \mathrm{Mol}(A)$. Then $\langle m \rangle^\perp$ is a proper ideal of A. Suppose that $x \wedge y \in \langle m \rangle^\perp$, that is, $x \wedge y \wedge m = 0$. If $x \notin \langle m \rangle^\perp$ and $y \notin \langle m \rangle^\perp$, then we have $0 < x \wedge m,\ y \wedge m \leq m$, and hence $x \wedge y \wedge m > 0$, a contradiction. Thus either $x \in \langle m \rangle^\perp$ or $y \in \langle m \rangle^\perp$, proving that $\langle m \rangle^\perp$ is prime. Conversely, suppose that $\langle m \rangle^\perp$ is a prime ideal of A. Suppose that $x,\ y \in A$ satisfy $0 < x,\ y \leq m$. Then $x = x \wedge m \neq 0$ and $y = y \wedge m \neq 0$, that is $x \notin \langle m \rangle^\perp$ and $y \notin \langle m \rangle^\perp$. This means that $x \wedge y \notin \langle m \rangle^\perp$, and hence $x \wedge y \wedge m \neq 0$. Thus $x \wedge y > 0$. ■

In [**8**, Theorem 4.14], it was shown that $I^\perp$ is prime if and only if I is linearly ordered and $\neq \{0\}$. Consequently, we have the following result.

Corollary 2.3. *$m \in \mathrm{Mol}(A)$ if and only if $\langle m \rangle$ is linearly ordered and $\neq \{0\}$.*

Theorem 2.4. *I is linearly ordered if and only if every non-zero element of I is a molecule.*

Proof: Suppose that I is linearly ordered and $0 \neq m \in I$. Then $\langle m \rangle$ is linearly ordered and $\neq \{0\}$. Hence $m \in \mathrm{Mol}(A)$. Conversely suppose that every non-zero element of I is a molecule of A. Let $m_1, m_2 \in I$. Then $(m_1 * m_2) \wedge (m_2 * m_1) = 0$. If m_1 and m_2 are not comparable, then $m_1 * m_2 > 0$ and $m_2 * m_1 > 0$. Now $0 < m_1 * m_2$, $m_2 * m_1 \leq m_1 + m_2$. Since $m_1 + m_2 \in \mathrm{Mol}(A)$, we have that $(m_1 * m_2) \wedge (m_2 * m_1) > 0$, a contradiction. Hence either $m_1 * m_2 = 0$ or $m_2 * m_1 = 0$, that is, either $m_1 \leq m_2$ or $m_2 \leq m_1$. ■

Corollary 2.5. *If $m \in \mathrm{Mol}(A)$, then $km \in \mathrm{Mol}(A)$ for each integer $k \geq 1$.*

Theorem 2.6. *Let $a \in At(A)$ and $m \in \mathrm{Mol}(A)$. If $a \wedge m \neq 0$, then $a + m \in \mathrm{Mol}(A)$.*

Proof: Suppose that $0 < x, y \leq a+m$. Then $x*m \leq a$ and $y*m \leq a$. ■

We have several possibilities:

(i) $x * m = 0$ and $y * m = 0$,
(ii) $x * m = a$, and $y * m = a$, and
(iii) $x * m = a$ and $y * m = 0$, or $y * m = a$ and $x * m = 0$.

In case (i), we have $0 < x$, $y \leq m$ and hence $x \wedge y > 0$. In case (ii), we have $(x \wedge y) * m = (x * m) \wedge (y * m) = a > 0$ and hence $x \wedge y > 0$. In case (iii) we need only consider the possibility $x * m = a$ and $y * m = 0$, that is, $y \leq m$. Then we also have $x * a \leq m$. We claim that $x * a > 0$. For if $x * a = 0$, then $0 < x \leq a$ and hence $x = a$. This means that $a * m = a$, that is, $a \wedge m = 0$, contradicting the assumption that $a \wedge m \neq 0$. Thus we have $0 < x * a$, $y \leq m$. Hence $x \wedge y \geq (x * a) \wedge y > 0$.

