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ON BILINEAR LITTLEWOOD-PALEY SQUARE FUNCTIONS

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Abstract

On the real line, let the Fourier transform of k_n be $\hat{k}_n(\xi) = \hat{k}(\xi - n)$ where $\hat{k}(\xi)$ is a smooth compactly supported function. Consider the bilinear operators $S_n(f,g)(x) = \int f(x+y)g(x-y)k_n(y)\,dy$. If $2 \le p, q \le \infty$, with $1/p + 1/q = 1/2$, I prove that

$$\sum_{n=-\infty}^{\infty} \|S_n(f,g)\|_2^2 \le C^2 \|f\|_p^2 \|g\|_q^2 .$$

The constant C depends only upon k.

1. The inequalities

The principal inequality of this paper has two motivations. In 1964, Alberto Calderón raised conjectures concerning the following operator acting on two functions f and g defined on the real line:

$$H(f,g)(x) = \frac{1}{\pi} \int f(x+y)g(x-y)\,dy/y.$$

This operator, which has come to be known as the bilinear Hilbert transform, serves as the quintessential example of a bilinear operator whose Fourier multiplier has a singularities. This operator will not be addressed in this paper.

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The second motivation is the inequality of Littlewood-Paley type below. Denote by $S_I f(x)$ the restriction of the Fourier transform of f to an interval $I \subset \mathbf{R}$. There is the inequality

$$\left\| \left(\sum_{n=-\infty}^{\infty} |S_{[n,n+1)} f|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p, \quad 2 \le p.$$

This is due to L. Carleson [**C**]. The restriction to $2 \le p$ is sharp, as can be seen by taking the Fourier transform of f to be the indicator of the interval $[0, N]$, for N large. The reader should consult Rubio de Francia's interesting generalization [**RdF**].

We extend the inequality above to a bilinear setting, in the case of $p = 2$. To set notation, consider a smooth bump function with $\hat{k}(\xi)$ supported on the unit cube of $\mathbf{R}^d$. For integers $n \in \mathbf{Z}^d$, let k_n be the function with Fourier transform $\widehat{k}_n(\xi) = \hat{k}(\xi - n)$ and define

$$S_n(f,g)(x) = \int_{\mathbf{R}^d} f(x+y) g(x-y) k_n(y)\, dy.$$

Theorem 1.1. *For all $2 \le p, q \le \infty$, with $1/p + 1/q = 1/2$,*

$$\sum_{n \in \mathbf{Z}^d} \|S_n(f,g)\|_2^2 \le C^2 \|f\|_p^2 \|g\|_q^2. \tag{1.2}$$

The constant C is independent of p and q.

The proof offered can be extended to other bilinear forms, where for instance the "$+y$" in the integral is replaced by "$-By$" where B is an invertible linear transformation on $\mathbf{R}^d$ which is not the identity.

Notice that we only prove the theorem in its most obvious possible formulation and even then the proof is curiously intricate. The question arises of the boundedness of the square function $(\sum_n |S_n(f,g)|^2)^{1/2}$ as a map into L^p for $p > 2$, but I do not know how to address this question.

It is natural to ask for possible extensions to a sharp cut off in frequency, namely by taking the k_n to have Fourier tranform equal to the indicator of the cube $[n, n+1)$. Such a result if true, would be deep as it would already entail the boundedness of the bilinear Hilbert transform as a map into L^2. Recent progress has been made on this conjecture in [**LT**].

2. Decomposition of the functions

For the proof f and g are decomposed in the space-frequency plane. Define the Fourier transform to be

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \mathrm{e}^{-2\pi i x\cdot\xi} f(x)\,dx, \quad \xi \in \mathbf{R}^d.$$

In the sequel, $\langle f, g\rangle$ will mean $\int f\bar{g}\,dx$, and we shall adopt the notation $\mathrm{e}_\xi(x) = \mathrm{e}^{2\pi i\xi\cdot x}$. Let φ be a smooth, rapidly decreasing function, and set $\varphi_{m,n}(x) = \mathrm{e}_m(x)\varphi(x-n)$ for integers $m, n \in \mathbf{Z}^d$.

The particular decomposition of f and g that is needed is written as

$$\Phi f(x) = \sum_{m,n\in\mathbf{Z}^d} \langle f, \varphi_{m,n}\rangle \varphi_{m,n}(x). \tag{2.1}$$

A curious feature of the problem is that a space-frequency decomposition seems to be required, although the definition of $S_n(f,g)$ would not seem to force it upon us.

