

L^2 -BOUNDEDNESS OF A SINGULAR INTEGRAL OPERATOR

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Abstract

In this paper we study a singular integral operator T with rough kernel. This operator has singularity along sets of the form $\{x = Q(|y|)y'\}$, where $Q(t)$ is a polynomial satisfying $Q(0) = 0$. We prove that T is a bounded operator in the space $L^2(\mathbb{R}^n)$, $n \geq 2$, and this bound is independent of the coefficients of $Q(t)$.

We also obtain certain Hardy type inequalities related to this operator.

1. Introduction

Let S^{n-1} be the unit sphere in \mathbb{R}^n , $n \geq 2$, with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)|x|^{-n}$ be a homogeneous function of degree $-n$, with $\Omega \in L^1(S^{n-1})$ and

$$(1.1) \quad \int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$.

Suppose $b(|x|)$ is an L^∞ function. We consider the distribution $K = p.v. b(|x|)\Omega(x)|x|^{-n}$ and study the boundedness of the singular integral operator $T_{Q,b}(f)$ defined by

$$(1.2) \quad T_{Q,b}(f)(x) = \int_{\mathbb{R}^n} K(y)f(x - Q(|y|)y') dy$$

where $y' = y/|y| \in S^{n-1}$ and $Q(t) = \sum_{k=1}^m b_k t^k$ is a polynomial of degree m .

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For the sake of simplicity, we denote $T_{Q,b} = T_b$ if $Q(t) = t$ and $T_{Q,b} = T$ if $Q(t) = t$ and $b(x) \equiv 1$.

The maximal operator $T_b^*(f)(x)$ now is defined by

$$(1.3) \quad T_b^*(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} K(y)f(x - y) dy \right|.$$

The singular integral operator Tf was first studied by Calderón and Zygmund in their pioneering papers [CZ1] and [CZ2]. In [CZ2], Calderón and Zygmund proved that if $\Omega \in L \text{Log}^+ L(S^{n-1})$ satisfies the mean zero condition (1.1) then the operator T with kernel $\Omega(x')|x|^{-n}$ is a bounded operator in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Below let us recall briefly the idea used in Calderón-Zygmund's proof.

Suppose that $\Omega \in L'(S^{n-1})$ is an odd function, then one can easily show that $Tf(x)$ is equal to

$$(1.4) \quad \int_{\mathbb{R}^n} f(x - y)\Omega(y')|y|^{-n} dy = \frac{1}{2} \int_{S^{n-1}} \Omega(y') \left\{ \int_{-\infty}^{\infty} f(x - ty')t^{-1} dt \right\} d\sigma(y').$$

By the method of rotation and the well-known L^p boundedness of the Hilbert transform one then obtains the L^p boundedness of T under the weak condition $\Omega \in L^1(S^{n-1})$.

For even kernels, the condition $\Omega \in L^1(S^{n-1})$ is insufficient. It turns out the right condition is $\Omega \in L \text{Log}^+ L(S^{n-1})$ (as far as the size of Ω is concerned). The idea of Calderón-Zygmund is to compose the operator T with the Riesz transform R_j , $1 \leq j \leq n$, and show that $R_j T$ is a singular integral operator with an appropriate odd kernel. Thus $\|R_j T \psi\|_p \leq C_p \|\psi\|_p$ for all test functions $\psi \in \mathcal{L}$. Furthermore, one can obtain

$$\begin{aligned} \|T\psi\|_p &= \left\| \left(\sum_{j=1}^n R_j^2 \right) T\psi \right\|_p \\ &\leq \sum_{j=1}^n \|R_j(R_j T\psi)\|_p \leq na_p \|R_j T\psi\|_p \leq na_p C_p \|\psi\|_p \end{aligned}$$

for all $\psi \in \mathcal{L}$, since $\sum_{j=1}^n R_j^2$ is equal to the identity map.

In [Fe], R. Fefferman generalized this Calderón-Zygmund singular integral by replacing the kernel $\Omega(x')|x|^{-n}$ by $b(|x|)\Omega(x')|x|^{-n}$, where b is an arbitrary L^∞ function. This allows the kernel to be rough not only on the sphere, but also in the radial direction. For the singular integral operator T_b with the kernel $K(x) = b(|x|)\Omega(x')|x|^{-n}$, the formula (1.4) now is

$$(1.4') \quad T_b f(x) = \int_{S^{n-1}} \Omega(y') \left\{ \int_0^\infty f(x - ty') b(t) t^{-1} dt \right\} d\sigma(y').$$

Clearly, the method by Calderón and Zygmund can no longer be used to estimate the above integral in (1.4') even if Ω is odd, since the integral in the parenthesis can not be reduced to the Hilbert transform for an arbitrary $b(t)$. Thus one needs to find a new approach.

Using a method which is different from Calderón and Zygmund, in [Fe] R. Fefferman showed that if Ω satisfies a Lipschitz condition then T_b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. J. Namazi [Na] improved Fefferman's theorem by using the assumption $\Omega \in L^q(S^{n-1})$. The same L^p result was also obtained by L. Chen for the maximal operator T_b^* (see [Ch]). In [Fa], one of the authors obtained the L^2 boundedness for T_b under the significantly weaker condition $\Omega \in H^1(S^{n-1})$, where $H^1(S^{n-1})$ is the Hardy space on S^{n-1} . The condition $b \in L^\infty$ is also replaced by a weaker condition

$$(1.5) \quad R^{-1} \int_0^R |b(\rho)|^q d\rho \leq A, \text{ for all } R > 0 \text{ and some } q > 1$$

(see also [St] or [DR]).

