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ON THE DEFINITION OF THE DUAL LIE COALGEBRA OF A LIE ALGEBRA

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Abstract

Let L be a Lie algebra over a field K. The dual Lie coalgebra L° of L has been defined by W. Michaelis to be the sum of all good subspaces V of the dual space L^* of L: V is good if ${}^t m(V) \subset V \otimes V$, where m is the multiplication of L. We show that $L^\circ = {}^t m^{-1}(L^* \otimes L^*)$ as in the associative case.

Let L be a Lie algebra over the field K with multiplication $m : L \otimes L \to L$: i.e. m is a linear map and setting $m(x \otimes y) = [x, y]$, one has

$$[x, x] = 0 \tag{1}$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0. \tag{2}$$

Let L^* be the dual vector space of L and ${}^t m : L^* \to (L \otimes L)^*$ be the transpose of m. We identify $L^* \otimes L^*$ with a subspace of $(L \otimes L)^*$ and we set $L^\odot = {}^t m^{-1}(L^* \otimes L^*)$.

Fix $f \in L^*$ and consider the linear map $\gamma_f : L \to L^*$ defined by

$$\langle \gamma_f(x), y \rangle = \langle f, [x, y] \rangle = \langle {}^t m(f), x \otimes y \rangle, \quad x,\, y \in L. \tag{3}$$

Setting, as usual, $adx(y) = [x, y]$, one has $\gamma_f(x) = {}^t(adx)(f)$. Sometimes, we shall write $\gamma_f(x) = x \cdot f$.

Lemma 1. *For $f \in L^*$, the following statements are equivalent*

(i) $f \in L^{\odot}$

(ii) *The linear map $\gamma_f : L \to L^*$ is of finite rank.*

Proof: The equivalence follows readily from (3).

Moreover, since $L^* \otimes L^*$ can be identified with the space of the linear maps of L into L^* of finite rank, via $\iota : L^* \otimes L^* \hookrightarrow \mathrm{Hom}(L, L^*)$ by setting $\iota(f \otimes g)(x) = f(x)g$, one has $f \in L^{\odot}$ iff $\gamma_f = \sum_{j=1}^{n} f_j \otimes g_j$ iff ${}^t m(f) = \sum_{j=1}^{n} f_j \otimes g_j$. ■

Lemma 2. *For $f \in L^{\odot}$ and $x \in L$, one has, $\gamma_f(x) = x \cdot f \in L^{\odot}$. Moreover $\gamma_f(L)$ is a vector subspace of $L^{\odot}$ of finite dimension.*

Proof: If $f \in L^{\odot}$, then for x, $y \in L$, one has $\langle f, [x,y] \rangle = \langle \gamma_f(x), y \rangle = \sum_{j=1}^{n} \langle f_j, x \rangle \, \langle g_j, y \rangle$. However, (2) can be written $[x,[y,z]] = -[y,[z,x]] - [z,[x,y]]$. Therefore, one has

$$
\begin{aligned}
\langle \gamma_{x \cdot f}(y), z \rangle &= \langle x \cdot f, [y,z] \rangle \\
&= \langle f, [x,[y,z]] \rangle \\
&= -\langle f, [y,[z,x]] \rangle - \langle f, [z,[x,y]] \rangle \\
&= -\sum_{j=1}^{n} \langle f_j, y \rangle \langle g_j, [z,x] \rangle - \sum_{j=1}^{n} \langle f_j, z \rangle \langle g_j, [x,y] \rangle \\
&= \sum_{j=1}^{n} \langle f_j, y \rangle \, \langle x \cdot g_j, z \rangle - \sum_{j=1}^{n} \langle x \cdot g_j, y \rangle \, \langle f_j, z \rangle.
\end{aligned}
$$

Hence $\gamma_{x \cdot f}(y) = \sum_{j=1}^{n} \langle f_j, y \rangle x \cdot g_j - \sum_{j=1}^{n} \langle x \cdot g_j, y \rangle f_j$. It follows that $\gamma_{x \cdot f}$ is of finite rank, that is $x \cdot f = \gamma_f(x) \in L^{\odot}$. Then, it is clear that $\gamma_f(L)$ is a vector subspace of $L^{\odot}$ and finite dimensional. ■

