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CALCULATING THE GENUS OF A DIRECT PRODUCT OF CERTAIN NILPOTENT GROUPS

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Abstract

The Mislin genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroup admits an abelian group structure. If N satisfies some additional conditions —we say that N belongs to $\mathcal{N}_1$— we know exactly the structure of $\mathcal{G}(N)$. Considering a direct product $N_1 \times \cdots \times N_k$ of groups in $\mathcal{N}_1$ takes us virtually always out of $\mathcal{N}_1$. We here calculate the Mislin genus of such a direct product.

1. Introduction.

By $\mathcal{N}_0$ we denote the class of finitely generated infinite nilpotent groups N with finite commutator subgroup $[N, N]$. From [**1**], [**2**] we know that the (Mislin) genus $\mathcal{G}(N)$, for $N \in \mathcal{N}_0$, may then be given the structure of a finite abelian group. Moreover, if N is a nilpotent group and we consider the short exact sequence $0 \to TN \to N \to FN \to 0$, where TN is the torsion subgroup of N and FN the torsionfree quotient, then $N \in \mathcal{N}_0$ if and only if TN is finite and FN is free abelian of finite rank. If additionally

(1) TN is abelian;
(2) $0 \to TN \to N \to FN \to 0$ splits on the right, so that N is the semidirect product for an action $\omega : FN \to \operatorname{Aut}(TN)$ of FN on TN;
(3) the action ω satisfies $\omega(FN) \subseteq Z \operatorname{Aut}(TN)$, where Z denotes the centre,

then we say that $N \in \mathcal{N}_1 \subset \mathcal{N}_0$. Note that (finite) direct products of members of $\mathcal{N}_1$ inherit properties (1) and (2) above, but not, in general, property (3).

Recall from [**3**] that, given (1), (3) is equivalent to requiring that for each $\xi \in FN$, there exists $u \in \mathbb{Z}$, prime to $\exp TN$, such that $\xi \cdot a = ua$ for all $a \in TN$ (TN is here written additively).

Now if t is the height of $\ker \omega$ in FN (meaning that t is the largest positive integer m such that $\ker \omega \subseteq mFN$), then we know from [**3**]

1.1. Theorem. *$\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}$, for $N \in \mathcal{N}_1$.*

Moreover it is proved in [**4**] that

1.2. Theorem. *For $N \in \mathcal{N}_1$ with FN not cyclic, $\mathcal{G}(N^k) = 0$ for any $k \geq 1$, where N^k is the k^{th} direct power of N.*

Now, if $\exp TN = n = p_1^{m_1} \dots p_s^{m_s}$, $p_1 < p_2 < \cdots < p_s$, $m_i \geq 1$, we know that t must have the form $t = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_s^{\lambda_s}$, with $0 \leq \lambda_i < m_i$ ($i = 1, 2, \dots, s$); we write $p_i^{\lambda_i} \| \, t$, $i = 1, 2, \dots, s$. We also write $T(N)$ for the collection of primes $(p_1, p_2, \dots, p_s)$.

In [**5**] the authors calculate $\mathcal{G}(N^k)$ for $N \in \mathcal{N}_1$ with FN cyclic and $k \geq 2$, obtaining the following theorem.

1.3. Theorem. *For $N \in \mathcal{N}_1$ with FN cyclic and for any $k \geq 2$, we obtain $\mathcal{G}(N^k)$ from $\mathcal{G}(N)$ by factoring out those residues $m \mod t$ such that*

$$m \equiv \epsilon_i \mod p_i^{\lambda_i}, \;\; \epsilon_i = \pm 1, \;\; i = 1, 2, \dots, s.$$

In this paper we will generalize these calculations to obtain a result for the genus of a direct product, $\mathcal{G}(N_1 \times \cdots \times N_k)$, where $N_1, \dots, N_k \in \mathcal{N}_1$. In the third section we will show that, if the direct product involves a group $N_j \in \mathcal{N}_1$ with a non-cyclic torsionfree quotient FN_j, then the genus of the direct product is trivial. Note that this is a generalization of Theorem 1.2. In fact, we prove

1.4. Theorem. *For $N_1 \in \mathcal{N}_1$ with FN_1 not cyclic and $N_2 \in \mathcal{N}_0$, we have*

$$\mathcal{G}(N_1 \times N_2) = 0.$$

In the case where the direct product only involves groups $N_1, N_2, \dots, N_k \in \mathcal{N}_1$, all with a cyclic torsionfree quotient FN_i, an important role is played by the so-called *generators that obstruct an isomorphism*. In the definition below, we write $|a|$ for the order of the element a in some given group.

1.5. Definition. Let $N_1, N_2 \in \mathcal{N}_1$ and $p \in T = T(N_1 \times N_2)$. Suppose that

$$\begin{aligned}(TN_1)_p &= \langle a_{1(1)}\rangle \oplus \langle a_{2(1)}\rangle \oplus \cdots \oplus \langle a_{s_1(1)}\rangle,\\ &\qquad \text{where } \exp(TN_1)_p = |a_{1(1)}| \geq |a_{2(1)}| \geq \cdots \geq |a_{s_1(1)}|;\\ (TN_2)_p &= \langle a_{1(2)}\rangle \oplus \langle a_{2(2)}\rangle \oplus \cdots \oplus \langle a_{s_2(2)}\rangle,\\ &\qquad \text{where } \exp(TN_2)_p = |a_{1(2)}| \geq |a_{2(2)}| \geq \cdots \geq |a_{s_2(2)}|.\end{aligned}$$

Let $2 \leq x \leq \min(s_1, s_2) + 1$, and suppose that

$$|a_{1(1)}| = |a_{1(2)}|,\ |a_{2(1)}| = |a_{2(2)}|, \dots, |a_{x-1(1)}| = |a_{x-1(2)}|.$$

Then we say that $a_{x(1)}$ *obstructs an isomorphism* between $(TN_1)_p$ and $(TN_2)_p$ if either $|a_{x(1)}| \neq |a_{x(2)}|$ or $x = s_2 + 1 \leq s_1$. Similarly we speak of $a_{x(2)}$ obstructing an isomorphism. We call the *order of obstruction* (of $(TN_1)_p, (TN_2)_p$) the maximum of the orders of all generators of $(TN_1)_p, (TN_2)_p$ obstructing an isomorphism. Of course, the order of obstruction is independent of the choice of direct sum decomposition of $(TN_1)_p, (TN_2)_p$.

In the course of the fourth section we will prove our main theorem, namely,

1.6. Theorem. *Let $N_1, N_2, \dots, N_k \in \mathcal{N}_1$ with $\mathcal{G}(N_i) \cong (\mathbb{Z}/t_i)^*/\{\pm 1\}$. Set*

$$t = \gcd(t_1, \dots, t_k) = p_1^{\lambda_1} \dots p_s^{\lambda_s}.$$

Let $FN_i = \langle \xi_i \rangle$ with $\xi_i \cdot a = u_i a$ for $a \in TN_i$. Define P to be the set of prime divisors p of t such that there are distinct r, $v \in \{1, \dots, k\}$ for which the following conditions hold:

1. $\exp(TN_r)_p = \exp(TN_v)_p$*;*
2. $u_v \in \langle u_r \rangle$*,* $u_r \in \langle u_v \rangle$*, where* u_r*,* u_v *are viewed as elements of* $(\mathbb{Z}/\exp(TN_v)_p)^*$*;*
3. *On those generators of $(TN_r)_p$ and $(TN_v)_p$ that obstruct an isomorphism between these two torsion groups, the actions of ξ_v, ξ_r are trivial. This means that*

$$u_v \equiv u_r \equiv 1 \textit{ modulo the order of obstruction.}$$

Then we obtain $\mathcal{G}(N_1 \times \cdots \times N_k)$ from $(\mathbb{Z}/t)^*$ by factoring out the residue class of -1 and those residues $m \mod t$ such that

$$\begin{cases} m \equiv 1 \mod p_i^{\lambda_i} & \textit{for all } p_i \notin P\\ m \equiv 1 \textit{ or } -1 \mod p_i^{\lambda_i} & \textit{for all } p_i \in P.\end{cases}$$

Note that this is indeed a generalization of Theorem 1.3. For if $N_1 = N_2 = \cdots = N_k$, then P would consist of all primes p_i dividing t, so that $\mathcal{G}(N^k)$ would be obtained from $(\mathbb{Z}/t)^*$ by dividing out those residues $m \mod t$ which are congruent to 1 or $-1 \mod p_i^{\lambda_i}$ for all p_i.