The definition of Hardy space will be reviewed in Section 2. But we should mention here that on S^{n-1} , it is well-known that for $q > 1$,

$$L^q \subseteq L \text{Log}^+ L \subseteq H^1(S^{n-1}) \subseteq L^1$$

and all inclusions are proper.

The main purpose of this paper is to study the L^2 boundedness for the more general singular integral operator $T_{Q,b}(f)$ defined in (1.2) as well as the maximal operator $T_b^*(f)$ with $\Omega \in H^1(S^{n-1})$. In a forthcoming paper, we will study the L^p boundedness for another singular integral T_Φ that also takes T_b as a model case.

The following is the main theorem in this paper:

Theorem 1. *Let $T_{Q,b}$ be the singular integral operator defined by (1.2) and T_b^* be the maximal operator defined in (1.3). If $\Omega \in H^1(S^{n-1})$ satisfies (1.1) then both these two operators are bounded in $L^2(\mathbb{R}^n)$.*

More precisely, we have

$$(1.6) \quad \|T_{Q,b}(f)\|_2 \leq C \|b\|_\infty \|\Omega\|_{H^1(S^{n-1})} \|f\|_2;$$

$$(1.7) \quad \|T_b^*(f)\|_2 \leq C \|b\|_\infty \|\Omega\|_{H^1(S^{n-1})} \|f\|_2,$$

where C is a constant independent of b, Ω, f and the coefficients of Q .

By the proof in Theorem 1, we can further obtain the following result:

Theorem 2 (Hardy-type inequalities).

(i) Let $Q(t) = \sum_{k=1}^m b_k t^k$ be a polynomial in \mathbb{R} and $\Omega \in H^1(S^{n-1})$ satisfy the mean zero property (1.1). Then we have

$$(1.8) \quad \int_{\mathbb{R}^n} |x|^{-n} \left| \int_{S^{n-1}} e^{iQ(|x|\langle x', \xi' \rangle)} \Omega(\xi') d\sigma(\xi') \right| dx \leq C \|\Omega\|_{H^1(S^{n-1})},$$

where C is a constant independent of Ω and the coefficients of Q .

(ii) If Ω is a distribution in the Hardy space $H^p(S^{n-1})$, $0 < p < 1$, with property (1.1) then

$$(1.9) \quad \int_{\mathbb{R}^n} |x|^{(1-n)(2-p)-1} \left| \int_{S^{n-1}} e^{i\langle x, \xi' \rangle} \Omega(\xi') d\sigma(\xi') \right|^p dx \leq C_p \|\Omega\|_{H^p(S^{n-1})}^p,$$

where C is a constant independent of Ω .

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but independent of the essential variables.

2. Definitions and Lemmas

Recall that the Poisson kernel on S^{n-1} is defined by

$$P_{ry'}(x') = (1 - r^2)/|ry' - x'|^n,$$

where $0 \leq r < 1$ and $x', y' \in S^{n-1}$.

For any $f \in \mathcal{L}'(S^{n-1})$, we define the radial maximal function $P^+ f(x')$ by

$$P^+ f(x') = \sup_{0 \leq r < 1} \left| \int_{S^{n-1}} f(y') P_{rx'}(y') d\sigma(y') \right|,$$

where $\mathcal{L}'(S^{n-1})$ is the space of Schwartz distributions on S^{n-1} . The Hardy space $H^p(S^{n-1})$, $0 < p \leq 1$, is the linear space of distribution $f \in \mathcal{L}'(S^{n-1})$ with the finite norm $\|f\|_{H^p(S^{n-1})} = \|P^+ f\|_{L^p(S^{n-1})} < \infty$. The space $H^p(S^{n-1})$ was studied in [Co] (see also [CTW]). In particular, it is known that

$$H^p(S^{n-1}) \supseteq L^1(S^{n-1}) \supseteq H^1(S^{n-1}) \supseteq L \log^+ L(S^{n-1}) \supseteq L^q(S^{n-1})$$

for any $q > 1 > p > 0$.

Another important property of $H^p(S^{n-1})$ is the atomic decomposition, which will be reviewed below.

An *exceptional atom* is an L^∞ function $E(x)$ satisfying $\|E\|_\infty \leq 1$. A *regular (p, q) atom* is an L^q ($1 < q \leq \infty$) function $a(\cdot)$ that satisfies

$$(2.1) \quad \text{supp}(a) \subset \{x' \in S^{n-1}, |x' - x'_0| < \rho\} \\ \text{for some } x'_0 \in S^{n-1} \text{ and } \rho > 0\};$$

$$(2.2) \quad \int_{S^{n-1}} a(\xi') Y(\xi') d\sigma(\xi') = 0,$$

for any spherical harmonic polynomial with degree $\leq N$, where N is any fixed integer larger than $[(n - 1)(1/p - 1)]$;

$$(2.3) \quad \|a\|_q \leq \rho^{(n-1)(1/q-1/p)}.$$

From [Co] or [CTW], we find that any $\Omega \in H^p(S^{n-1})$ has an atomic decomposition $\Omega = \sum \lambda_j a_j$, where the a_j 's are either exceptional atoms or regular (p, q) atoms and $\sum |\lambda_j|^p \leq C \|\Omega\|_{H^p(S^{n-1})}^p$. In particular, if $\Omega \in H^p(S^{n-1})$ has the mean zero property (1.1) then all the atoms a_j in the atomic decomposition can be chosen to be regular (p, q) atoms.

