Publicacions Matemàtiques, Vol **40** (1996), 127–133.

OSCILLATION OF SOLUTIONS OF SOME NONLINEAR DIFFERENCE EQUATIONS

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Abstract

Sufficient conditions for the oscillation of some nonlinear difference equations are established.

1. Introduction

In this note we consider the nonlinear difference equation of the form

(1) $$\Delta(r_n\Delta x_n) + q_n\, f(x_{n-\tau_n}) = 0, \quad n = 0, 1, 2, \dots ,$$

where Δ denotes the forward difference operator: $\Delta v_n = v_{n+1} - v_n$ for any sequence (v_n) of real numbers; (q_n) is a sequence of real numbers, (τ_n) is a sequence of integers such that

$$\lim_{n\to\infty}(n - \tau_n) = \infty,$$

(r_n) is a sequence of positive numbers and

$$R_n = \sum_{k=0}^{n-1}\frac{1}{r_k} \to \infty, \text{ as } n \to \infty.$$

$f : R \to R$ is a continuous with $u\, f(u) > 0$ $(u \neq 0)$.

By a solution of Equation (1) we mean a sequence (x_n) which is defined for

$$n \geq \min_{i\geq 0}(i - \tau_i)$$

and satisfies Equation (1) for all large n.

A nontrivial solution (x_n) of (1) is said to be oscillatory if for every $n_0 > 0$ there exists an $n \geq n_0$ such $x_n x_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

In several recent papers the oscillatory behaviour of solutions of nonlinear difference equations have been discussed e.g. see [**1**]-[**6**].

Our purpose in this paper is to give the sufficient conditions for the oscillation of solutions of Equation (1). The results obtained here extend those in [**6**].

2. Main results

Theorem 1. *Assume that*

(i) $q_n \geq 0$ *and* $\sum^{\infty} q_n = \infty$,
(ii) $\lim_{|u|\to\infty} \inf |f(u)| > 0$.

Then every solution of Equation (1) is oscillatory.

Proof: Assume, that Equation (1) has nonoscillatory solution (x_n), and we assume that (x_n) is eventually positive. Then there is a positive integer n_0 such that

$$x_{n-\tau_n > 0} \text{ for } n \geq n_0. \tag{2}$$

From the Equation (1) we have

$$\Delta(r_n \Delta x_n) = -q_n f(x_{n-\tau_n}) \leq 0, \quad n \geq n_0,$$

and so $(r_n \Delta x_n)$ is an eventually nonincreasing sequence. We first show that

$$r_n \Delta x_n \geq 0 \text{ for } n \geq n_0.$$

In fact, if there is an $n_1 \geq n_0$ such that $r_{n_1} \Delta x_{n_1} = c < 0$ and $r_n \Delta x_n \leq c$ for $n \geq n_1$ that is

$$\Delta x_n \leq \frac{c}{r_n}$$

and hence

$$x_n \leq x_{n_1} + c \sum_{k=n_1}^{n-1} \frac{1}{r_k} \to -\infty \text{ as } n \to \infty$$

which contradicts the fact that $x_n > 0$ for $n \geq n_1$. Hence $r_n \Delta x_n \geq 0$ for $n \geq n_0$. Therefore we obtain

$$x_{n-\tau_n} > 0, \quad \Delta x_n \geq 0, \quad \Delta(r_n \Delta x_n) \leq 0 \text{ for } n \geq n_0.$$

Let

$$L = \lim_{n\to\infty} x_n.$$

Then $L > 0$ is finite or infinite.

Case 1. $L > 0$ is finite.