1.7. Corollary. *Assume further that $t = p^\lambda$. Then, with no further hypothesis,*

$$\mathcal{G}(N_1 \times \cdots \times N_k) \cong (\mathbb{Z}/t)^*/\{\pm 1\}.$$

It is also interesting to note that the condition (2), namely, $u_r \in \langle u_v \rangle$, $u_v \in \langle u_r \rangle$ is in fact equivalent to $|u_r| = |u_v|$, if the group $(\mathbb{Z}/\exp(TN_v)_p)^*$ is cyclic. This group is indeed cyclic if $p \neq 2$. However if $p = 2$ and $m \geq 3$, the group $(\mathbb{Z}/2^m)^*$ is not cyclic, and in this case we cannot replace the given condition by the weaker condition $|u_r| = |u_v|$, as Example 4.4 will show.

We anticipate that the notions of *generators obstructing an isomorphism* and the *order of obstruction to an isomorphism* may prove to be of interest beyond the scope of this paper. Notice that we only apply these notions to groups N_1, N_2 such that $\exp(TN_1) = \exp(TN_2)$, since we insist in Definition 1.5 that $x \geq 2$.

2. Some preliminary results.

Recall from [**2**], [**3**] the following exact sequence (where $N \in \mathcal{N}_0$)

$$T\text{-}\operatorname{Aut} N \xrightarrow{\theta} (\mathbb{Z}/e)^*/\{\pm 1\} \to \mathcal{G}(N) \to 0.$$

Here $T = T(N)$ is the set of prime divisors of $n = \exp TN$, $QN = N/FZN$, FZN being the free center of N, $e = \exp QN_{ab}$, and T-Aut N is the semigroup of self T-equivalences of N. Recall also how θ acts. For any T-automorphism φ, $\theta(\varphi)$ is the residue class modulo ± 1 of $\det \varphi$, φ being restricted to FZN. (In [**1**], [**2**] it is shown that a T-automorphism sends FZN to itself). Moreover in [**5**] the authors show the following.

2.1. Lemma. *Let $\varphi : N \to N$ be an endomorphism. Then φ induces $\psi : FN \to FN$. If $\varphi(FZN) \subseteq FZN$, then* $\det(\varphi|FZN) = \det \psi$.

So, for a T-automorphism φ of N, $\theta(\varphi)$ is in fact the residue class of $\det \psi$. For $N \in \mathcal{N}_0$ satisfying conditions (1) and (2) of $\mathcal{N}_1$, we also have the following ([**5**], [**6**]).

2.2. Lemma. *An endomorphism φ of N induces a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TN & \longrightarrow & N & \longrightarrow & FN & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & TN & \longrightarrow & N & \longrightarrow & FN & \longrightarrow & 0 \end{array}$$

and φ is a T-automorphism if and only if α is an automorphism and ψ is a T-automorphism.

2.3. Lemma.

(i) *For all $\xi \in FN$ and for all $a \in TN$, we have $\alpha(\xi \cdot a) = \psi(\xi) \cdot \alpha(a)$.*

(ii) *Suppose that a diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TN & \longrightarrow & N & \longrightarrow & FN & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & TN & \longrightarrow & N & \longrightarrow & FN & \longrightarrow & 0 \end{array}$$

is given, such that $\alpha(\xi \cdot a) = \psi(\xi) \cdot \alpha(a)$, for all $\xi \in FN$ and for all $a \in TN$. Then we may find $\varphi : N \to N$ making a commutative diagram as in the previous lemma.

We call (i) above the *compatibility condition*.

3. The genus of a direct product, involving a group in $\mathcal{N}_1$ with a non-cyclic torsionfree quotient.

Proof of Theorem 1.4: Set $T = T(N_1) \cup T(N_2)$. Since $N_1 \times N_2 \in \mathcal{N}_0$, we have the following exact sequence:

$$T\text{-}\operatorname{Aut}(N_1 \times N_2) \xrightarrow{\ \theta\ } (\mathbb{Z}/e)^*/\{\pm 1\} \longrightarrow \mathcal{G}(N_1 \times N_2) \longrightarrow 0$$

where $e = \operatorname{lcm}(e_1, e_2)$. We show that we can realize the residue class of any m, prime to e, by some T-automorphism ϕ of $N_1 \times N_2$. In other words we show that for any m prime to e, there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TN_1 \times TN_2 & \longrightarrow & N_1 \times N_2 & \longrightarrow & FN_1 \times FN_2 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & TN_1 \times TN_2 & \longrightarrow & N_1 \times N_2 & \longrightarrow & FN_1 \times FN_2 & \longrightarrow & 0 \end{array}$$

where α is an automorphism and ϕ, ψ are T-automorphisms, such that $\det \psi = m$.

Choose a basis for FN_1 such that

$$FN_1 = \langle \xi_1, \xi_2, \dots, \xi_r \rangle, \ker \omega_1 = \langle t_1\xi_1, t_2\xi_2, \dots, t_r\xi_r \rangle$$

where $t = t_1 \mid t_2 \mid \dots \mid t_r$ and ω_1 is the action of FN_1 on TN_1. Let $\xi_i \cdot a = u_i a$ for $a \in TN_1$. Remark that the order of u_i modulo $\exp(TN_1)$ is then t_i. Now set

$$\alpha = \mathrm{Id}_{TN_1 \times TN_2}$$

and (in additive notation)

$$\begin{cases} \psi(\xi_1) = m\xi_1 + l\xi_2, \text{ where } l \text{ remains to be determined} \\ \psi(\xi_j) = \xi_j \quad (j \neq 1) \\ \psi | FN_2 = \mathrm{Id}_{FN_2}\,. \end{cases}$$

Then we have only to verify the compatibility condition (Lemma 2.3) for ξ_1. Now

$$\alpha(\xi_1 \cdot a) = \psi(\xi_1) \cdot \alpha(a) \text{ for all } a \in TN_1 \times TN_2$$

if and only if

$$u_1 a = u_1^m u_2^l a \text{ for all } a \in TN_1,$$

which is equivalent to

$$u_1^{m-1} u_2^l \equiv 1 \mod \exp(TN_1). \tag{3.1}$$

We now have one of the following three possibilities:

(1) If e is even and t_1 is even, then m is odd, $m-1$ is even and thus $u_1^{m-1} \in \langle u_1^2 \rangle$;
(2) If e is even and t_1 is odd, then $u_1 \in \langle u_1^2 \rangle$, since t_1 is odd;
(3) If e is odd, then t_1 is odd (because $t_1 | e_1 | e$) and $u_1 \in \langle u_1^2 \rangle$, since t_1 is odd.