In the rest of the paper, for any non-zero $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, we write $\xi/|\xi| = \xi' = (\xi'_1, \dots, \xi'_n) = (\zeta_1, \dots, \zeta_n) = \zeta$. Thus $\zeta \in S^{n-1}$. Also we use ζ_* to denote $(\zeta_2, \dots, \zeta_n)$ and use ξ_* to denote (ξ_2, \dots, ξ_n) .

The following lemma is essentially Proposition 2.5 in [FP].

Lemma 2.1. *Suppose $n \geq 3$ and $a(\cdot)$ is a $(1, \infty)$ atom on S^{n-1} supported in $S^{n-1} \cap B(\zeta, \rho)$, where $B(\zeta, \rho)$ is the ball with radius ρ and center $\zeta = \xi' \in S^{n-1}$. Let*

$$F_a(s) = (1 - s^2)^{(n-3)/2} \chi_{(-1,1)}(s) \int_{S^{n-2}} a(s, (1 - s^2)^{1/2} \tilde{y}) d\sigma(\tilde{y}).$$

Then, up to a constant multiplier independent of $a(\cdot)$, $F_a(s)$ is a $(1, \infty)$ atom on \mathbb{R} . More precisely, there are $s_0 \in \mathbb{R}$ and a constant C which is independent of $a(\cdot)$ such that

$$(2.4) \quad \text{supp}(F_a) \subseteq (s_0 - 2r, s_0 + 2r);$$

$$(2.5) \quad \|F_a\|_\infty \leq C/r;$$

$$(2.6) \quad \int_{\mathbb{R}} F_a(s) ds = 0,$$

where $r = r(\xi') = |\xi|^{-1}|A_\rho \xi|$ and $A_\rho \xi = (\rho^2 \xi_1, \rho \xi_2, \dots, \rho \xi_n)$.

Proof: If $\rho < 1/4$, the proof can be found in [FP]. Suppose $\rho \geq 1/4$, then, clearly $\text{supp}(F_a) \subseteq (-1, 1)$ and $\|F_a\|_\infty \leq C$. It is also easy to see that F_a satisfies (2.6). ■

Lemma 2.2. *Suppose $n = 2$ and that $a(\cdot)$ is a $(1, \infty)$ atom supported in $S^1 \cap B(\zeta, \rho)$ and satisfies (2.1)-(2.3) with $p = 1$. Let*

$$F_a(s) = (1 - s^2)^{-1/2} \chi_{(-1,1)}(s) \left(a(s, (1 - s^2)^{1/2}) + a(s, -(1 - s^2)^{1/2}) \right).$$

Then, up to a constant multiplier independent of $a(\cdot)$, $F_a(s)$ is a $(1, q)$ atom on \mathbb{R} , where q is any fixed number in the interval $(1, 2)$. The radius of $\text{supp}(F_a)$ is equal to $r = r(\xi') = |\xi|^{-1}(\rho^4 \xi_1^2 + \rho^2 \xi_2^2)^{1/2}$.

Proof: By the discussion in Lemma 2.1, without loss of generality, we may assume that $a(\cdot)$ is supported in $S^{n-1} \cap B(\zeta, \rho)$ with a sufficiently small ρ , where $\zeta \in S^1$. Let $\zeta = (\zeta_1, (1 - \zeta_1^2)^{1/2} \sigma)$ for $\sigma \in \{\pm 1\}$. If $F_a(s) \neq 0$ then $(s, (1 - s^2)^{1/2} \delta) \in B(\zeta, \rho)$ for some $\delta \in \{\pm 1\}$. Therefore we have

$$(s - \zeta_1)^2 + \{\delta(1 - s^2)^{1/2} - \sigma(1 - \zeta_1^2)^{1/2}\}^2 < \rho^2.$$

Noting that either $\delta = \sigma$ or $\delta = -\sigma$, we easily see that

$$(2.7) \quad (s - \zeta_1)^2 + |(1 - s^2)^{1/2} - (1 - \zeta_1^2)^{1/2}|^2 \leq \rho^2$$

which implies that

$$(2.8) \quad |s - \zeta_1| \leq \rho;$$

$$(2.9) \quad |(1 - s^2)^{1/2} - (1 - \zeta_1^2)^{1/2}| \leq \rho;$$

and

$$(2.10) \quad |s - \zeta_1| \leq \rho^2 + 2\rho(1 - \zeta_1^2)^{1/2}.$$

Inequalities (2.8) and (2.9) follow from (2.7) trivially. To see (2.10) we shall consider the following two cases.