Note. One deduces from the above proof that if $f \in L^{\odot}$ and ${}^t m(f) = \sum_{j=1}^{n} f_j \otimes g_j$ then for $x \in L$, one has

$$ {}^t m(x \cdot f) = \sum_{j=1}^{n} f_j \otimes (x \cdot g_j) - \sum_{j=1}^{n} (x \cdot g_j) \otimes f_j. $$

Theorem 1. *$L^{\odot}$ is a good subspace of L^* i.e. ${}^t m(L^{\odot}) \subset L^{\odot} \otimes L^{\odot}$. Moreover, one has $L^{\odot} = L^{\circ}$.*

Proof: Let $f \in L^{\odot}$ and let $(g_j)_{1 \le j \le n}$ be a base of $\gamma_f(L) \subset L^{\odot}$. One has $g_j = x_j \cdot f$ and for any $x \in L$, $\gamma_f(x) = \sum_{j=1}^{n} f_j(x) g_j$; hence ${}^t m(f) = \sum_{j=1}^{n} f_j \otimes g_j$, $f_j \in L^*$. However $m = -m \circ \tau$ (skew-symmetry), therefore, one has ${}^t m = -{}^t\tau \circ {}^t m$ and ${}^t m(f) = \sum_{j=1}^{n} f_j \otimes g_j = -{}^t\tau(\sum_{j=1}^{n} f_j \otimes g_j) = -\sum_{j=1}^{n} g_j \otimes f_j$. Since $(g_j)_{1 \le j \le n}$ is free in L^*, there exists for $1 \le \ell \le n$ $y_\ell \in L$ such that $\langle g_j, y_\ell \rangle = \delta_{j\ell}$. Hence $(y_\ell \otimes 1_{L^*})({}^t m(f)) = \sum_{j=1}^{n} \langle f_j, y_\ell \rangle g_j = -\sum_{j=1}^{n} \langle g_j, y_\ell \rangle f_j = -f_\ell$, that is $f_\ell = -\sum_{j=1}^{n} \langle f_j, y_\ell \rangle g_j \in \gamma_f(L)$. It follows that ${}^t m(f) = \sum_{j=1}^{n} f_j \otimes g_j \in \gamma_f(L) \otimes \gamma_f(L) \subset L^{\odot} \otimes L^{\odot}$ and $L^{\odot} \subset L^{\circ}$.

On the other hand, it is clear that any good subspace V of L^* is contained in $L^{\odot}$, therefore $L^{\circ} \subset L^{\odot}$. We have proved that $L^{\odot} = L^{\circ}$. ■

Note. If $f \in L^{\odot}$ and if $(x_j \cdot f)_{1 \le j \le n}$ is a base of $\gamma_f(L)$, one has ${}^t m(f) = \sum_{j=1}^{n} (y_j \cdot f) \otimes (x_j \cdot f)$ where, for $1 \le j \le n$, y_j is such that $\langle f, [x_\ell, y_j] \rangle = \delta_{\ell j}$.

Furthermore, if $x \in L$, one has

$$ (4) \quad {}^t m(x \cdot f) = \sum_{j=1}^{n} (y_j \cdot f) \otimes [x \cdot (x_j \cdot f)] - \sum_{j=1}^{n} [x \cdot (x_j \cdot f)] \otimes (y_j \cdot f). $$

Remark. Put $\Delta = {}^t m|_{L^\odot} : L^\odot \to L^\odot \otimes L^\odot$. Following W. Michaelis [**1**], (see also [**2**], [**3**], [**4**], [**5**] and [**6**]) one obtains a Lie coalgebra $(L^\odot, \Delta)$, that is :

$$\Delta = -\tau \circ \Delta \tag{5}$$

if the characteristic of K is different from 2 and $\operatorname{Im}\Delta \subset \operatorname{Im}(1_{L^\odot} - \tau)$ otherwise $[\tau(f \otimes g) = g \otimes f]$.

$$(id_3 + \sigma + \sigma^2) \circ (1_{L^\odot} \otimes \Delta) \circ \Delta = 0 \tag{6}$$

where $\sigma(f \otimes g \otimes h) = h \otimes f \otimes g$.

This follows from (1) and (2). Notice that (2) is equivalent to $m \circ (1_L \otimes m) \circ (id_3 + \rho + \rho^2) = 0$ where $\rho(x \otimes y \otimes z) = z \otimes x \otimes y$ and one has ${}^t\rho^2|_{L^\odot} = \sigma$.