Theorem 2.7. *Let $m \in \mathrm{Mol}(A)$. Then for each $e \in B(A)$, either $m \leq e$ or $m \leq \overline{e}$.*

Proof: We may assume that $e \neq 0, 1$. If both $me > 0$ and $m\overline{e} > 0$, we have $0 < me$, $m\overline{e} \leq m$ and hence $(me) \wedge (m\overline{e}) > 0$, that is, $m(e \wedge \overline{e}) > 0$. But $e \wedge \overline{e} = 0$, a contradiction. Hence either $me = 0$ or $m\overline{e} = 0$, that is, either $m \leq \overline{e}$ or $m \leq e$. ■

Theorem 2.8. *If $B(A) \neq \{0, 1\}$, then $1 \notin \mathrm{Mol}(A)$.*

Proof: By hypothesis, there exists $e \in B(A)$ with $e \neq 0, 1$ and hence $\overline{e} \neq 0, 1$. Then $0 < e$, $\overline{e} < 1$, and hence if $1 \in \mathrm{Mol}(A)$, this is a contradiction because $e \wedge \overline{e} = 0$. ■

Recall that the order of an element $x \in A$, $\operatorname{ord} x$, was defined in [**4**] as the smallest positive integer n such that $nx = 1$. If no such integer exists, then $\operatorname{ord} x = \infty$.

Corollary 2.9. *If* $B(A) \neq \{0, 1\}$, *then for each* $m \in \mathrm{Mol}(A)$, $\operatorname{ord} m = \infty$.

Proof: If $\operatorname{ord} m = n$ then $nm = 1$. Hence $1 \in \mathrm{Mol}(A)$, a contradiction. ■

Definition 2.10. Let $B_2(A) = \{a \in A \mid 2a = 3a\}$.

Proposition 2.11. $B_2(A) = \{a \in A \mid 2a \in B(A)\}$.

Proof: If $a \in B_2(A)$, then $2a = 3a$ and hence $2a \in B(A)$. Conversely if $2a \in B(A)$, then $2a = 4a$. Hence $(3a) * (2a) = (3a) * (4a) = 0$. This means that $2a = 3a$. ■

Note. Of course $B(A) \subset B_2(A)$.

Theorem 2.12. $\mathrm{Mol}(A) \subset \mathrm{Inf}(A) \cup B_2(A)$.

Proof: Let $m \in \mathrm{Mol}(A)$. If $m \notin \mathrm{Inf}(A)$ and $m \notin B_2(A)$, then $m^2 > 0$ and $2m \neq 3m$, that is, $\overline{m}^2 \neq \overline{m}^3$. Thus $\overline{m}^2 \wedge m = \overline{m}^2 * \overline{m}^3 \neq 0$. Hence we have $0 < m^2, \overline{m}^2 \wedge m \leq m$. This means that $m^2 \wedge \overline{m}^2 \wedge m > 0$, that is, $m^2 \wedge \overline{m}^2 > 0$. But $m^2 \wedge \overline{m}^2 = (m \wedge \overline{m})^2 = 0$, a contradiction. ■

Remark. Theorem 2.12 means that $\mathrm{Mol}(A) \cap B(A) = \emptyset$ if and only if $\mathrm{Mol}(A) \subset \mathrm{Inf}(A)$.

Theorem 2.13. *Let* $a \in At(A)$. *Then* $a < m$ *for some* $m \in \mathrm{Mol}(A)$ *if and only if* $a \in \mathrm{Inf}(A)$.

Proof: Suppose that $a < m$ for some $m \in \mathrm{Mol}(A)$. Now $At(A) \subset \mathrm{Inf}(A) \cup B(A)$. If $a \notin \mathrm{Inf}(A)$, then $a \in B(A)$, and by Theorem 2.7, either $m \leq a$ or $m \leq \overline{a}$. Hence we must have $m \leq \overline{a}$, which means that $a < m \leq \overline{a}$, giving $a^2 = 0$, a contradiction. Conversely, suppose that $a \in \mathrm{Inf}(A)$. Now, $a < a + a \in \mathrm{Mol}(A)$ by Corollary 2.5. ■

Theorem 2.14. $\mathrm{Mol}(A) \cap \mathrm{Inf}(A) = \emptyset$ *if and only if* $\mathrm{Mol}(A) \subset B(A)$.