Case A: $|\zeta_1| > 3/4$. Then by (2.8) and (2.9) we have

$$|s + \zeta_1| \geq 2|\zeta_1| - |s - \zeta_1| > 1$$

and

$$\begin{aligned} |s - \zeta_1| &\leq |s^2 - \zeta_1^2| \\ &= |(1 - s^2)^{1/2} - (1 - \zeta_1^2)^{1/2}| \\ &\quad |2(1 - \zeta_1^2)^{1/2} + (1 - s^2)^{1/2} - (1 - \zeta_1^2)^{1/2}| \\ &\leq \rho(\rho + 2(1 - \zeta_1^2)^{1/2}) = \rho^2 + 2\rho(1 - \zeta_1^2)^{1/2}. \end{aligned}$$

Case B: $|\zeta_1| \leq 3/4$. Then $(1 - \zeta_1^2)^{1/2} \geq 1/2$. By (2.8) we find

$$|s - \zeta_1| \leq \rho < \rho^2 + 2\rho(1 - \zeta_1^2)^{1/2},$$

which proves (2.10).

Recalling $\xi' = \zeta$, we easily see that in both Case A and Case B,

$$|s - \zeta_1| \leq 2|\xi|^{-1}|A_\rho\xi|.$$

By letting $s_0 = \zeta_1$, $r = r(\xi') = |\xi|^{-1}|A_\rho\xi|$, we see that (2.4) and (2.6) are satisfied.

It remains to show, for $1 < q < 2$, $\|F_a\|_q \leq Cr^{-1+1/q}$. To this end, we first assume that $(1 - \zeta_1^2)^{1/2} > 99\rho$. By (2.9) we find

$$(2.11) \quad 1/2(1 - \zeta_1^2)^{1/2} \leq (1 - s^2)^{1/2} \leq 2(1 - \zeta_1^2)^{1/2}.$$

Thus by the definition of F_a we have $\|F_a\|_\infty \leq C\rho^{-1}(1 - \zeta_1^2)^{-1/2} \leq Cr^{-1}$. Now by the support condition (2.4) we have $\|F_a\|_q \leq Cr^{-1+1/q}$.

If $(1 - \zeta_1^2)^{1/2} = |\zeta_2| \leq 99\rho$, then by (2.9) we know $(1 - s^2)^{1/2} \leq 100\rho$. So by the definition of F_a we have

$$\|F_a\|_q \leq C\rho^{-1} \left\{ \int_{1-s^2 \leq 10000\rho^2} |1 - s^2|^{-q/2} ds \right\}^{1/q}.$$

Noting that we can assume that ρ is sufficiently small so that $10000\rho^2 \leq 1/16$, thus we easily show $\|F_a\|_q \leq C\rho^{2(-1+1/q)} \leq C(\rho^2|\zeta_1| + \rho|\zeta_2|)^{-1+1/q} \leq Cr^{-1+1/q}$. Lemma 2.2 is proved. ■

Remarks.

1. Let $a(\cdot)$ be a (p, ∞) atom with support in a ball of radius ρ and let F_a be defined in Lemma 2.1. Since $\rho^{(1/p-1)(n-1)}a(\cdot)$ is a $(1, \infty)$ atom, by Lemma 2.1 we have

$$(2.12) \quad \|F_a\|_\infty \leq r^{-1}\rho^{(1-1/p)(n-1)}.$$

2. Since any spherical harmonic polynomial is the restriction to S^{n-1} of a polynomial in \mathbb{R}^n , from the condition (2.2) for $a(\cdot)$ we easily see

$$(2.13) \quad \int_{\mathbb{R}} F_a(s)s^k ds = 0 \text{ for any integer } k \in [0, N],$$

where N is an integer larger than $[(n-1)(1/p-1)]$.

3. Proof of Theorem 1

We first prove (1.6) in Theorem 1. By Fubini's Theorem we easily see that the Fourier transform of $T_{Q,b}(f)$ is equal to $\hat{f}(\xi)\tilde{K}_\Omega(\xi)$, where

$$(3.1) \quad \tilde{K}_\Omega(\xi) = \int_{\mathbb{R}^n} |y|^{-n}b(|y|)\Omega(y')e^{-iQ(|y|)|\xi|\langle y', \xi' \rangle} dy.$$

By Plancherel's Theorem, we only need to prove that

$$(3.2) \quad \|\tilde{K}_\Omega\|_\infty \leq C\|b\|_\infty \|\Omega\|_{H^1(S^{n-1})}.$$

Since $\Omega \in H^1(S^{n-1})$ satisfies the mean zero property (1.1), we can write $\Omega = \sum \lambda_j a_j$, where $\sum |\lambda_j| \leq C\|\Omega\|_{H^1(S^{n-1})}$ and each a_j is a $(1, \infty)$ atom. Therefore to prove (3.2), it suffices to show

$$(3.3) \quad \|\tilde{K}_{a_j}\|_\infty \leq C\|b\|_\infty$$

for any atom $a_j = a$, where C is independent of the coefficients of Q and the atom $a(\cdot)$. By the method of rotation, we can assume $x' = (1, 0, \dots, 0)$. Let $y' = (s, y_2, y_3, \dots, y_n)$. Then it is easy to see that

$$(3.4) \quad \tilde{K}_a(x) = \int_0^\infty b(t)t^{-1} \int_{\mathbb{R}} F_a(s)e^{-iQ(t)|x|s} ds dt$$

where $F_a(s)$ is the function defined in Lemma 2.1 or Lemma 2.2. By Lemma 2.1 and Lemma 2.2, without loss of generality, we may assume that F_a is a $(1, q)$ atom with support in $(-r, r)$ for $1 < q < 2$. Thus $A(\cdot) = rF_a(r\cdot)$ is a $(1, q)$ atom with support in the interval $(-1, 1)$.