The subsequent section will be devoted to a proof of

Lemma 2.2. *Adopt the notation of Theorem 1.1, and let $\Delta(f,g)$ denote the square function of (1.2). Let φ be a function of L^2 norm 1, with $\hat{\varphi}$ supported in a translate of the cube $[-1/4, 1/4]^d$, and*

$$\sup_x (1+|x|)^{10d}|\varphi(x)| = B < \infty. \tag{2.3}$$

Let Φf be as in (2.1). And let φ' be a second function satisfying these same attributes, with $\Phi' g$ being the corresponding decomposition of g. Then, there is a constant C_B so that for all $2 \le p, q \le \infty$, with $1/p + 1/q = 1/2$,

$$\|\Delta(\Phi f, \Phi' g)\|_2 \le C_B \|f\|_p \|g\|_q. \tag{2.4}$$

To conclude the theorem, we need to replace Φf by f, and likewise for g. This can be done by way of a general principle, which we formulate in this way.

Lemma 2.5. *Let T be a sublinear map of bounded smooth functions on $\mathbf{R}^d$ into a Banach space X. If $T \circ \Phi$ is a bounded operator from L^p, $1 \le p < \infty$, into X, with norm $N(\varphi)$, then T maps L^p into X with norm at most*

$$\int_{[0,1)^d}\int_{[0,1)^d} N(\mathrm{e}^{2\pi i s\cdot}\varphi(\cdot - t))\,ds\,dt.$$

Proof: The basis of the proof resides in the following resolution of the identity

$$f(x) = \|\varphi\|_2^{-2} \iint \langle f, \varphi_{m,n}\rangle \varphi_{m,n}(x)\, dm\, dn.$$

This equality is initially understood in the sense of inner products, as we show. Let If denote the right hand side above. Then $\langle If, g\rangle = \langle f, g\rangle$, indeed

$$\langle If, g\rangle = \|\varphi\|_2^{-2} \iint \langle f, \varphi_{m,n}\rangle \overline{\langle g, \varphi_{m,n}\rangle}\, dm\, dn.$$

Now, the Fourier transform is unitary, so that

$$\langle f, \varphi_{m,n}\rangle = \langle \hat{f}(\xi), \mathrm{e}^{2\pi i n\cdot\xi}\hat{\varphi}(\xi - m)\rangle,$$

which is the Fourier transform of the function $F_m(\xi) = \hat{f}(\xi)\overline{\hat{\varphi}(\xi - m)}$ evaluated at $\xi = -n$. Thus, $\langle f, \varphi_{m,n}\rangle = \widehat{F_m}(-n)$. Likewise, $\langle g, \varphi_{m,n}\rangle$ is the Fourier transform of the function $G_m(\xi) = \hat{g}(\xi)\overline{\hat{\varphi}(\xi - m)}$ evaluated at $-n$.

From these considerations, it follows that

$$\begin{aligned}
\langle If, g\rangle &= \|\varphi\|_2^{-2} \int \langle \widehat{F_m}, \widehat{G_m}\rangle\, dm \\
&= \|\varphi\|_2^{-2} \int \langle F_m, G_m\rangle\, dm \\
&= \|\varphi\|_2^{-2} \int \hat{f}(\xi)\overline{\hat{g}(\xi)}|\hat{\varphi}(\xi - m)|^2\, d\xi\, dm \\
&= \langle \hat{f}, \hat{g}\rangle \\
&= \langle f, g\rangle,
\end{aligned}$$

as claimed.

Recalling the definition of Φ, we then see that it is a discrete form of I. To be more explicit, let $\varphi^{s,t}$ be the expansion of (2.1) associated to the function $\mathrm{e}_s(x)\varphi(x - t)$, and assume $\|\varphi\|_2 = 1$. Then

$$If(x) = \int_{[0,1]^d} \int_{[0,1]^d} \Phi^{s,t} f(x)\, ds\, dt.$$

Moreover, for f assumed smooth and compactly supported, the expansions $If(x)$ and $\Phi^{s,t} f$ are absolutely convergent. Hence we can interpret