Moreover, by the same argument as in Theorem 1.1 of [**4**], we can show that in any case $u_1^2 \in \langle u_2 \rangle$. Thus in each of the three cases it is clear that we can always solve (3.1) for l. Moreover $\det \psi = m$, which completes our proof. ■

4. The genus of a direct product of $N_1, \dots, N_k$ in $\mathcal{N}_1$, each N_i having a cyclic torsionfree quotient FN_i.

Let $N_1, N_2, \dots, N_k \in \mathcal{N}_1$ with $\mathcal{G}(N_i) \cong (\mathbb{Z}/t_i)^*/\{\pm 1\}$, t_i being defined as in Section 1. Set $t = \gcd(t_1, \dots, t_k) = p_1^{\lambda_1} \dots p_s^{\lambda_s}$ and set $T = T(N_1 \times N_2 \times \cdots \times N_k)$. Suppose that, for $p \mid t$ and for $i = 1, \dots, k$,

$$(TN_i)_p = \langle a_{1(i)} \rangle \oplus \langle a_{2(i)} \rangle \oplus \cdots \oplus \langle a_{s_i(i)} \rangle$$

with $\exp(TN_i)_p = |a_{1(i)}| \geq |a_{2(i)}| \geq \cdots \geq |a_{s_i(i)}|$. Let $FN_i = \langle \xi_i \rangle$ with $\xi_i \cdot a = u_i a$ for $a \in TN_i$.

To calculate $\mathcal{G}(N_1 \times \cdots \times N_k)$ for $N_1, \dots, N_k \in \mathcal{N}_1$, we will use the exact sequence

$$T\text{-}\operatorname{Aut}(N_1 \times \cdots \times N_k) \xrightarrow{\theta} (\mathbb{Z}/e)^*/\{\pm 1\} \to \mathcal{G}(N_1 \times \cdots \times N_k) \to 0,$$

which is valid, since clearly $N_1 \times \cdots \times N_k \in \mathcal{N}_0$. In the following two propositions we will give a description of $\operatorname{im}\theta$. From these we can then conclude how to use Theorem 1.6 to obtain $\mathcal{G}(N_1 \times \cdots \times N_k)$.

4.1. Proposition. *Consider the following commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \to & TN_1 \times \cdots \times TN_k & \to & N_1 \times \cdots \times N_k & \to & FN_1 \times \cdots \times FN_k & \to & 0 \\
& & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\
0 & \to & TN_1 \times \cdots \times TN_k & \to & N_1 \times \cdots \times N_k & \to & FN_1 \times \cdots \times FN_k & \to & 0
\end{array}$$

where α is an automorphism and φ, ψ are T-automorphisms. Let $p \mid t$.

Let $\psi(\xi_m) = \sum_{j=1}^{k} \beta_{mj}\xi_j$ for $m = 1, \dots, k$, and let

$$\alpha_p(a_{i(v)}) = \sum_{\ell=1}^{s_1} \alpha_{i(v)}^{\ell(1)} a_{\ell(1)} + \sum_{\ell=1}^{s_2} \alpha_{i(v)}^{\ell(2)} a_{\ell(2)} + \cdots + \sum_{\ell=1}^{s_k} \alpha_{i(v)}^{\ell(k)} a_{\ell(k)}$$

for $v \in \{1, \dots, k\}$ and $i \in \{1, \dots, s_v\}$.

Then there exists a bijection $f : \{1, \dots, k\} \longrightarrow \{1, \dots, k\} : j \longmapsto f(j)$, where $f(j)$ is the unique index such that

$$p \nmid \alpha_{q(f(j))}^{1(j)} \text{ for some } q \in \{1, \dots, s_{f(j)}\}.$$

Moreover, we also have

(1) $\exp(TN_j)_p = \exp(TN_{f(j)})_p$*;*

(2) $u_j \in \langle u_{f(j)} \rangle$*,* $u_{f(j)} \in \langle u_j \rangle$*, with* u_j*,* $u_{f(j)}$ *viewed as elements of* $(\mathbb{Z}/\exp(TN_j)_p)^*$*;*

(Note that, of course, (1) and (2) become trivial if $j = f(j)$.)

(3) $u_{f(j)} \equiv u_{f(r)} \equiv 1 \mod |\alpha_{q(f(j))}^{\ell(r)} a_{\ell(r)}|$ *for all $r \neq j$, for all $\ell \in \{1, \dots, s_r\}$ and for all $q \in \{1, \dots, s_{f(j)}\}$.*

Further, if $f(j) = j$ for all $j \in \{1, \dots, k\}$, then $\det\psi \equiv 1 \mod p^\lambda$*; and if there exists $j \in \{1, \dots, k\}$ such that $f(j) \neq j$, then* $\det\psi \equiv \pm 1 \mod p^\lambda$*. In the latter case we need, moreover, the condition*

$$u_j \equiv u_{f(j)} \equiv 1 \text{ modulo the order of obstruction of } (TN_j)_p, (TN_{f(j)})_p.$$

Proof: The compatibility condition (Lemma 2.3) tells us that

$$\psi(\xi_m) \cdot \alpha(a_{i(v)}) = \alpha(\xi_m \cdot a_{i(v)})$$

for $m \in \{1, \dots, k\}$, $v \in \{1, \dots, k\}$, $i \in \{1, \dots, s_v\}$. This yields the following.

If $m = v$, then

$$\begin{aligned}&\sum_\ell u_1^{\beta_{v1}} \alpha_{i(v)}^{\ell(1)} a_{\ell(1)} + \cdots + \sum_\ell u_k^{\beta_{vk}} \alpha_{i(v)}^{\ell(k)} a_{\ell(k)} \\ =&\sum_\ell u_v \alpha_{i(v)}^{\ell(1)} a_{\ell(1)} + \cdots + \sum_\ell u_v \alpha_{i(v)}^{\ell(k)} a_{\ell(k)}.\end{aligned}$$

If $m \neq v$, then

$$\begin{aligned}&\sum_\ell u_1^{\beta_{m1}} \alpha_{i(v)}^{\ell(1)} a_{\ell(1)} + \cdots + \sum_\ell u_k^{\beta_{mk}} \alpha_{i(v)}^{\ell(k)} a_{\ell(k)} \\ =&\sum_\ell \alpha_{i(v)}^{\ell(1)} a_{\ell(1)} + \cdots + \sum_\ell \alpha_{i(v)}^{\ell(k)} a_{\ell(k)}.\end{aligned}$$

This means that, for all m, v, $r \in \{1, \dots, k\}$, for all $i \in \{1, \dots, s_v\}$, and for all $\ell \in \{1, \dots, s_r\}$:

$$\begin{aligned}&\text{If } m = v, \text{ then } u_r^{\beta_{vr}} \alpha_{i(v)}^{\ell(r)} a_{\ell(r)} = u_v \alpha_{i(v)}^{\ell(r)} a_{\ell(r)}; \\ &\text{If } m \neq v, \text{ then } u_r^{\beta_{mr}} \alpha_{i(v)}^{\ell(r)} a_{\ell(r)} = \alpha_{i(v)}^{\ell(r)} a_{\ell(r)}.\end{aligned}$$

Thus

$$\text{If } m = v, \text{ then } u_r^{\beta_{vr}} \equiv u_v \mod |\alpha_{i(v)}^{\ell(r)} a_{\ell(r)}|; \tag{4.1}$$

$$\text{If } m \neq v, \text{ then } u_r^{\beta_{mr}} \equiv 1 \mod |\alpha_{i(v)}^{\ell(r)} a_{\ell(r)}|. \tag{4.2}$$

We now assert that

$$\forall\, j \in \{1, \dots, k\} \quad \exists!\, f(j) \in \{1, \dots, k\} \text{ such that } p \nmid \alpha_{q(f(j))}^{1(j)} \text{ for some } q.$$