For the polynomial

$$Q(t)|x| = \sum_{k=1}^m |x|b_k t^k \text{ with } b_m \neq 0,$$

we let

$$\beta_k = (|x|rb_k) \text{ and } \tilde{Q}(t) = -\sum_{k=1}^m \beta_k t^k.$$

Then after changing variables we have

$$(3.4') \quad \tilde{K}_a(x) = \int_0^\infty t^{-1}b(t) \int_{\mathbb{R}} A(s)e^{i\tilde{Q}(t)s} ds dt.$$

Let $|\beta_\kappa|^{1/\kappa} = \max\{|\beta_k|^{1/k}, k = 1, 2, \dots, m\}$ and $\beta = |\beta_\kappa|^{-1/\kappa}$.

Then $\tilde{K}_a(x)$ is bounded by

$$\begin{aligned} \|b\|_\infty \int_0^\beta \left| \int_{\mathbb{R}} A(s)\{e^{i\tilde{Q}(t)s} - 1\} ds \right| t^{-1} dt \\ + \|b\|_\infty \int_\beta^\infty t^{-1} \left| \int_{\mathbb{R}} A(s)e^{i\tilde{Q}(t)s} ds \right| dt = I_1 + I_2. \end{aligned}$$

By the choice of β , it is easy to see that I_1 is bounded by

$$C\|b\|_\infty \sum_{k=1}^m |\beta_k| \int_0^\beta t^{-1+k} dt \leq C\|b\|_\infty$$

where C is independent of β_k 's.

To estimate I_2 , we let $R_j = [2^j, 2^{j+1})$ for any integer j and let $\Psi \in C^\infty(\mathbb{R})$ satisfy

$$\begin{aligned} \Psi(t) &\equiv 1 && \text{for } |t| \leq 1 \\ \Psi &\equiv 0 && \text{for } |t| \geq 2. \end{aligned}$$

Define T_j by

$$(T_j f)(t) = \chi_{R_j}(t) \int_{\mathbb{R}} e^{is\tilde{Q}(t)} \Psi(s) f(s) ds.$$

From the estimate on page 60 in [Pa], we can find an $N > 0$ such that

$$\|T_j\|_{L^2 \rightarrow L^2} \leq C 2^{j/2} |\beta_\kappa|^{-1/2N} 2^{-j\kappa/2N}.$$

By the trivial estimate $\|T_j\|_{L^1 \rightarrow L^\infty} \leq C$ and interpolation we now have

$$(3.5) \quad \|T_j\|_{L^p \rightarrow L^q} \leq C 2^{j/q} |\beta_\kappa|^{-1/qN} 2^{-j\kappa/qN}$$

where $1/p + 1/q = 1$ and $q \geq 2$.

Choosing an integer J such that $2^J \leq \beta < 2^{J+1}$, then, we have

$$\begin{aligned} I_2 &\leq \|b\|_\infty \int_\beta^\infty t^{-1} \left| \int_{\mathbb{R}} A(s) e^{i\tilde{Q}(t)s} ds \right| dt \\ &\leq C \|b\|_\infty \sum_{j \geq J} \int_{2^j}^{2^{j+1}} t^{-1} |T_j(A)(t)| dt \\ &\leq C \|b\|_\infty \sum_{j \geq J} \left\{ \int_{2^j}^{2^{j+1}} t^{-p} dt \right\}^{1/p} \|T_j(A)\|_{L^q}. \end{aligned}$$

Thus by (3.5) we have

$$I_2 \leq C \|b\|_\infty \sum_{j \geq J} 2^{-j/q} 2^{j/q} \|A\|_{L^p} 2^{-j\kappa/qN} |\beta_\kappa|^{-1/qN} \leq C \|b\|_\infty,$$

because $2^J \geq (1/2) |\beta_\kappa|^{-1/\kappa}$.

The first part in Theorem 1 is proved.

To prove (1.7) in Theorem 1, we notice that

$$\begin{aligned} T_b^* f(x) &= \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} b(|y|) |y|^{-n} \Omega(y') f(x-y) dy \right| \\ &\leq \sum |\lambda_j| \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} b(|y|) |y|^{-n} a_j(y') f(x-y) dy \right| \end{aligned}$$

where $\sum |\lambda_j| \leq C \|\Omega\|_{H^1(S^{n-1})}$ and all a_j 's are $(1, \infty)$ atoms.

So it suffices to show that for any $(1, \infty)$ atom $a(\cdot)$ on S^{n-1}

$$(3.6) \quad \left\| \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} b(|y|) |y|^{-n} a(y') f(\cdot - y) dy \right| \right\|_2 \leq C \|b\|_\infty \|f\|_2$$

with a constant C independent of b, f and $a(\cdot)$. Without loss of generality, we may assume $\text{supp } a(\cdot) \subseteq B(\mathbf{1}, \rho) \cap S^{n-1}$ where $\mathbf{1} = (1, 0, \dots, 0)$.