From the continuity of function $f(u)$ we have

$$\lim_{n\to\infty} f(x_{n-\tau_n}) = f(L) > 0.$$

Thus, we may choose a positive integer n_3 ($\geq n_0$) such that

$$f(x_{n-\tau_n}) > \frac{1}{2} f(L) \quad n \geq n_3. \tag{3}$$

By substituting (3) into Equation (1) we obtain

$$\Delta(r_n \Delta x_n) + \frac{1}{2} f(L) q_n \leq 0, \quad n \geq n_3. \tag{4}$$

Summing up both sides of (4) from n_3 to n ($\geq n_3$), we obtain

$$r_{n+1}\Delta x_{n+1} - r_{n_3}\Delta x_{n_3} + \frac{1}{2} f(L) \sum_{i=n_3}^{n} q_i \leq 0$$

and so

$$\frac{1}{2} f(L) \sum_{i=n_3}^{n} q_i \leq r_{n_3}\Delta x_{n_3}, \quad n \geq n_3,$$

which contradicts (i).

Case 2. $L = \infty$.

For this case, from the condition (ii) we have

$$\lim_{n\to\infty} \inf f(x_{n-\tau_n}) > 0$$

and so we may choose a positive constant c and a positive integer n_4 sufficiently large such that

$$f(x_{n-\tau_n}) \geq c \text{ for } n \geq n_4. \tag{5}$$

Substituting (5) into Equation (1) we have

$$\Delta(r_n \Delta x_n) + c q_n \leq 0, \quad n \leq n_4.$$

Using the similar argument as that of Case 1 we may obtain a contradiction to the condition (i). This completes the proof. ■

Theorem 2. *Assume, that*

(iii) $q_n \geq 0$ *and* $\sum^{\infty} R_n q_n = \infty$, *then every bounded solution of (1) is oscillatory.*

Proof: Proceeding as in the proof of Theorem 1 with assumption that (x_n) is a bounded nonoscillatory solution of (1) we get the inequality (4) and so we obtain

$$R_n\Delta(r_n\Delta x_n) + \frac{1}{2}f(L)R_nq_n \le 0, \quad n \ge n_3. \tag{6}$$

It is easy to see that

$$R_n\Delta(r_n\Delta x_n) \ge \Delta(R_nr_n\Delta x_n) - r_n\Delta x_n\Delta R_n. \tag{7}$$

From inequalities (6) and (7) we deduce

$$\sum_{k=n_3}^{n}\Delta(R_kr_k\Delta x_k) - \sum_{k=n_3}^{n}\Delta x_k + \frac{1}{2}f(L)\sum_{k=n_3}^{n}R_qq_k \le 0 \quad n \ge n_3,$$

which implies

$$\frac{1}{2}f(L)\sum_{k=n_3}^{n}R_kq_k \le x_{n+1} + R_{n_3}r_{n_3}\Delta x_{n_3} - x_{n_3}, \quad n \ge n_3.$$

Hence there exists a constant c such that

$$\sum_{k=n_3}^{n}R_kq_k \le c \text{ for all } n \ge n_3,$$

contrary to the assumption of the theorem. ■

Theorem 3. *Assume that*

(iv) $(n-\tau_n)$ *is nondecreasing, where* $\tau_n \in \{0,1,2,\dots\}$,

(v) *there is a subsequence of* (r_n)*, say* (r_{n_k}) *such that* $r_{n_k} \le 1$ *for* $k = 0,1,2,\dots$,

(vi) $\sum_{n=0}^{\infty} q_n = \infty$,

(vii) f *is nondecreasing and there is a nonnegative constant* M *such that*

$$\limsup_{u\to 0}\frac{u}{f(u)} = M. \tag{8}$$

Then the difference (Δx_n) *of every solution* (x_n) *of Equation (1) oscillates.*

Proof: If not, then Equation (1) has a solution (x_n) such that its difference (Δx_n) is nonoscillatory. Assume first that the sequence (Δx_n) is eventually negative. Then there is a positive integer n_0 such that

$$\Delta x_n < 0 \quad n > n_0$$

and so (x_n) is decreasing for $n \geq n_0$ which implies that (x_n) is also nonoscillatory. Set

$$w_n = \frac{r_n \Delta x_n}{f(x_{n-\tau_n})}, \quad n \geq n_1 \geq n_0. \tag{9}$$