Proof: Clearly, if $\mathrm{Mol}(A) \subset B(A)$, then if there exists $m \in \mathrm{Mol}(A) \cap \mathrm{Inf}(A)$, we would have $m \in B(A) \cap \mathrm{Inf}(A) = \{0\}$, a contradiction. Conversely, suppose that $\mathrm{Mol}(A) \cap \mathrm{Inf}(A) = \emptyset$. Let $m \in \mathrm{Mol}(A)$. If $\langle m \rangle \cap At(A) \neq \emptyset$, then we have an element $a \in \langle m \rangle \cap At(A)$. Since $\langle m \rangle$ is linearly ordered, we must have $a \leq m$. Now, $At(A) \cap \mathrm{Inf}(A) \subset$

$\mathrm{Mol}(A) \cap \mathrm{Inf}(A) = \emptyset$, that is, $At(A) \subset B(A)$. Thus $a \in B(A) \cap \langle m \rangle$ and hence a is the largest element of $\langle m \rangle$ by [**8**, Lemma 5.1]. This means that $m \leq a$, that is, $m = a \in B(A)$. On the other hand, if $\langle m \rangle \cap At(A) = \emptyset$ then by [**8**, Theorem 5.21] there exists a non-zero element $y \in \langle m \rangle$ such that $2y \leq m$. Then $y \in \mathrm{Mol}(A)$ and hence $y \in B_2(A)$ by Theorem 2.12. This means that $2y \in B(A)$, and hence $2y$ is the largest element of $\langle m \rangle$. Thus $m \leq 2y$, that is, $m = 2y \in B(A)$. ∎

Lemma 2.15. *If $m \in \mathrm{Inf}(A)$, then $\langle \overline{m} \rangle$ is essential.*

Proof: We have $m^2 = 0$, that is, $m \leq \overline{m}$. Hence $\langle m \rangle \subset \langle \overline{m} \rangle$ and hence $\langle \overline{m} \rangle^{\perp} \subset \langle m \rangle^{\perp}$. But if $x \in \langle m \rangle^{\perp}$, then $x \wedge m = 0$. Therefore, $x = x\overline{m} \leq \overline{m}$, that is, $\langle m \rangle^{\perp} \subset \langle \overline{m} \rangle$. This means that $\langle \overline{m} \rangle^{\perp} \subset \langle m \rangle^{\perp} \subset \langle \overline{m} \rangle$ and hence $\langle \overline{m} \rangle^{\perp} = \{0\}$. ∎

Theorem 2.16. *If $\mathrm{Mol}(A) \subset \mathrm{Inf}(A)$, then $At(A) \subset \mathrm{Mol}(A) \subset \mathrm{Rad}(A)$ and for each $m \in \mathrm{Mol}(A)$, $\langle \overline{m} \rangle$ is essential.*

Proof: Let $m \in \mathrm{Mol}(A)$. Then $m^2 = 0$ and $m \leq \overline{m}$. In general, if $n \geq 1$ is an integer, we have $nm \in \mathrm{Mol}(A)$ and hence $nm \leq \overline{m}^n \leq \overline{m}$. Thus $m \in \mathrm{Rad}(A)$ (see [**6**], [**7**] and [**11**] for this characterization of $\mathrm{Rad}(A)$). The rest of the theorem follows from Lemma 2.15. ∎

The hypothesis that $\mathrm{Mol}(A) \subset \mathrm{Inf}(A)$ is extremely strong, as is evidenced by the next two results.

Corollary 2.17. *Suppose that $\mathrm{Mol}(A) \subset \mathrm{Inf}(A)$ and I is a linearly ordered ideal. Then $I \subset \mathrm{Rad}(A)$, and hence if $I \neq \{0\}$, it cannot have a largest element.*

Proof: We have $I - \{0\} \subset \mathrm{Mol}(A) \subset \mathrm{Rad}(A)$ and hence $I \subset \mathrm{Rad}(A)$. If $I \neq \{0\}$ and if it has a largest element e, then $e \in B(A) \cap \mathrm{Rad}(A) = \{0\}$. ∎

Corollary 2.18. *Suppose that $\mathrm{Mol}(A) \subset \mathrm{Inf}(A)$. If there exists an implicative linearly ordered ideal I, then $\mathrm{Mol}(A) \subset I = \mathrm{Rad}(A) = \mathrm{Inf}(A) = N(A) = I\,\mathrm{Rad}(A)$, and hence every maximal ideal of A is implicative.*

Proof: Suppose that I is linearly ordered and implicative. Then $\mathrm{Inf}(A) \subset I \subset \mathrm{Rad}(A)$ and hence $I\,\mathrm{Rad}(A) = \langle \mathrm{Inf}(A) \rangle \subset I \subset \mathrm{Rad}(A) = \mathrm{Inf}(A) \subset N(A) \subset I\,\mathrm{Rad}(A)$. Hence if M is a maximal ideal of A, then $\mathrm{Inf}(A) = I\,\mathrm{Rad}(A) = \mathrm{Rad}(A) \subset M$ and hence M is implicative. ∎