In order to prove (3.6) we need to prove the following:

Proposition 3.1. *Suppose that $b \in L^\infty$ and $a(\cdot)$ is a $(1, \infty)$ atom supported in $B(\mathbf{1}, \rho) \cap S^{n-1}$. For $\varepsilon > 0$, let*

$$T_{b,\varepsilon}f(x) = (a(\cdot)b(|\cdot|)|\cdot|^{-n}\chi_{\{|y|>\varepsilon\}}(\cdot) * f)(x).$$

We have

$$(3.7) \quad \left\| \sup_{0 < s < \infty} \frac{1}{s} \int_0^s |T_{b,\varepsilon}f(\cdot)| d\varepsilon \right\|_2 \leq C \|b\|_\infty \|f\|_2$$

where C is independent of b, f and $a(\cdot)$.

Proof: The Fourier transform of $T_{b,\varepsilon}f$ is $m_\varepsilon(\xi)\hat{f}(\xi)$, where

$$m_\varepsilon(\xi) = \int_\varepsilon^\infty b(t)t^{-1} \int_{S^{n-1}} a(y')e^{-it\langle \xi, y' \rangle} d\sigma(y') dt.$$

For each fixed $\xi \neq 0$, we choose a rotation O such that $O(\xi) = |\xi|\mathbf{1}$. Thus

$$m_\varepsilon(\xi) = \int_\varepsilon^\infty b(t)t^{-1} \int_{S^{n-1}} a(O^{-1}(y'))e^{-it|\xi|\langle \mathbf{1}, y' \rangle} d\sigma(y') dt.$$

Now $a(O^{-1}(y'))$ is an atom supported in $B(\zeta, \rho) \cap S^{n-1}$ so that

$$m_\varepsilon(\xi) = \int_\varepsilon^\infty b(t)t^{-1} \int_{\mathbb{R}} F_a(s)e^{-it|\xi|s} ds dt,$$

where F_a is the function as in Lemma 2.1 if $n > 2$ and in Lemma 2.2 if $n = 2$. Without loss of generality, we assume $\text{supp}(F_a) \subseteq (-r, r)$.

For the above $r = r(\xi')$, we take a radial function $\Phi \in C^\infty(\mathbb{R}^n)$ such that its Fourier transform $\hat{\Phi}$ satisfies $\hat{\Phi}(\xi) = 1$ if $|\xi| \leq 1$ and $\hat{\Phi}(\xi) = 0$ if $|\xi| > 2$ and define Φ_ε by $\hat{\Phi}_\varepsilon(\xi) = \hat{\Phi}(\varepsilon r(\xi')|\xi|)$. It is easy to see that the maximal function $\Phi^*(f) = \sup_{\varepsilon > 0} |\Phi_\varepsilon * f|$ is bounded in $L^p(\mathbb{R}^n)$. Now we define a g -function

$$g(f)(x) = \left\{ \int_0^\infty |T_{b,\varepsilon}f(x) - \Phi_\varepsilon * T_b f(x)|^2 \varepsilon^{-1} d\varepsilon \right\}^{1/2}.$$

Then

$$\frac{1}{s} \int_0^s |T_{b,\varepsilon}f(x)| d\varepsilon \leq g(f)(x) + \frac{1}{s} \int_0^s |\Phi_\varepsilon * T_b f(x)| d\varepsilon.$$

Thus

$$\sup_{\varepsilon>0} \frac{1}{s} \int_0^s |T_{b,\varepsilon} f(x)| d\varepsilon \leq g(f)(x) + \Phi^*(T_b f)(x).$$

By (1.6) we have

$$\|\Phi^*(T_b f)\|_2 \leq C \|T_b f\|_2 \leq C \|b\|_\infty \|f\|_2.$$

So it remains to show

$$(3.8) \quad \|g(f)\|_2 \leq C \|b\|_\infty \|f\|_2.$$

By Plancherel's Theorem we know that

$$\|g(f)\|_2^2 = C \int_{\mathbb{R}^n} \int_0^\infty |m_\varepsilon(\xi) - \hat{\Phi}(\varepsilon r|\xi|) m_0(\xi)|^2 |\hat{f}(\xi)|^2 \varepsilon^{-1} d\varepsilon d\xi.$$

So we only need show

$$(3.9) \quad \int_0^\infty |m_\varepsilon(\xi) - \hat{\Phi}(\varepsilon r(\xi')|\xi|) m_0(\xi)|^2 \varepsilon^{-1} d\varepsilon \leq C \|b\|_\infty^2$$

where C is a constant independent of b , ξ and r .