Indeed, since p does not divide the determinant of α_p, there certainly exists such a $f(j)$. And if we suppose that there exist v, v' such that

$$p \nmid \alpha_{i(v)}^{1(j)} \text{ for some } i \in \{1, \dots, s_v\}$$

and

$$p \nmid \alpha_{i'(v')}^{1(j)} \text{ for some } i' \in \{1, \dots, s_{v'}\},$$

then it follows from (4.2) that

$$\forall\, m \neq v \quad u_j^{\beta_{mj}} \equiv 1 \mod |a_{1(j)}| = \exp(TN_j)_p$$

and

$$\forall\, m \neq v' \quad u_j^{\beta_{mj}} \equiv 1 \mod |a_{1(j)}| = \exp(TN_j)_p.$$

If $v \neq v'$, this would imply that

$$\forall\, m \quad \beta_{mj} \equiv 0 \mod (t_j)_p,$$

where $(t_j)_p$ stands for the p-part of t_j. Hence it would follow that $\det \psi = \det(\beta_{ij}) \equiv 0 \mod (t)_p$. However, this is impossible, since $p \nmid \det \psi$ (ψ being a T-automorphism). The assertion assures us that the matrix of α_p, reduced $\mod p$, looks like

$$a_{1(j)}\alpha(TN_{f(j)})_p \begin{pmatrix} & & 0 & & & \\ & & 0 & & & \\ & & 0 & & & \\ & & 0 & & & \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & * & & & \\ & & * & & & \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & 0 & & & \\ & & 0 & & & \\ & & 0 & & & \\ & & \vdots & & & \\ & & 0 & & & \end{pmatrix}$$

Note that the above also implies that $\exp(TN_j)_p \leq \exp(TN_{f(j)})_p$.

Thus we have set up a map $f : \{1, \ldots, k\} \longrightarrow \{1, \ldots, k\} : j \longmapsto f(j)$. We claim that this map f is a bijection. Indeed, if we suppose that $f(j) = f(j') = v$, meaning that $p \nmid \alpha_{q(v)}^{1(j)}$ for some $q \in \{1, \ldots, s_v\}$ and that $p \nmid \alpha_{q'(v)}^{1(j')}$ for some $q' \in \{1, \ldots, s_v\}$, then we would get from (4.2) that

$$\begin{aligned} \forall\, m \neq v \quad & u_j^{\beta_{mj}} \equiv 1 \mod |a_{1(j)}| \\ \forall\, m \neq v \quad & u_{j'}^{\beta_{mj'}} \equiv 1 \mod |a_{1(j')}|, \end{aligned}$$

and thus

$$\begin{aligned} \forall\, m \neq v \quad & \beta_{mj} \equiv 0 \mod (t_j)_p \\ \forall\, m \neq v \quad & \beta_{mj'} \equiv 0 \mod (t_{j'})_p. \end{aligned}$$

If $j \neq j'$, this would again imply that $\det \psi \equiv 0 \mod (t)_p$, yielding a contradiction. So we conclude that f is a bijection. The above means that the following situation is impossible for the matrix of α_p (each column being reduced $\mod p$):

$$\begin{array}{cc}
 & \begin{array}{cccccc} & & a_{1(j)} & & a_{1(j')} & \end{array} \\
\alpha(TN_v)_p & \begin{pmatrix}
 & & 0 & & 0 & \\
 & & 0 & & 0 & \\
 & & 0 & & 0 & \\
 & & 0 & & 0 & \\
\cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
 & & * & & * & \\
 & & * & & * & \\
\cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
 & & 0 & & 0 & \\
 & & 0 & & 0 & \\
 & & 0 & & 0 & \\
 & & \vdots & & \vdots & \\
 & & 0 & & 0 &
\end{pmatrix}
\end{array}$$

Now note that $p \nmid \alpha^{1(j)}_{q(f(j))}$ for some q implies (using (4.1)) that

$$\text{(4.3)} \quad (\text{taking } m = f(j)) \quad u_j^{\beta_{f(j)j}} \equiv u_{f(j)} \mod |a_{1(j)}| = \exp(TN_j)_p;$$

and by (4.2) that

$$\forall\, m \neq f(j) \quad u_j^{\beta_{mj}} \equiv 1 \mod |a_{1(j)}| = \exp(TN_j)_p;$$

and thus

$$\text{(4.4)} \qquad \forall\, m \neq f(j) \quad \beta_{mj} \equiv 0 \mod (t_j)_p.$$

Moreover, restating (4.1), (4.2), we get that, for all $r \in \{1, \dots, k\}$, for all $\ell \in \{1, \dots, s_r\}$, and for all $q \in \{1, \dots, s_{f(j)}\}$,

$$\text{(4.5)} \qquad (\text{for } m = f(j)) \quad u_r^{\beta_{f(j)r}} \equiv u_{f(j)} \mod |\alpha^{\ell(r)}_{q(f(j))} a_{\ell(r)}|;$$

$$\text{(4.6)} \qquad \forall\, m \neq f(j) \quad u_r^{\beta_{mr}} \equiv 1 \mod |\alpha^{\ell(r)}_{q(f(j))} a_{\ell(r)}|.$$

If we take $r \neq j$ in (4.5), then we know that $\beta_{f(j)r} \equiv 0 \mod (t_r)_p$ (see (4.4)) and thus, again using (4.5),

$$\text{(4.7)} \qquad 1 \equiv u_{f(j)} \mod |\alpha^{\ell(r)}_{q(f(j))} a_{\ell(r)}|.$$

If we take $r \neq j$ and $m = f(r)$ in (4.6) (note that we then still have $m \neq f(j)$, since $r \neq j$), then we get

$$u_r^{\beta_{f(r)r}} \equiv 1 \mod |\alpha_{q(f(j))}^{\ell(r)} a_{\ell(r)}|.$$

But of course

$$u_r^{\beta_{f(r)r}} \equiv u_{f(r)} \mod |\alpha_{q(f(j))}^{\ell(r)} a_{\ell(r)}|,$$

since, due to (4.3), this congruence is true $\mod \exp(TN_r)_p$. Thus we get

$$u_{f(r)} \equiv 1 \mod |\alpha_{q(f(j))}^{\ell(r)} a_{\ell(r)}|. \tag{4.8}$$

Let us reformulate what we have already proved. We have a bijection

$$f : \{1, \dots, k\} \longrightarrow \{1, \dots, k\} : j \longmapsto f(j)$$

such that

$$p \nmid \alpha_{q(f(j))}^{1(j)} \text{ for some } q \in \{1, \dots, s_{f(j)}\},$$

and f is uniquely determined by this property. Indeed,

$$\forall\, m \neq f(j) \quad \beta_{mj} \equiv 0 \mod (t_j)_p; \tag{4.9}$$

$$u_j^{\beta_{f(j)j}} \equiv u_{f(j)} \mod \exp(TN_j)_p; \tag{4.10}$$

$$\forall\, r \neq j, \forall q \in \{1, \dots, s_{f(j)}\}, \forall\, \ell \in \{1, \dots, s_r\} \quad u_{f(j)} \equiv 1 \mod |\alpha_{q(f(j))}^{\ell(r)} a_{\ell(r)}|; \tag{4.11}$$

$$\forall\, r \neq j, \forall\, q \in \{1, \dots, s_{f(j)}\}, \forall\, \ell \in \{1, \dots, s_r\} \quad u_{f(r)} \equiv 1 \mod |\alpha_{q(f(j))}^{\ell(r)} a_{\ell(r)}|; \tag{4.12}$$

$$\exp(TN_j)_p \leq \exp(TN_{f(j)})_p. \tag{4.13}$$

So we have already established (3) in the statement of the proposition, which is simply (4.11) and (4.12).