3. Essential and Implicative Ideals

If $m \in \mathrm{Mol}(A)$, we have seen that $\langle m \rangle \neq \{0\}$ and is linearly ordered, and hence $\langle m \rangle^\perp$ is prime. We wish to consider when $\langle m \rangle^\perp$ is implicative as well. It was shown in [**8**, Theorems 3.7 and 3.8] that I is prime and implicative if and only if it is maximal and implicative. Generally, if $I \neq \{0\}$ is linearly ordered, then $I^\perp$ is prime. We shall answer our question for this general situation. To do so, we recall that in [**15**, Theorem 2.5] we proved the following result.

Theorem 3.1. *Suppose that I is prime and implicative. If I is not essential, then there exists $x \in B(A) \cap At(A) \subset At\big(B(A)\big)$ such that $I^\perp = \{0, x\} = \langle x \rangle$ and $I = \{y \in A \mid y \leq \overline{x}\} = \langle \overline{x} \rangle$.*

Theorem 3.2. *Suppose that $I \neq \{0\}$ is linearly ordered. Then $I^\perp$ is implicative if and only if $I = \{0, a\}$ where $a \in B(A) \cap At(A)$.*

Proof: Suppose that $I^\perp$ is implicative. Since it is prime and $0 \neq I \subset I^{\perp\perp}$, it follows by Theorem 3.1 that $I = I^{\perp\perp} = \{0, a\} = \langle a \rangle$ for some $a \in B(A) \cap At(A)$. Conversely, suppose that $I = \{0, a\}$ where $a \in B(A) \cap At(A)$. Let $x \in \mathrm{Inf}(A)$. Then $x \wedge a = 0$ or a. If $x \wedge a = a$, then $a \leq x$ and hence $a = a^2 \leq 0$, a contradiction. Hence $x \wedge a = 0$, that is, $x \in I^\perp$. Thus $\mathrm{Inf}(A) \subset I^\perp$.

For each $a \in A$, let $C(a) = \{x \in A \mid x = xa + x\overline{a}\}$. It is easily verified that $C(a) = A$ if and only if $a \in B(A)$. It was shown in [**14**, Theorem 2.3] that $x \in C(a)$ if and only if $x \wedge \overline{x} \wedge a \wedge \overline{a} = 0$, and in [**14**, Theorem 2.10] that $C(a)$ is a subalgebra of A containing $B(A)$. If $m \in \mathrm{Mol}(A)$, we can identify the subalgebra $C(m)$ of A in terms of the element m. To do so, let us recall that an ideal I generates a subalgebra $A_I = I \cup \overline{I}$, where $x \in \overline{I}$ if and only if $\overline{x} \in I$ (see [**2**] and [**8**]). ■

Theorem 3.3. *Let $m \in \mathrm{Mol}(A)$. Then $C(m) = A_{\langle m \wedge \overline{m} \rangle^\perp}$.*

Proof: If $m \in B(A)$, then $C(m) = A$ as observed above. Also $m \wedge \overline{m} = 0$ and hence $\langle m \wedge \overline{m} \rangle^\perp = A$. Thus $A_{\langle m \wedge \overline{m} \rangle^\perp} = A$. We need therefore only consider the case $m \wedge \overline{m} > 0$. Then $m \wedge \overline{m} \in \mathrm{Mol}(A)$, and $\langle m \wedge \overline{m} \rangle \neq \{0\}$ and is linearly ordered. Hence $\langle m \wedge \overline{m} \rangle^\perp$ is prime. Let $x \in C(m)$. Then $x \wedge \overline{x} \wedge m \wedge \overline{m} = 0$, that is, $x \wedge \overline{x} \in \langle m \wedge \overline{m} \rangle^\perp$, and hence $x \in \langle m \wedge \overline{m} \rangle^\perp$ or $\overline{x} \in \langle m \wedge \overline{m} \rangle^\perp$. This means that $x \in A_{\langle m \wedge \overline{m} \rangle^\perp}$. Conversely, if $x \in A_{\langle m \wedge \overline{m} \rangle^\perp}$, then $x \in \langle m \wedge \overline{m} \rangle^\perp$ or $\overline{x} \in \langle m \wedge \overline{m} \rangle^\perp$. Thus $x \wedge \overline{x} \in \langle m \wedge \overline{m} \rangle^\perp$, proving that $x \wedge \overline{x} \wedge m \wedge \overline{m} = 0$, and hence $x \in C(m)$. ■

Remark. If $m \in \mathrm{Mol}(A)$, then $\langle \overline{m} \rangle$ may not be proper. For example, if $m \in \mathrm{Inf}(A)$ then $m^2 = 0$, and hence $2\overline{m} = 1$. Thus $\langle \overline{m} \rangle = A$. However, if $m \notin N(A)$, then $m \in B_2(A)$ and we have the following result.