$If(x)$ in a pointwise sense. For such an f, it follows from our assumptions that

$$\begin{aligned}\|Tf(x)\|_X &= \|T(If)\|_X \\ &\le \int_{[0,1]^d}\int_{[0,1]^d} \left\|T\Phi^{s,t}f\right\|_X \, ds\, dt \\ &\le \|f\|_p \int_{[0,1]^d}\int_{[0,1]^d} N(\mathrm{e}_s(\cdot)\varphi(\cdot - t))\, ds\, dt.\end{aligned}$$

This is the bound claimed in the lemma. Smooth and compactly supported function are dense in L^p, for $1 \le p < \infty$, proving the lemma. ■

The previous two lemmas prove Theorem 1.1 for $2 < p < \infty$ as the smooth functions are dense in L^p. That leaves the inequality for the square function Δ as a map on $L^2 \times L^\infty$. Here, one can take $f \in L^2$ to be bounded smooth and compactly supported. For an arbitrary $g \in L^\infty$, one can take bounded smooth and compactly supported g_n so that $\|g_n\|_\infty \le \|g\|_\infty$ and

$$\Delta(f, g_n) \xrightarrow{L^2} \Delta(f, g) \text{ on all compact subsets of } \mathbf{R}^d.$$

Moreover, as the proof shows, each g_n can be written as an average of expansions Φ. Hence the limiting case of $L^2 \times L^\infty$ follows.

3. The Proof of Lemma 2.2

One needs a clear understanding of S_μ as a Fourier multiplier, obtained by expanding $S_\mu(f,g)$ in frequency in a formal way. This requires that f and g be expanded in different frequency variables, say α and β respectively.

$$\begin{aligned}S_\mu(f,g)(x) &= \int f(x+y)\left\{\int \mathrm{e}_\beta(x-y)\mathcal{F}_\beta g(\beta)\, d\beta\right\} k_\mu(y)\, dy \\ &= \mathcal{F}_\beta^{-1}\left\{\int \mathrm{e}_\beta(-y) f(x+y)\, k_\mu(y)\, dy\, \mathcal{F}_\beta g(\beta)\right\}(x) \\ &= \mathcal{F}_\beta^{-1}\left\{\mathcal{F}_\alpha^{-1}\left\{\int \mathrm{e}_{\beta-\alpha}(-y) k_\mu(y)\, dy\, \mathcal{F}_\alpha f(\alpha)\right\}(x)\mathcal{F}_\beta g(\beta)\right\}(x) \\ &= \mathcal{F}_\beta^{-1}\left\{\mathcal{F}_\alpha^{-1}\left\{\widehat{k_\mu}(\beta-\alpha)\mathcal{F}_\alpha f(\alpha)\right\}(x)\, \mathcal{F}_\beta g(\beta)\right\}(x)\end{aligned} \tag{3.1}$$

where $\mathcal{F}_\alpha$ denotes the Fourier transform with frequency variable α. The interchange of integrals in this formal calculation can be rigorously verified for smooth and compactly supported functions f and g.

To recap, we are to establish the inequality (2.4) above. A central part of this is to diagonalize the sum

$$S_\mu(\Phi f, \Phi' g)(x) = \sum_{m,n\in\mathbf{Z}^d} \sum_{m',n'\in\mathbf{Z}^d} \langle f, \varphi_{m,n}\rangle \langle g, \varphi_{m',n'}\rangle S_\mu(\varphi_{m,n}, \varphi'_{m',n'})(x). \tag{3.2}$$

To simplify the notation of the proof, we specialize to the case in which $\varphi = \varphi'$, and the Fourier transform of φ is supported in $[-1/4, 1/4]^d$. The function φ satisfies in addition the remaining hypotheses of the lemma. The reader will easily supply the necessary changes in the proof for the general case, as they are only evident in the next paragraph.

It follows from (3.1) that for $\mu, m, m' \in \mathbf{Z}^d$,

$$S_\mu(\varphi_{m,n}, \varphi_{m',n'})(x) \equiv 0 \text{ if } m' - m \notin \{\mu + e \mid e \in \{0,1\}^d\}. \tag{3.3}$$

This will diagonalize the sums in (3.2) in the frequency variables, which is to say m and m'. Below, for notational convienence, we specialize to the case where $m' = m + \mu$.