We now distinguish two cases.

Case 1: $f(j) = j$ for all $j \in \{1, \dots, k\}$.

We then see, from (4.9) (4.10), that, for all j,

$$\begin{cases} \beta_{mj} \equiv 0 \mod (t_j)_p, & \text{for } m \neq j \\ \beta_{jj} \equiv 1 \mod (t_j)_p. \end{cases}$$

Thus the matrix of ψ, reduced $\mod (t)_p = p^\lambda$, looks like the identity matrix, and thus

$$\det \psi \equiv 1 \mod p^\lambda.$$

Case 2: $f(j) \neq j$ for some $j \in \{1, \dots, k\}$.
Suppose that

$$\begin{aligned} f(j) &= y_1 \\ f(y_1) &= y_2 \\ f(y_2) &= y_3 \\ &\dots \\ f(y_{s-1}) &= y_s \\ f(y_s) &= j. \end{aligned}$$

Note that, since f is a bijection, we are certainly able to form a "closed chain" for f as above. Note also that the above implies that

$$\exp(TN_j)_p \leq \exp(TN_{y_1})_p \leq \cdots \leq \exp(TN_{y_s})_p \leq \exp(TN_j)_p.$$

Thus all exponents are equal, meaning that all p-torsion groups appearing in a "closed chain" have the same p-exponent. This already establishes (1) in the statement of the proposition. Moreover, in this chain, we also have

$$\begin{aligned}
&\begin{cases} \beta_{mj} \equiv 0 \mod (t_j)_p & \text{for } m \neq y_1 \\ u_j^{\beta_{y_1 j}} \equiv u_{y_1} \mod \exp(TN_j)_p, & \text{since } f(j) = y_1 \end{cases} \\
&\begin{cases} \beta_{my_1} \equiv 0 \mod (t_{y_1})_p & \text{for } m \neq y_2 \\ u_{y_1}^{\beta_{y_2 y_1}} \equiv u_{y_2} \mod \exp(TN_{y_1})_p, & \text{since } f(y_1) = y_2 \end{cases} \\
&\dots \\
&\begin{cases} \beta_{my_{s-1}} \equiv 0 \mod (t_{y_{s-1}})_p & \text{for } m \neq y_s \\ u_{y_{s-1}}^{\beta_{y_s y_{s-1}}} \equiv u_{y_s} \mod \exp(TN_{y_{s-1}})_p, & \text{since } f(y_{s-1}) = y_s \end{cases} \\
&\begin{cases} \beta_{my_s} \equiv 0 \mod (t_{y_s})_p & \text{for } m \neq j \\ u_{y_s}^{\beta_{j y_s}} \equiv u_j \mod \exp(TN_{y_s})_p, & \text{since } f(y_s) = j. \end{cases}
\end{aligned}$$

So we get

$$u_j \equiv u_{y_s}^{\beta_{j y_s}} \equiv u_{y_{s-1}}^{\beta_{j y_s}\beta_{y_s y_{s-1}}} \equiv \cdots \equiv u_j^{\beta_{j y_s}\beta_{y_s y_{s-1}}\cdots\beta_{y_1 j}} \mod \exp(TN_j)_p.$$

From this it follows that

$$\beta_{jy_s}\beta_{y_sy_{s-1}}\dots\beta_{y_1j} \equiv 1 \mod (t_j)_p, \tag{4.14}$$

while all the other β's in the columns $j, y_1, y_2, \dots, y_s$ are congruent to 0, modulo $(t)_p$. Moreover, from the above, it is also clear that $u_{f(j)} = u_{y_1} \in \langle u_j \rangle$ and that $u_j \in \langle u_{f(j)} = u_{y_1} \rangle$ if we view these as elements of $(\mathbb{Z}/\exp(TN_j)_p)^*$. This establishes (2) in the statement of the proposition.

Repeating this process of constructing "closed chains" for f, until we have exhausted the whole of $\{1, \dots, k\}$, we obtain a number of congruences of the form (4.14), while all the other β's are congruent to zero modulo $(t)_p$.

Combining all this together, we get

$$\det\psi \equiv \pm \prod_{\text{all non-zero } \beta' s} \beta \equiv \pm 1 \mod (t)_p = p^{\lambda}.$$

It only remains to verify that

$$u_j \equiv u_{f(j)} \equiv 1 \text{ modulo the order of obstruction of } (TN_j)_p, (TN_{f(j)})_p.$$

Of course, since $u_j \in \langle u_{f(j)} \rangle \in (\mathbb{Z}/\exp(TN_{f(j)})_p)^* = (\mathbb{Z}/\exp(TN_j)_p)^*$, it is sufficient to prove that

$$u_{f(j)} \equiv 1 \text{ modulo the order of obstruction of } (TN_j)_p,\ (TN_{f(j)})_p.$$

We now have two cases. Either the order of obstruction is the order of some $a_{x(j)}$ or else it is the order of some $a_{x(f(j))}$. Suppose that the order of obstruction is equal to $|a_{x(j)}|$. Then it is sufficient to prove that there exists $v \neq f(j)$ such that $p \nmid \alpha_{i(v)}^{x(j)}$, for some $i \in \{1, \dots, s_v\}$. Indeed, this would imply that $|\alpha_{i(v)}^{x(j)} a_{x(j)}| = |a_{x(j)}|$, which in turn would imply, by (4.12), that $u_{f(j)} \equiv 1 \mod |a_{x(j)}|$, as required. Reduce the matrix of α_p modulo p, and consider the columns of $a_{1(j)}, a_{2(j)}, \dots, a_{x-1(j)}$. We may suppose that

$$p \mid \alpha_{i(v)}^{1(j)},\ p \mid \alpha_{i(v)}^{2(j)}, \dots, p \mid \alpha_{i(v)}^{x-1(j)}, \text{ for all } v \neq f(j) \text{ and all } i \in \{1, \dots, s_v\}.$$

Indeed, otherwise we would obtain that $|\alpha_{i(v)}^{\ell(j)} a_{\ell(j)}| = |a_{\ell(j)}|$ for some $\ell < x$, which would lead by (4.12) to $u_{f(j)} \equiv 1 \mod |a_{\ell(j)}|$, and thus $\mod |a_{x(j)}|$. But of course, $p \nmid \det \alpha_p$. So we can find $q_1, q_2, \dots, q_{x-1}$ such that $q_1 \neq q_2 \neq \dots \neq q_{x-1}$ and such that

$$p \nmid \alpha^{1(j)}_{q_1(f(j))},\, p \nmid \alpha^{2(j)}_{q_2(f(j))}, \dots, p \nmid \alpha^{x-1(j)}_{q_{x-1}(f(j))}.$$

Moreover, we then know that

$$\begin{gathered}
|a_{q_1(f(j))}| \geq |a_{1(j)}| \\
|a_{q_2(f(j))}| \geq |a_{2(j)}| \\
\dots \\
|a_{q_{x-1}(f(j))}| \geq |a_{x-1(j)}|.
\end{gathered}$$

Of course, due to the supposition on the orders and the order of obstruction, we are now unable to find for $a_{x(j)}$ a generator $a_{q_x(f(j))}$, such that $p \nmid \alpha^{x(j)}_{q_x(f(j))}$ with $q_x \neq q_1, q_2, \dots, q_{x-1}$, implying that $|a_{q_x(f(j))}| \geq |a_{x(j)}|$. This means that, since $p \nmid \det \alpha_p$, there exists $v \neq f(j)$ such that

$$p \nmid \alpha^{x(j)}_{i(v)} \text{ for some } i \in \{1, \dots, s_v\}.$$

In other words, in the column of $a_{x(j)}$ in the matrix of α_p, reduced modulo p, we must have a non-zero number on a row outside $\alpha(TN_{f(j)})_p$, which is what we had to prove.