Theorem 3.4. *Let $m \in \mathrm{Mol}(A)$. If m is not nilpotent, then $\langle \overline{m} \rangle$ is a prime ideal.*

Proof: Since m is not nilpotent, then $\langle \overline{m} \rangle$ is proper. Let $x \in \langle m \rangle^\perp$. Then $x \wedge m = 0$, that is, $x = x * m = x\overline{m} \leq \overline{m}$. Thus $\langle m \rangle^\perp \subset \langle \overline{m} \rangle$. Since $\langle m \rangle^\perp$ is prime, it follows from the prime extension property for MV-algebras (see [**10**, Corollary 2.11]) that $\langle \overline{m} \rangle$ is prime. ■

We now characterize weakly essential ideals,

Theorem 3.5. *$I \neq \{0\}$ is weakly essential if and only if $I^\perp \subset B(A)$.*

Proof: If I is weakly essential, we have $I^\perp \subset B(A)$ by [**15**, Theorem 3.11]. Conversely, suppose that $I^\perp \subset B(A)$. Let J be an ideal such that $J \cap \{A - B_1(A)\} \neq \emptyset$, that is, there exists $x \in J$ such that $x \in A - B_1(A)$. If $x = 1$, then $J = A$ and hence $I \cap J = I \neq \{0\}$. We may therefore assume that $x < 1$. Then $x \notin B(A)$ and hence $x \notin I^\perp$. This means that we can find $y \in I$ such that $x \wedge y > 0$. Thus we have $0 \neq x \wedge y \in I \cap J$, proving that I is weakly essential. ■

We showed in [**15**, Corollary 2.9] the following result.

Theorem 3.6. *Suppose that I is prime and $\mathrm{Rad}(A) \subset I$. Then there exists a unique $e \in B(A)$ such that $I^\perp = \langle e \rangle$.*

As a result we have the following.

Theorem 3.7. *Suppose that $\mathrm{Mol}(A) \subset \mathrm{Inf}(A)$. Then every maximal ideal $I \neq \{0\}$ is essential.*

Proof: By Theorem 3.6, we have $I^\perp = \langle e \rangle$ for some $e \in B(A)$, and $I^\perp$ is linearly ordered since I is prime. Hence $e \in \mathrm{Mol}(A)$, and hence $e \in \mathrm{Mol}(A) \cap B(A) \subset \mathrm{Inf}(A) \cap B(A) = \{0\}$. Thus I is essential. ■

Theorem 3.8. *Suppose that $I \neq \{0\}$ is not essential. If I is weakly essential and prime, then $I^\perp = \langle e \rangle$ and $I = \langle \overline{e} \rangle$ for some $e \in B(A) \cap At(A) \subset At\big(B(A)\big)$.*

Proof: We have $I^\perp \subset B(A)$. Since $I^\perp$ is linearly ordered and $\neq \{0\}$, we have $I^\perp = \{0, e\}$ for some $e \in B(A)$ with $e \neq 0$. Clearly $e \in At(A)$

since if $0 < x \in A$ is such that $0 < x \leq e$, then $x \in I^{\perp}$ and hence $x = e$. Thus $e \in B(A) \cap At(A) \subset At\big(B(A)\big)$. Now, $I \subset I^{\perp\perp} = \langle \overline{e} \rangle$. But $e \wedge \overline{e} = 0 \in I$ and since $e \neq 0$, we have $e \notin I$. Hence $\overline{e} \in I$. Thus $\langle \overline{e} \rangle \subset I$, proving that $I = \langle \overline{e} \rangle$. ∎

Corollary 3.9. *Suppose that $At(A) \cap B(A) = \emptyset$. If $I \neq \{0\}$ is weakly essential and prime, then it is essential.*

Theorem 3.10. *Suppose that $I \neq \{0\}$ is weakly essential and maximal. Then I is either essential or implicative.*