For the diagonalization in space, let $n, n' \in \mathbf{Z}^d$, and observe that

$$S_\mu(\varphi_{m,n}, \varphi_{m+\mu,n'})(x) = \mathrm{e}_{2m+\mu}(x) \int \mathrm{e}_\mu(-y) k_\mu(y) \varphi(x+y-n)\varphi(x-y-n')\, dy.$$

In this integral, $k_\mu(y)$ is the convolution kernel associated to the operation S_μ. Hence, $\tilde{k}(y) = \mathrm{e}_\mu(-y)k_\mu(y)$ is independent of μ. The integral above is then

$$\psi_{n,n'-n}(x) = \int \tilde{k}(y)\varphi(x+y-n)\varphi(x-y-n')\, dy.$$

Below, we will denote the difference $n' - n$ by η.

The Fourier transform of $\tilde{k}(y)$ is smooth, hence

$$|\tilde{k}(y)| \le C_m (1 \wedge |y|^{-d})^m, \quad m > 1.$$

With the assumption stated on the decay of φ,

$$|\varphi(x)| \le B_\varphi (1 + |x|)^{-10d}, \tag{3.4}$$

it follows that for $n, \eta \in \mathbf{Z}^d$, $\psi_{n,\eta}$ satisfies

$$|\psi_{n,\eta}(x)| \le B_\varphi (1 + |\eta|)^{-5d} (1 + |x-n| + |x-n-\eta|)^{-5d}.$$

Provided a function ψ has a sufficently fast decay, the map of $L^2(\mathbf{R}^d)$ into $\ell^2(\mathbf{Z}^d)$ given by $f \to \{\langle f, \mathrm{e}_m(\cdot)\psi(\cdot - n)\rangle \mid m, n \in \mathbf{Z}^d\}$ is bounded and invertible. See for instance Section 3.4 of [**D**] for a discussion of this. In particular, with the estimate on $\psi_{n,\eta}$, we have the inequality below.

$$\left\| \sum_{m,n\in\mathbf{Z}^d} a_{m,n}\mathrm{e}_m(x)\psi_{n,\eta}(x) \right\|_2 \leq B_\varphi(1+|\eta|)^{-4d} \left(\sum_{m,n\in\mathbf{Z}^d} |a_{m,n}|^2 \right)^{1/2}. \tag{3.5}$$

The rapid decay in η permits us to diagonalize the sum in (3.2) in the space variables n and n'. Let us consider, with $\mu \in \mathbf{Z}^d$ fixed,

$$\begin{aligned}
&\left\| \sum_{\eta\in\mathbf{Z}^d} \sum_{m,n\in\mathbf{Z}^d} \langle f, \varphi_{m,n}\rangle\langle g, \varphi_{m+\mu,n+\eta}\rangle S_\mu(\varphi_{m,n}, \varphi_{m+\mu,n+\eta}) \right\|_2 \\
&\quad = \left\| \sum_{\eta\in\mathbf{Z}^d} \sum_{m,n\in\mathbf{Z}^d} \langle f, \varphi_{m,n}\rangle\langle g, \varphi_{m+\mu,n+\eta}\rangle \mathrm{e}_{2m+\mu}(x)\psi_{n,\eta}(x) \right\|_2 \\
&\quad \leq \sum_{\eta\in\mathbf{Z}^d} \left\| \sum_{m,n\in\mathbf{Z}^d} \langle f, \varphi_{m,n}\rangle\langle g, \varphi_{m+\mu,n+\eta}\rangle \mathrm{e}_{2m+\mu}(x)\psi_{n,\eta}(x) \right\|_2 \\
&\quad \leq C\, B_\varphi \sum_{\eta\in\mathbf{Z}^d} (1+|\eta|)^{-4d} \left(\sum_{m,n\in\mathbf{Z}^d} |\langle f, \varphi_{m,n}\rangle\langle g, \varphi_{m+\mu,n+\eta}\rangle|^2 \right)^{1/2} \\
&\quad \leq C\, B_\varphi \left(\sum_{\eta\in\mathbf{Z}^d} (1+|\eta|)^{-6d} \sum_{m,n\in\mathbf{Z}^d} |\langle f, \varphi_{m,n}\rangle\langle g, \varphi_{m+\mu,n+\eta}\rangle|^2 \right)^{1/2}.
\end{aligned} \tag{3.6}$$

We have invoked (3.5), and used a Cauchy-Schwartz inequality to control the sum over η. With rapid decay in $|\eta|$, that variable no longer plays an interesting role.