Suppose on the other hand that the order of obstruction is equal to $|a_{x(f(j))}|$. It is then sufficient to prove that there exists $r \neq j$ such that

$$|\alpha^{\ell(r)}_{x(f(j))} a_{\ell(r)}| = |a_{x(f(j))}| \text{ for some } \ell.$$

Indeed, by (4.11), it would then follow that $u_{f(j)} \equiv 1 \mod |a_{x(f(j))}|$. We will use the following notations:

$$\begin{aligned}
\widehat{(TN_j)}_p &= \text{ the direct product of all } (TN_i)_p, \text{ except } (TN_j)_p \\
\mathrm{pr}_j &= \text{ the projection onto the } (TN_j)_p\text{-component} \\
\mathrm{pr} &= \text{ the projection onto the rest, that is, onto } \widehat{(TN_j)}_p.
\end{aligned}$$

Thus $\alpha_p = (\mathrm{pr}_j \circ \alpha_p) + (\mathrm{pr} \circ \alpha_p)$. Now

$$\begin{cases}
\alpha_p(a_{1(f(j))}) = (\mathrm{pr}_j \circ \alpha_p)(a_{1(f(j))}) + (\mathrm{pr} \circ \alpha_p)(a_{1(f(j))}) \\
\alpha_p(a_{2(f(j))}) = (\mathrm{pr}_j \circ \alpha_p)(a_{2(f(j))}) + (\mathrm{pr} \circ \alpha_p)(a_{2(f(j))}) \\
\dots \\
\alpha_p(a_{x-1(f(j))}) = (\mathrm{pr}_j \circ \alpha_p)(a_{x-1(f(j))}) + (\mathrm{pr} \circ \alpha_p)(a_{x-1(f(j))}).
\end{cases}$$

We may suppose that the second components in these sums have smaller order than $|a_{1(f(j))}|, \dots, |a_{x-1(f(j))}|$ in $\widehat{(TN_j)}_p$, respectively. Indeed, otherwise there would exist $r \neq j$ such that

$$|\alpha^{\ell(r)}_{q(f(j))} a_{\ell(r)}| = |a_{q(f(j))}|$$

for some $q \in \{1, \dots, x-1\}$ and some ℓ, which would imply by (4.11) that

$$u_{f(j)} \equiv 1 \mod |a_{q(f(j))}|$$

and thus also $\mod |a_{x(f(j))}|$, which is what we wished to prove. But then we need that the first components of these sums have orders $|a_{1(f(j))}|, |a_{2(f(j))}|, \dots, |a_{x-1(f(j))}|$ respectively in $(TN_j)_p$. Now set

$$\begin{cases} H_1 = \langle (\mathrm{pr}_j \circ \alpha_p)(a_{1(f(j))}) \rangle \\ H_2 = \langle (\mathrm{pr}_j \circ \alpha_p)(a_{2(f(j))}) \rangle \\ \dots \\ H_{x-1} = \langle (\mathrm{pr}_j \circ \alpha_p)(a_{x-1(f(j))}) \rangle. \end{cases}$$

So H_q (for $q = 1, \dots, x-1$) is a subgroup of $(TN_j)_p$ of order $|a_{q(f(j))}|$. Moreover, we claim that $H_{q_1} \cap H_{q_2} = \{0\}$ if $q_1 \neq q_2$. Indeed, suppose that $0 \neq x \in H_{q_1} \cap H_{q_2}$. Then there exist λ_1, λ_2 (where we may assume that either $p \nmid \lambda_1$ or $p \nmid \lambda_2$) such that

$$\begin{aligned} x &= \lambda_1 (\mathrm{pr}_j \circ \alpha_p)(a_{q_1(f(j))}) \\ &= \lambda_2 (\mathrm{pr}_j \circ \alpha_p)(a_{q_2(f(j))}). \end{aligned}$$

But then it is easy to see that this yields a contradiction with the injectivity of α_p. We now have

$$H_1 \oplus H_2 \oplus \dots \oplus H_{x-1} \rightarrowtail (TN_j)_p \twoheadrightarrow (TN_j)_p / H_1 \oplus \dots \oplus H_{x-1}.$$

We see that in the quotient we have to factor out cyclic subgroups of orders $|a_{1(f(j))}| = |a_{1(j)}|$, $|a_{2(f(j))}| = |a_{2(j)}|, \dots, |a_{x-1(f(j))}| = |a_{x-1(j)}|$. This quotient thus has exponent $< |a_{x(f(j))}|$. It is then easily seen that, since

$$\alpha_p(a_{x(f(j))}) = (\mathrm{pr}_j \circ \alpha_p)(a_{x(f(j))}) + (\mathrm{pr} \circ \alpha_p)(a_{x(f(j))}),$$

the order of the first component in the above sum in $(TN_j)_p$ is $< |a_{x(f(j))}|$. This means that the second component of the sum has order equal to $|a_{x(f(j))}|$ in $\widehat{(TN_j)}_p$, so that there exists $r \neq j$ such that $|\alpha^{\ell(r)}_{x(f(j))} a_{\ell(r)}| = |a_{x(f(j))}|$, which is what we wished to prove. This concludes the proof of Proposition 4.1. ■

Thus we know that any m in $\operatorname{im} \theta$ must fulfil the conditions given in Theorem 1.6. We now proceed to the converse; that is, we show that any such m is realizable.

4.2. Proposition. *Let $m \in (\mathbb{Z}/t)^*$ with $m \equiv \epsilon_i \mod p_i^{\lambda_i}$, for all $i \in \{1, \dots, k\}$ where $\epsilon_i = 1$ or -1. Additionally if $\epsilon_i = -1$, then suppose that there exist r, $v \in \{1, \dots, k\}$ such that $r \neq v$ and*

(1) $\exp(TN_r)_p = \exp(TN_v)_p$ *(that is $|a_{1(r)}| = |a_{1(v)}|$);*
(2) $u_v \in \langle u_r \rangle$, $u_r \in \langle u_v \rangle$, *where u_r, u_v are viewed as elements of $(\mathbb{Z}/\exp(TN_v)_p)^*$;*
(3) $u_v \equiv u_r \equiv 1$ *modulo the order of obstruction of $(TN_r)_p$, $(TN_v)_p$.*

Then we can realize m, that is, $[m] \in \operatorname{im}\theta$.

Proof: We will construct an automorphism $\alpha \in \operatorname{Aut}(TN_1 \times \cdots \times TN_k)$ and a T-automorphism $\psi \in T\text{-}\operatorname{Aut}(FN_1 \times \cdots \times FN_k)$, which satisfy the compatibility condition of Lemma 2.3, such that $\det \psi \equiv m \mod t$. It will follow that any endomorphism φ of $(N_1 \times \cdots \times N_k)$, compatible with α and ψ, will be a T-automorphism realizing m. We will determine α completely, but we will only determine the matrix of $\psi \mod t$.