Proof: We have that $I^{\perp} \subset B(A)$ and $I^{\perp}$ is linearly ordered. Hence $I^{\perp} \cap \mathrm{Rad}(A) = \{0\}$. In [**15**, Theorem 2.17], we showed the following result. Suppose that J is a linearly ordered ideal such that $J \cap \mathrm{Rad}(A) = \{0\}$. Then every element $x \in A$ can be written as $x = x_1 + x_2$ for a unique $x_1 \in J$ and $x_2 \in J^{\perp}$, and there exists a unique $e \in B(A)$ such that $J = \{x \mid x \leq e\} = \langle e \rangle$ and $J^{\perp} = \{x \mid x \leq \overline{e}\} = \langle \overline{e} \rangle$. Thus in our case, we can find $e \in B(A)$ such that $I^{\perp} = \langle e \rangle$ and $I^{\perp\perp} = \langle \overline{e} \rangle$. Let $x \in \mathrm{Inf}(A)$. Then $x = x_1 + x_2$ where $x_1 \in I^{\perp} \subset B(A)$ and $x_2 \in I^{\perp\perp}$. Thus $0 = x^2 = (x_1 + x_2)^2 = (x_1 \vee x_2)^2$ since $x_1 \wedge x_2 = 0$. Hence $0 = x_1^2 \vee x_2^2 = x_1^2 + x_2^2 = x_1 + x_2^2$ using the fact that $x_1^2 \wedge x_2^2 = 0$. Hence $x_1 = 0$ and $x_2^2 = 0$, that is, $x = x_2 \in I^{\perp\perp}$. Thus $\mathrm{Inf}(A) \subset I^{\perp\perp}$. This means that $I^{\perp\perp}$ is implicative. Also since $I \subset I^{\perp\perp}$ and I is maximal, then either $I = I^{\perp\perp}$ or $I^{\perp\perp} = A$ which means that either I is implicative or I is essential.

Using these results and the earlier mentioned result that the set of ideals that are both prime and implicative is precisely the set of ideals that are both maximal and implicative, we can state the following. ∎

Theorem 3.11. *Suppose that $I \neq \{0\}$ is not essential. Then the following are equivalent:*

(1) *I is weakly essential and maximal*
(2) *I is implicative and maximal*
(3) *I is implicative and prime.*

We have seen that if I is implicative, then $I^{\perp} \subset B(A)$. To get the converse, we need to consider only those maximal ideals which are not essential.

Theorem 3.12. *Suppose that I is maximal but not essential. Then I is implicative if and only if $I^{\perp} \subset B(A)$.*

Proof: If I is implicative, we have already shown that $I^{\perp} \subset B(A)$ in [**15**, Lemma 2.1], as mentioned above. Conversely, suppose that $I^{\perp} \subset$

$B(A)$. Since I is maximal and hence prime, we have that $I^\perp$ is linearly ordered. Also $I^\perp \neq \{0\}$. Hence $I^\perp$ can have at most one non-zero element, that is, there exists $0 \neq e \in B(A)$ such that $I^\perp = \{0, e\} = \langle e \rangle$. It is easily verified that $e \in At(A) \cap B(A) \subset At\big(B(A)\big)$. Now $I^{\perp\perp} = \langle \overline{e} \rangle \neq A$. Since $I \subset I^{\perp\perp}$ and I is maximal, it follows that $I = I^{\perp\perp} = \langle \overline{e} \rangle$. Now, let $x \in \mathrm{Inf}(A)$. Then $x \wedge e \in I^\perp \cap \mathrm{Inf}(A) \subset B(A) \cap \mathrm{Inf}(A) = \{0\}$. Thus $xe \in 0$ and $x \leq \overline{e} \in I$. This means that $\mathrm{Inf}(A) \subset I$, and hence I is implicative. ■

We have seen that if $I \neq \{0\}$ is implicative then it is weakly essential. To get a converse, we again need to consider only those maximal ideals which are not essential.

Theorem 3.13. *Suppose that $I \neq \{0\}$ is maximal but not essential. Then I is weakly essential if and only if it is implicative.*

Proof: In one direction, it is clear. In the other direction, we need only apply Theorem 3.10. ■

Putting these results together, we can state the following:

Theorem 3.14. *Suppose that $I \neq \{0\}$ is maximal but not essential. Then the following are equivalent:*

(1) *I is implicative*

(2) *I is weakly essential*

(3) *$I^\perp \subset B(A)$.*

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