By the definition of m_ε and changing of variables we have

$$\begin{aligned} & \int_0^\infty |m_\varepsilon(\xi) - \hat{\Phi}(\varepsilon r|\xi|) m_0(\xi)|^2 \varepsilon^{-1} d\varepsilon \\ &= C \int_0^\infty \left| \int_{\varepsilon|\xi|}^\infty b(t/|\xi|) t^{-1} \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') dt \right. \\ & \quad \left. - \hat{\Phi}(\varepsilon r|\xi|) \int_0^\infty b(t/|\xi|) t^{-1} \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') dt \right|^2 \varepsilon^{-1} d\varepsilon \\ &= C \int_0^\infty \left| \int_\varepsilon^\infty b(t/|\xi|) t^{-1} \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') dt \right. \\ & \quad \left. - \hat{\Phi}(r\varepsilon) \int_0^\infty b(t/|\xi|) t^{-1} \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') dt \right|^2 \varepsilon^{-1} d\varepsilon \\ &\leq C \int_0^{1/r} \left| \int_0^\varepsilon b(t/|\xi|) t^{-1} \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') dt \right|^2 \varepsilon^{-1} d\varepsilon \\ & \quad + C \|b\|_\infty^2 \int_{1/r}^{2/r} \varepsilon^{-1} d\varepsilon \left\{ \int_0^\infty t^{-1} \left| \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') \right| dt \right\}^2 \\ & \quad + C \int_{2/r}^\infty \left| \int_\varepsilon^\infty b(t/|\xi|) t^{-1} \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') dt \right|^2 \varepsilon^{-1} d\varepsilon \\ & \hspace{15em} = I_1 + I_2 + I_3. \end{aligned}$$

Using the method of rotation again and the cancellation condition of F_a we have

$$I_1 \leq C \|b\|_\infty^2 \int_0^{1/r} \left| \int_0^\varepsilon t^{-1} \left| \int_{\mathbb{R}} F_a(s) \{e^{-its} - 1\} ds \right| dt \right|^2 \varepsilon^{-1} d\varepsilon$$

$$\leq C \|b\|_\infty^2 \int_0^{1/r} r^2 \varepsilon d\varepsilon = C \|b\|_\infty^2.$$

$$I_2 \leq C \|b\|_\infty^2 \left\{ \int_0^\infty t^{-1} \left| \int_{\mathbb{R}} F_a(s) e^{-its} ds \right| dt \right\}^2$$

$$\leq C \|b\|_\infty^2 \left\{ \int_0^\infty t^{-1} |\hat{F}_a(t)| dt \right\}^2 \leq C \|b\|_\infty^2.$$

The last inequality is the classical Hardy inequality (page 128 in [St]), since F_a is an atom on \mathbb{R}

$$I_3 \leq C \|b\|_\infty^2 \int_{2/r}^\infty \left\{ \int_\varepsilon^\infty |t^{-1} \hat{F}_a(t)| dt \right\}^2 \varepsilon^{-1} d\varepsilon$$

$$\leq C \|b\|_\infty^2 \int_{2/r}^\infty \left\{ \int_\varepsilon^\infty t^{-p} dt \right\}^{2/p} \varepsilon^{-1} d\varepsilon \|\hat{F}_a\|_q^2.$$

Thus by the Hausdorff-Young inequality we have

$$I_3 \leq C \|b\|_\infty^2 r^{2-2/p} \|F_a\|_p^2 \leq C \|b\|_\infty^2.$$

Clearly the constant C in the above estimates is independent of the essential variables and functions. The proposition is proved.

Now we return to prove (3.6). Let

$$W(t, \xi) = \int_{S^{n-1}} a(y') e^{-it|\xi|\langle \xi', y' \rangle} d\sigma(y') b(t).$$

Then

$$m_\varepsilon(\xi) = C \int_\varepsilon^\infty W(t, \xi) t^{-1} dt.$$

Thus

$$\begin{aligned} \frac{1}{s} \int_0^s m_\varepsilon(\xi) d\varepsilon &= C \frac{1}{s} \int_0^s \int_\varepsilon^\infty W(t, \xi) t^{-1} dt d\varepsilon \\ &= \frac{C}{s} \left\{ \int_0^s \left(\int_0^t W(t, \xi) d\varepsilon \right) t^{-1} dt \right. \\ &\quad \left. + \int_0^s \int_s^\infty W(t, \xi) t^{-1} dt d\varepsilon \right\} \\ &= \frac{C}{s} \int_0^s W(t, \xi) dt + C m_s(\xi). \end{aligned}$$

Thus

$$\begin{aligned} & \sup_{\varepsilon>0} |T_{b,\varepsilon}f(x)| \\ & \leq \sup_{s>0} \frac{1}{s} \int_0^s |T_{b,\varepsilon}f(x)| d\varepsilon + C \left\{ \int_0^\infty |(W(t, \cdot)\hat{f}(\cdot))^v(x)|^2 t^{-1} dt \right\}^{1/2}, \end{aligned}$$

where $f^v(x)$ is the Fourier inverse of f . By Proposition (3.1) we only need to prove

$$J = \left\| \left\{ \int_0^\infty |(W(t, \cdot)\hat{f}(\cdot))^v(x)|^2 t^{-1} dt \right\}^{1/2} \right\|_2 \leq C \|b\|_\infty \|f\|_2.$$

Using Plancherel's Theorem, we have

$$J \leq C \left\| \left\{ \int_0^\infty |W(t, \cdot)|^2 t^{-1} dt \right\}^{1/2} \hat{f}(\cdot) \right\|_2.$$

So it suffices to show

$$(3.10) \quad R(\xi) = \int_0^\infty |W(t, \xi)|^2 t^{-1} dt \leq C \|b\|_\infty^2$$

with C independent of ξ , $b(\cdot)$ and $a(\cdot)$.