Fix a particular p among the prime divisors of t and let

$$p^{\ell_1} \| \, t_1, \quad p^{\ell_2} \| \, t_2, \dots, p^{\ell_k} \| \, t_k, \quad p^{\lambda} \| \, t.$$

Set

$$\begin{cases} \psi(\xi_1) = \beta_{11}\xi_1 + \cdots + \beta_{1k}\xi_k \\ \cdots \\ \psi(\xi_k) = \beta_{k1}\xi_1 + \cdots + \beta_{kk}\xi_k. \end{cases}$$

The idea is the following. If $m \equiv 1 \mod p^{\lambda}$, we will construct α_p as the identity on $(TN_1 \times \cdots \times TN_k)_p$ and the matrix of ψ, reduced $\mod p^{\lambda}$, should look like the identity matrix. If $m \equiv -1 \mod p^{\lambda}$, then α_p should map $(TN_r)_p$ to $(TN_v)_p$ and vice-versa as much as possible. This means that we map the respective generators with the same order (for example $a_{1(r)}$ and $a_{1(v)}$) on each other. On the generators of $(TN_r)_p$ obstructing an isomorphism, and on later generators, we define α_p to be the identity, and likewise for $(TN_v)_p$. On all other p-torsion subgroups $(TN_j)_p$ for $j \neq r$, v, we also define α_p to be the identity. The matrix of ψ, reduced $\mod p^{\lambda}$, will look like the identity matrix outside the r^{th} and v^{th} columns. These two columns contain β_{rv} and β_{vr} such that $u_v \equiv u_r^{\beta_{vr}}$, $u_r \equiv u_v^{\beta_{rv}} \mod \exp(TN_v)_p$. Then, as we will show, $\det \psi$ will be congruent to $-1 \mod p^{\lambda}$.

Case 1: $m \equiv 1 \mod p^{\lambda}$.
Define $\alpha_p = \mathrm{Id} : (TN_1 \times \cdots \times TN_k)_p \to (TN_1 \times \cdots \times TN_k)_p$ and let

$$\begin{cases} \beta_{ii} \equiv 1 \mod p^{\ell_i} & \text{for all } i \in \{1, \dots, k\} \\ \beta_{ij} \equiv 0 \mod p^{\ell_j} & \text{if } j \neq i. \end{cases}$$

Case 2: $m \equiv -1 \mod p^\lambda$.

We then know that there exist $r, v \in \{1, \dots, k\}$, $r \neq v$ such that

(1) $\exp(TN_r)_p = \exp(TN_v)_p$ (that is $|a_{1(r)}| = |a_{1(v)}|$);
(2) $u_v \in \langle u_r \rangle$, $u_r \in \langle u_v \rangle$ viewed as elements of $(\mathbb{Z}/\exp(TN_v)_p)^*$;
(3) $u_v \equiv u_r \equiv 1$ modulo the order of obstruction of $(TN_r)_p$, $(TN_v)_p$.

Define $\alpha_p : (TN_1 \times \cdots \times TN_k)_p \to (TN_1 \times \cdots \times TN_k)_p$ as follows

$$\begin{cases} \alpha_p = \mathrm{Id} \quad \text{outside } (TN_r \times TN_v)_p; \\ \alpha_p = \mathrm{Id} \quad \text{for the generators of } (TN_r)_p \text{ and } (TN_v)_p \\ \quad \text{obstructing an isomorphism and for later generators;} \\ \alpha_p(a_{j(r)}) = a_{j(v)} \text{ and } \alpha_p(a_{j(v)}) = a_{j(r)} \\ \qquad\qquad \text{for the other generators of } (TN_r \times TN_v)_p; \end{cases}$$

and let

$$\begin{cases} \beta_{ii} \equiv 1 \mod p^{\ell_i} \quad \text{for } i \neq r,\, v \\ \beta_{vv} \equiv 0 \mod p^{\ell_v} \\ \beta_{rr} \equiv 0 \mod p^{\ell_r} \\ \beta_{rv} \text{ and } \beta_{vr} \text{ be chosen such that} \\ \quad u_v \equiv u_r^{\beta_{vr}},\, u_r \equiv u_v^{\beta_{rv}} \mod \exp(TN_v)_p \\ \quad \text{(which is always possible, by hypothesis)} \\ \beta_{ij} \equiv 0 \mod p^{\ell_j} \quad \text{otherwise .} \end{cases}$$

Remark that in both cases we can solve all the congruences (by the Chinese Remainder Theorem) and that β_{ij} will be determined $\mod t_j$, so that the entries of the matrix of ψ will be determined $\mod \gcd(t_1, \dots, t_k) = t$. We will now check that α and ψ, as constructed above, satisfy the compatibility condition (Lemma 2.3).

Case 1: $m \equiv 1 \mod p^\lambda$

$$\alpha(\xi_s \cdot a_q) = \psi(\xi_s) \cdot \alpha(a_q) \quad (a_q \in TN_q) \quad (q \neq s)$$
$$\Longleftrightarrow a_q = u_q^{\beta_{sq}} a_q, \text{ and the latter holds since } \beta_{sq} \equiv 0 \mod p^{\ell_q}$$
$$\alpha(\xi_s \cdot a_s) = \psi(\xi_s) \cdot \alpha(a_s)$$
$$\Longleftrightarrow u_s a_s = u_s^{\beta_{ss}} a_s, \text{ and the latter holds since } \beta_{ss} \equiv 1 \mod p^{\ell_s}.$$

Case 2: $m \equiv -1 \mod p^\lambda$

If $\{q, s\} \neq \{v, r\}$, we get similar equations to those above. If $\{q, s\} = \{v, r\}$, we have for generators $a_{j(r)}$, $a_{j(v)}$ that are mapped under α on each other:

$$\begin{aligned}
\alpha(\xi_v \cdot a_{j(r)}) = \psi(\xi_v) \cdot \alpha(a_{j(r)}) &\Longleftrightarrow a_{j(v)} = a_{j(v)}\\
\alpha(\xi_v \cdot a_{j(v)}) = \psi(\xi_v) \cdot \alpha(a_{j(v)}) &\Longleftrightarrow u_v a_{j(r)} = u_r^{\beta vr} a_{j(r)}\\
\alpha(\xi_r \cdot a_{j(r)}) = \psi(\xi_r) \cdot \alpha(a_{j(r)}) &\Longleftrightarrow u_r a_{j(v)} = u_v^{\beta rv} a_{j(v)}\\
\alpha(\xi_r \cdot a_{j(v)}) = \psi(\xi_r) \cdot \alpha(a_{j(v)}) &\Longleftrightarrow a_{j(r)} = a_{j(r)},
\end{aligned}$$

and the latter relations all hold.

If $\{q, s\} = \{v, r\}$, we have for generators $a_{j(r)}$, $a_{j(v)}$ on which α is defined as the identity (that is, generators obstructing an isomorphism or later generators):

$$\begin{aligned}
\alpha(\xi_v \cdot a_{j(r)}) = \psi(\xi_v) \cdot \alpha(a_{j(r)}) &\Longleftrightarrow a_{j(r)} = u_r^{\beta_{vr}} a_{j(r)}\\
\alpha(\xi_v \cdot a_{j(v)}) = \psi(\xi_v) \cdot \alpha(a_{j(v)}) &\Longleftrightarrow u_v a_{j(v)} = a_{j(v)}\\
\alpha(\xi_r \cdot a_{j(r)}) = \psi(\xi_r) \cdot \alpha(a_{j(r)}) &\Longleftrightarrow u_r a_{j(r)} = a_{j(r)}\\
\alpha(\xi_r \cdot a_{j(v)}) = \psi(\xi_r) \cdot \alpha(a_{j(v)}) &\Longleftrightarrow a_{j(v)} = u_v^{\beta_{rv}} a_{j(v)},
\end{aligned}$$

and the latter relations all hold, by (3) above.