Then

$$\begin{aligned} \Delta w_n &= \frac{r_{n+1}\Delta x_{n+1}}{f(x_{n+1-\tau_{n+1}})} - \frac{r_n \Delta x_n}{f(x_{n-\tau_n})} \\ &= \frac{\Delta(r_n \Delta x_n)}{f(x_{n-\tau_n})} + r_{n+1}\Delta x_{n+1} \frac{f(x_{n-\tau_n}) - f(x_{n+1-\tau_{n+1}})}{f(x_{n+1-\tau_{n+1}}) f(x_{n-\tau_n})} \\ &\leq \frac{\Delta(r_n \Delta x_n)}{f(x_{n-\tau_n})} = -q_n, \quad n \geq n_1. \end{aligned} \tag{10}$$

Summing up both sides of (10) from n_1 to n, we have

$$w_{n+1} - w_{n_1} \leq -\sum_{i=n_1}^{n} q_i$$

and, by (vi), we get

$$\lim_{n\to\infty} w_n = -\infty, \tag{11}$$

which implies that eventually

$$f(x_{n-\tau_n}) > 0 \text{ and therefore } x_{n-\tau_n} > 0. \tag{12}$$

By (11), we can choose n_2 $(\geq n_1)$ such that

$$w_n \leq -(M+1), \quad n \geq n_2.$$

That is

$$r_n \Delta x_n + (M+1) f(x_{n-\tau_n}) \leq 0, \quad n \geq n_2. \tag{13}$$

Set

$$\lim_{n\to\infty} x_n = L.$$

Then $L \geq 0$. Now we prove that $L = 0$. If $L > 0$, then we have

$$\lim_{n\to\infty} f(x_{n-\tau_n}) = f(L) > 0,$$

by the continuity of $f(u)$. Choosing an n_3 sufficiently large, such that

$$f(x_{n-\tau_n}) > \frac{1}{2} f(L), \quad n \geq n_3 \tag{14}$$

and substituting (14) into (13), we have

$$\Delta x_n + \frac{1}{2r_n}(M+1)f(L) \leq 0, \quad n \geq n_3. \tag{15}$$

Summing up both sides of (15) from n_3 to n we get

$$x_{n+1} - x_{n_3} + \frac{1}{2}(M+1)f(L) \sum_{i=n_3}^{n} \frac{1}{r_i} \leq 0$$

which implies that

$$\lim_{n\to\infty} x_n = -\infty.$$

This contradicts (12). Hence

$$\lim_{n\to\infty} x_n = 0.$$

By the assumptions we have

$$\lim_{n\to\infty} \sup \frac{x_{n-\tau_n}}{f(x_{n-\tau_n})} \leq M.$$

From this we can choose n_4, such that

$$\frac{x_{n-\tau_n}}{f(x_{n-\tau_n})} < M + 1, \quad n \geq n_4.$$

That is

$$x_{n-\tau_n} < (M+1) f(x_{n-\tau_n}), \quad n \geq n_4,$$

and so from (13) we get

$$r_n \Delta x_n + x_{n-\tau_n} < 0, \quad n \geq n_4. \tag{16}$$

In particular, from (16) for a subsequence (r_{n_k}) satisfying the condition (v), we have

$$x_{n_k+1} - x_{n_k} + x_{n_k - \tau_{n_k}} \leq r_{n_k}(x_{n_k+1} - x_{n_k}) + x_{n_k - \tau_{n_k}} < 0,$$

for k sufficiently large, which implies that

$$0 < x_{n_k+1} + (x_{n_k - \tau_{n_k}} - x_{n_k}) < 0$$

for all large k. This is a contradiction.

The case that (Δx_n) is eventually positive can be treated in a similar fashion and so the proof of Theorem 3 is completed. ■

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Primera versió rebuda el 15 de Maig de 1995,
darrera versió rebuda el 18 d'Octubre de 1995