Do not forget that the square of the term above must be summed over μ as well. Thus we fix η, sum over μ and observe that the summing

variables m and μ separate.

$$\begin{aligned}
&\sum_{n\in\mathbf{Z}^d}\sum_{m,\mu\in\mathbf{Z}^d}|\langle f,\varphi_{m,n}\rangle\langle g,\varphi_{m+\mu,n+\eta}\rangle|^2\\
&\quad=\sum_{n\in\mathbf{Z}^d}\sum_{m,\mu\in\mathbf{Z}^d}|\langle f,\varphi_{m,n}\rangle\langle g,\varphi_{\mu,n+\eta}\rangle|^2 \qquad (3.7)\\
&\quad=\sum_{n\in\mathbf{Z}^d}\int_U\int_U\left|\sum_m \mathrm{e}_m(x)\langle f,\varphi_{m,n}\rangle\sum_\mu \mathrm{e}_\mu(y)\langle g,\varphi_{\mu,n+\eta}\rangle\right|^2 dx\,dy.
\end{aligned}$$

In the last line $U=[0,1)^d$.

From the definition of $\varphi_{m,n}$ observe that

$$\sum_{m\in\mathbf{Z}^d}\mathrm{e}_m(x)\langle f,\varphi_{m,n}\rangle=\sum_{m'\in\mathbf{Z}^d}\varphi(x-m'-n)f(x-m'),$$

where on the right, we have periodized the function $\varphi(x-n)f(x)$. It is clear that the right hand side has bounded $L^p([0,1)^d)$ norm, if $f\in L^p(\mathbf{R}^d)$. To be unambigious, let p' be the conjugate index to p, thus $1<p'<2$. Then, to simplify notation, we set $n=0$ and estimate as follows.

$$\begin{aligned}
&\left\|\sum_{m\in\mathbf{Z}^d}\mathrm{e}_m(x)\langle f,\varphi_{m,0}\rangle\right\|_{L^p([0,1)^d)}\\
&\qquad\le\sum_{m'\in\mathbf{Z}^d}\|f(x-m')\varphi(x-m')\|_{L^p([0,1)^d)}\\
&\qquad\le A\sum_{m\in\mathbf{Z}^d}(1+|m|)^{-10d}\|f(x-m)\|_{L^p([0,1)^d)}\\
&\qquad\le A\left(\sum_{m\in\mathbf{Z}^d}(1+|m|)^{-p'd}\right)^{1/p'}\\
&\qquad\quad\times\left(\sum_{m\in\mathbf{Z}^d}\|(1+|m|)^{-9d}f(x-m)\|^p_{L^p([0,1)^d)}\right)^{1/p}\\
&\qquad\le A\|(1+|x|)^{-9d}f(x)\|_{L^p(\mathbf{R}^d)}.
\end{aligned}$$

Clearly the same observations apply for any other choice of $n \neq 0$, and to g as well.

As $1/p + 1/q = 1/2$, it follows that we can bound the expression in (3.7), yielding the inequalities

$$\begin{aligned}
&\sum_{n\in\mathbf{Z}^d} \sum_{m,\mu\in\mathbf{Z}^d} |\langle f, \varphi_{m,n}\rangle\langle g, \varphi_{m+\mu,n+\eta}\rangle|^2 \\
&\qquad \le C \sum_{n\in\mathbf{Z}^d} \|f(x)(1+|x-n|)^{-9d}\|_p^2 \times \|g(x)(1+|x-n-\eta|)^{-9d}\|_q^2 \\
&\qquad \le C \left(\sum_{n\in\mathbf{Z}^d} \|f(x)(1+|x-n|)^{-9d}\|_p^p \right)^{2/p} \\
&\qquad\quad \times \left(\sum_{n\in\mathbf{Z}^d} \|g(x)(1+|x-n-\eta|)^{-9d}\|_q^q \right)^{2/q} \\
&\qquad \le C\|f\|_p^2 \|g\|_q^2.
\end{aligned} \tag{3.8}$$

Recall that we were to bound the sum over μ of $\|S_\mu(\Phi f, \Phi g)\|_2^2$, as expanded in (3.2). As pointed out in (3.3), the sum in (3.2) diagonalized in m and m'. The discussion leading to (3.5) shows that the sum effectively diagonalizes in the variables n and n' as well. Finally, (3.6) and (3.8) conclude the proof.

The constants that interceed depend only on the decay condition on φ, (3.4), which is as Lemma 2.2 requires.

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