Finally we look at $\det \psi$. For each $p|t$, we have

$$\det \psi = \det(\beta_{ij}) \equiv \begin{cases} 1 \mod p^\lambda & \text{if } m \equiv 1 \mod p^\lambda \\ -\beta_{rv}\beta_{vr} \mod p^\lambda & \text{if } m \equiv -1 \mod p^\lambda. \end{cases}$$

However in the second case we know

$$u_r \equiv u_v^{\beta_{rv}} \equiv (u_r^{\beta_{vr}})^{\beta_{rv}} \mod \exp(TN_r)_p.$$

From this it follows that $\beta_{vr}\beta_{rv} \equiv 1 \mod p^{\ell_r} = p^{\ell_v}$, so in either case we have

$$\det \psi \equiv m \mod p^\lambda.$$

Thus

$$\det \psi \equiv m \mod t,$$

which concludes the proof of Proposition 4.2. With these two propositions our main result, Theorem 1.6, is established. ■

We now give an example of how one can use Theorem 1.6 to calculate the genus of a direct product of groups in $\mathcal{N}_1$.

4.3. Example.

Let $N_1 \in \mathcal{N}_1$ with $TN_1 = \mathbb{Z}/9 \oplus \mathbb{Z}/49$, $FN_1 = \langle \xi_1 \rangle$ and $\xi_1 \cdot a = 22a$ for all $a \in TN_1$. So $u_1 = 1 + 3 \cdot 7$ and $t_1 = 3 \cdot 7 = 21$. Let $N_2 \in \mathcal{N}_1$

with $TN_2 = \mathbb{Z}/9 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/343$ and $FN_2 = \langle \xi_2 \rangle$ where $\xi_2 \cdot b = 148b$ for all $b \in TN_2$. So $u_2 = 1 + 3 \cdot 7^2$ and $t_2 = 3 \cdot 7 = 21$. Thus $t = 21$. Then $P = \{3\}$ (see Theorem 1.6). Note that $22 \equiv 148 \equiv 1 \mod 3$ (3 being the order of obstruction of $(TN_1)_3, (TN_2)_3$), but plainly $7 \notin P$. Thus we have to factor out of $(\mathbb{Z}/21)^*$ the residue classes of $1, -1$ and of those m such that $m \equiv 1 \mod 7$, $m \equiv \pm 1 \mod 3$. This means factoring out the group H generated by $\{-1, m\}$, where $m \equiv 1 \mod 7$, $m \equiv -1 \mod 3$. Thus $H = \langle -1, 8 \rangle$. Thus

$$\mathcal{G}(N_1 \times N_2) \cong (\mathbb{Z}/21)^*/H \cong \mathbb{Z}/3.$$

Of course, we can explicitly describe the groups in the genus of $N_1 \times N_2$, using the descriptions of the groups in $\mathcal{G}(N)$, for $N \in \mathcal{N}_1$, given in [**3**].

Finally we give the promised example to show that the condition $u_r \in \langle u_v \rangle$, $u_v \in \langle u_r \rangle$ cannot be replaced by the weaker condition $|u_r| = |u_v|$ in Theorem 1.6. That is, we will give an example where $2 \notin P$, although the prime 2 satisfies (1), (3) and the weaker form of (2); and where the compatibility condition of Lemma 2.3 excludes the condition $\det \psi \equiv -1 \mod 2^\lambda$.

4.4. Example.
Let

$$N_1 = \langle x, y \mid x^{16} = 1, yxy^{-1} = x^3 \rangle$$
$$N_2 = \langle x, y \mid x^{16} = 1, yxy^{-1} = x^5 \rangle.$$

Then

$$N_1 \in \mathcal{N}_1, \text{ with } TN_1 = \mathbb{Z}/16 = \langle a_1 \rangle,\ FN_1 = \mathbb{Z} = \langle \xi_1 \rangle \text{ and } \xi_1 \cdot a_1 = 3a_1$$
$$N_2 \in \mathcal{N}_1, \text{ with } TN_2 = \mathbb{Z}/16 = \langle a_2 \rangle,\ FN_2 = \mathbb{Z} = \langle \xi_2 \rangle \text{ and } \xi_2 \cdot a_2 = 5a_2.$$

Moreover $t_1 = 4$ and $t_2 = 4$, so $t = \gcd(t_1, t_2) = 4$. Let $\psi \in T\text{-}\operatorname{Aut}(FN_1 \times FN_2)$ be given by

$$\psi(\xi_i) = \beta_{i1}\xi_1 + \beta_{i2}\xi_2, \text{ for } i = 1, 2,$$

and let $\alpha \in \operatorname{Aut}(TN_1 \times TN_2)$ be given by

$$\alpha(a_j) = \alpha_{j1}a_1 + \alpha_{j2}a_2, \text{ for } j = 1, 2.$$

Expressing the condition $\alpha(\xi_i \cdot a_j) = \psi(\xi_i) \cdot \alpha(a_j)$ for $i,\, j = 1, 2$ yields the following equations :

$$3\alpha_{11}a_1 + 3\alpha_{12}a_2 = \alpha_{11}3^{\beta_{11}}a_1 + \alpha_{12}5^{\beta_{12}}a_2 \tag{1}$$

$$\alpha_{21}a_1 + \alpha_{22}a_2 = \alpha_{21}3^{\beta_{11}}a_1 + \alpha_{22}5^{\beta_{12}}a_2 \tag{2}$$

$$\alpha_{11}a_1 + \alpha_{12}a_2 = \alpha_{11}3^{\beta_{21}}a_1 + \alpha_{12}5^{\beta_{22}}a_2 \tag{3}$$

$$5\alpha_{21}a_1 + 5\alpha_{22}a_2 = \alpha_{21}3^{\beta_{21}}a_1 + \alpha_{22}5^{\beta_{22}}a_2 \tag{4}$$

Now, we have at least one of two cases; either $2 \nmid \alpha_{11}$ or $2 \nmid \alpha_{21}$. If $2 \nmid \alpha_{11}$, then we need $2|\alpha_{21}$ (otherwise (1) and (2) contradict) so that $2 \nmid \alpha_{22}$ (because $2 \nmid \det \alpha$) and so $2|\alpha_{12}$ (otherwise (3) and (4) contradict). However, in this case we obtain

$$\begin{cases} \beta_{11} \equiv 1 & \mod 4 \\ \beta_{12} \equiv 0 & \mod 4 \\ \beta_{21} \equiv 0 & \mod 4 \\ \beta_{22} \equiv 1 & \mod 4 \end{cases}$$

So that $\det \psi \equiv 1 \mod 4$.

If $2 \nmid \alpha_{21}$, then analogously $2|\alpha_{11}, 2 \nmid \alpha_{12}$ and $2|\alpha_{22}$. Here we need

$$\begin{cases} \beta_{11} \equiv 0 & \mod 4 \\ \beta_{22} \equiv 0 & \mod 4 \\ 3 \equiv 5^{\beta_{12}} & \mod 16 \\ 5 \equiv 3^{\beta_{21}} & \mod 16 \end{cases}$$

The two last congruences however have no solution. We thus can conclude that for each $\psi \in T\text{-}\operatorname{Aut}(FN_1 \times FN_2)$ we have $\det \psi \equiv 1 \mod 4$; and $\det \psi \equiv -1 \mod 4$ is impossible.

Of course, Corollary 1.7 gives us the simple formula for $\mathcal{G}(N_1 \times N_2)$ in this case, since only the prime 2 is involved; and the value of $\mathcal{G}(N_1 \times N_2)$ is unaffected by whether we can find $\psi \in T\text{-Aut}(N_1 \times N_2)$ with $\det \psi \equiv -1 \mod 4$. To obtain a counterexample to the statement of Theorem 1.6 with the weaker version of condition (2), we need to complicate our Example 4.4 by involving another prime p as a factor of t, in addition to the prime 2, and arranging that $p \notin P$. We would thereby obtain an example in which all the hypotheses of Theorem 1.6 were verified, except that condition (2) is replaced by the weaker version, but the conclusion of the theorem is false.

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