For $A \subset L^\odot$, let span(A) be the vector subspace of $L^\odot$ spaned by A.

Theorem 2. *Let $f \in L^\odot$. Put*

$V_0 = K \cdot f$
$V_1 = \gamma_f(L) = \{x_1 \cdot f, x_1 \in L\}$
$V_2 = \operatorname{span}\{x_2 \cdot f_1,\ x_2 \in L,\ f_1 \in V_1\}$
....
$V_n = \operatorname{span}\{x_n \cdot f_{n-1},\ x_n \in L,\ f_{n-1} \in V_{n-1}\}$
....

Then $W = \sum_{n\geq 0} V_n$ is a Lie subcoalgebra of $L^\odot$ and is the smallest Lie subcoalgebra of $L^\odot$ that contains f.

Proof: We have seen that if $f \in L^\odot$, then $\Delta(f) = \sum_{j=1}^{n} (y_j \cdot f) \otimes (x_j \cdot f)$. It follows that $\Delta(V_0) \subset V_1 \otimes V_1$. Furthermore $V_1 \subset L^\odot$, and by induction one has $V_n \subset L^\odot$. On the other hand, if $x_n \in L$, $f_{n-1} \in V_{n-1}$, $n \geq 1$, one has by (4)

$$\Delta(x_n \cdot f_{n-1}) = \sum_{j=1}^{m} (y_{nj} \cdot f_{n-1}) \otimes [x_n \cdot (x_{nj} \cdot f_{n-1})]$$
$$- \sum_{j=1}^{m} [x_n \cdot (x_{nj} \cdot f_{n-1})] \otimes (y_{nj} \cdot f_{n-1}) \in V_n \otimes V_{n+1} + V_{n+1} \otimes V_n.$$

Therefore, $\Delta(V_n) \subset V_n \otimes V_{n+1} + V_{n+1} \otimes V_n \subset W \otimes W$, $n \geq 1$, and since $\Delta(V_0) \subset V_1 \otimes V_1 \subset W \otimes W$, one has $\Delta(W) \subset W \otimes W$, i.e. W is a Lie subcoalgebra of $L^\odot$.

Let V be a Lie subcoalgebra of $L^{\odot}$. For any $h \in V$ and $x \in L$, one has $\Delta(h) = \sum_{j=1}^{n} h_j^1 \otimes h_j^2 \in V \otimes V$ and $x \cdot h = \gamma_h(x) = \sum_{j=1}^{n} \langle h_j^1, x\rangle h_j^2 \in V$. Therefore, if V contains f, one has $V_0 \subset V$ and $V_1 = \gamma_f(L) \subset V$. It is readily seen by induction that $V_n \subset V$ for all $n \geq 0$. It follows that $W = \sum_{n\geq 0} V_n$ is contained in V. ■

Note.

(i) One can prove, by induction, that the above V_n, $n \geq 0$, are finite dimensional.

(ii) One has for $n \geq 1$, $V_n = \mathrm{span}\{{}^t ad\ x_1 \circ {}^t ad\ x_2 \circ \ldots \circ {}^t ad\ x_n(f), x_1, \ldots, x_n \in L\}$.

Therefore, if L is *nilpotent* of class k, then for any $f \in L^{\odot}$, the associated sequence of subspaces $(V_n)_{n\geq 0}$ is such that $V_n = (0)$, for $n \geq k$. It follows that f belongs to the finite dimensional Lie subcoalgebra $W = \sum_{n=0}^{k-1} V_n$ of $L^{\odot}$. Hence, one has $L^{\odot} = \mathrm{Loc}(L^{\odot})$ the sum of the finite dimensional Lie subcoalgebras of $L^{\odot}$. In particular, if L is abelian, one has $V_n = (0)$, $n \geq 1$, and $L^{\odot} = L^*$.

More generally, one sees that $L^{\odot} = \mathrm{Loc}(L^{\odot})$ iff for each $f \in L^{\odot}$ the above associated Lie subcoalgebra W of $L^{\odot}$ is finite dimensional; in this case, there exists k such that $W = \sum_{n=0}^{k} V_n$. Question : what is the class of all Lie algebras L such that $L^{\odot} = \mathrm{Loc}(L^{\odot})$?

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