In fact,

$$R(\xi) \leq C \|b\|_\infty^2 \int_0^\infty \left| \int_{S^{n-1}} a(y') e^{-it\langle \xi', y' \rangle} d\sigma(y') \right|^2 t^{-1} dt.$$

So using the same argument as in estimating (3.4), we have

$$\begin{aligned} R(\xi) & \leq C \|b\|_\infty^2 \int_0^\infty \left| \int_{\mathbb{R}} F_a(s) e^{-its} ds \right|^2 t^{-1} dt \\ & = C \|b\|_\infty^2 \left\{ \int_{1/r}^\infty t^{-1} |\hat{F}_a(t)|^2 dt \right. \\ & \quad \left. + \int_0^{1/r} t^{-1} \left| \int_{\mathbb{R}} F_a(s) \{e^{-its} - 1\} ds \right| dt \right\} \\ & \leq C \|b\|_\infty^2 \{1 + r^{1/2} \|\hat{F}_a\|_4^2\} \\ & \leq C \|b\|_\infty^2 \{1 + r^{1/2} \|F_a\|_{4/3}^2\} \leq C \|b\|_\infty^2 \end{aligned}$$

where C is independent of all the essential variables and functions. The proof of Theorem is complete. ■

Proof of Theorem 2

To prove (1.8), using the atomic decomposition it suffices to show

$$(4.1) \quad \int_{\mathbb{R}^n} |x|^{-n} \left| \int_{S^{n-1}} e^{iQ(|x|\langle x', \xi' \rangle)} a(\xi') d\sigma(\xi') \right| dx \leq C$$

where C is independent of the atom $a(\cdot)$. By the polar coordinate, we can see that the above integral (4.1) is equal to

$$\int_{S^{n-1}} \int_0^\infty t^{-1} \left| \int_{S^{n-1}} e^{iQ(t)\langle x', \xi' \rangle} a(\xi') d\sigma(\xi') \right| dt d\sigma(x').$$

So similar to the proof in Theorem 1, by the method of rotation, the last integral is equal to

$$\int_{S^{n-1}} \int_0^\infty t^{-1} \left| \int_{\mathbb{R}} F_a(s) e^{isQ(t)} ds \right| dt d\sigma(x')$$

where F_a is the function defined in Lemma 2.1 or Lemma 2.2, that depends on $x' \in S^{n-1}$. Now following the estimate of (3.4'), we easily obtain that

$$\int_0^\infty t^{-1} \left| \int_{\mathbb{R}} F_a(s) e^{iQ(t)s} ds \right| dt \leq C$$

with the constant C independent of x' .

To prove the second part of Theorem 2, for simplicity, we only show the case of $n > 2$. In this case, we do not need to use Lemma 2.2 to estimate the L^q norm of F_a .

By the atomic decomposition of $\Omega \in H^p$, it suffices to show

$$(4.2) \quad \int_{\mathbb{R}^n} |x|^{-1+(p-2)(n-1)} \left| \int_{S^{n-1}} e^{i\langle x, \xi' \rangle} a(\xi') d\sigma(\xi') \right|^p dt \leq C,$$

where the constant C is independent of any (p, ∞) atom $a(\cdot)$.

Using the polar coordinate and the method of rotation, we only need to prove

$$(4.3) \quad I = \int_0^\infty t^{(n-1)p-n} \left| \int_{\mathbb{R}} F_a(s) e^{its} ds \right|^p dt \leq C.$$

Without loss of generality, we may assume $\text{supp}(F_a) \subseteq (-r, r)$. Now we write

$$I = \left\{ \int_0^{1/r} + \int_{1/r}^\infty \right\} t^{(n-1)p-n} \left| \int_{\mathbb{R}} F_a(s) e^{its} ds \right|^p dt = I_1 + I_2.$$

In I_1 , choosing an integer $N > (1/p - 1)(n - 1)$ and using the conditions (2.12) and (2.13), we have

$$\left| \int_{\mathbb{R}} F_a(s) e^{its} ds \right| \leq C(tr)^N \rho^{(1-1/p)(n-1)}.$$

Noting $r \leq C\rho$, so we further have

$$\left| \int_{\mathbb{R}} F_a(s) e^{its} ds \right| \leq Ct^N r^{N+(1-1/p)(n-1)}.$$

Now it is easy to see that

$$I_1 \leq Cr^{p\{N+(n-1)(1-1/p)\}} \int_0^{1/r} t^{-1} t^{p\{N+(1-1/p)(n-1)\}} dt \leq C.$$

To estimate I_2 , by Hölder's inequality,

$$I_2 \leq \|\hat{F}_a\|_2^p \left(\int_{1/r}^{\infty} t^{2(n-np+p)/(p-2)} dt \right)^{(2-p)/2}.$$

By (2.12) we have

$$\|\hat{F}_a\|_2^p \leq C\|F_a\|_2^p \leq Cr^{-1/2} \rho^{p(n-1)-(n-1)}.$$

So we easily obtain that $I_2 \leq C$ since $r \leq C\rho$. The theorem is proved. ■

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