Publicacions Matemàtiques, Vol **39** (1995), 201–214.

RINGS WHOSE MODULES HAVE MAXIMAL SUBMODULES

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Dedicated to Laci Fuchs on his 70th birthday

Abstract

A ring R is a **right max ring** if every right module $M \neq 0$ has at least one maximal submodule. It suffices to check for maximal submodules of a single module and its submodules in order to test for a max ring; namely, any cogenerating module E of mod-R; also it suffices to check the submodules of the injective hull $E(V)$ of each simple module V (Theorem 1). Another test is transfinite nilpotence of the radical of E in the sense that $\operatorname{rad}^\alpha E = 0$; equivalently, there is an ordinal α such that $\operatorname{rad}^\alpha(E(V)) = 0$ for each simple module V. This holds iff each $\operatorname{rad}^\beta(E(V))$ has a maximal submodule, or is zero (Theorem 2). If follows that R is right max iff every nonzero (subdirectly irreducible) quasi-injective right R-module has a maximal submodule (Theorem 3.3). We characterize a right max ring R via the endomorphism ring Λ of any injective cogenerator E of mod-R; namely, Λ/L has a minimal submodule for any left ideal $L = \operatorname{ann}_\Lambda M$ for a submodule (or subset) $M \neq 0$ of E (Theorem 8.8). Then Λ/L_0 has socle $\neq 0$ for: (1) any finitely generated left ideal $L_0 \neq \Lambda$; (2) each annihilator left ideal $L_0 \neq \Lambda$; and (3) each proper left ideal $L_0 = L + L'$, where $L = \operatorname{ann}_\Lambda M$ as above (e.g. as in (2)) and L' finitely generated (Corollary 8.9A).

HAMSHER MODULES

A module M is a **Hamsher module** provided each submodule $S \neq 0$ has a maximal submodule.[1]

[1]Hamsher modules are called max modules by Shock [**S**].

1. One-Module Theorem. *A ring R is a right max ring iff R has a cogenerating right Hamsher module E. A n.a.s.c. for this is that the injective hull $E(V)$ of each simple right R-module V is a Hamsher module.*

Proof: A module E cogenerates the category mod-R of all right R-modules iff for every module $M \neq 0$, there is a nonzero map $h : M \to E$ ([**F1**, pp. 91, 148 & 165]). Then $h(M) = M'$ is a nonzero submodule of E. Thus, when E is a Hamsher module, then M' has a maximal submodule M'', so $h^{-1}(M'')$ is a maximal submodule of M.

This proves the first statement in Theorem 1. Next let $E = \oplus E(V)$, as V range over all simple R-modules. Then E is a cogenerator module for mod-R ([**F1**, p. 167, prop. 3.55]). Let P_V be the projection $E \to E(V)$. Then, in the above, $0 \neq M' = h(M) \subseteq E$ implies $0 \neq P_V h(M) = M_V \subseteq E(V)$ is a nonzero submodule of $E(V)$ for some V, and so M has a maximal submodule, as before, whenever $E(V)$ is a Hamsher module for all V. ■

Note. $E(V)$ is direct summand of any cogenerator E of mod-R, hence the Hamsher condition on $E(V)$ is a consequence of that on E in Theorem 1. Moreover, this is sufficient for E to be Hamsher.

1.1. Corollary. *If R is a ring such that each simple module V has Noetherian injective hull $E(V)$, then R is a right max ring.*

To illustrate when $E(V)$ is not only Noetherian, but simple we will cite a theorem of Kaplansky, but first we recall some terminology:

R is **right V-ring** in case R has the equivalent properties. (See [**F1**, p. 356, 7.32A].)

(V1) Every simple right R-module V is injective, that is, $E(V)$ is simple.
(V2) $\operatorname{rad} M = 0$ for each right R-module M.
(V3) Every right ideal $I \neq R$ is the intersection of maximal right ideals, that is, $\operatorname{rad}(R/I)_R = 0$.

Note. A right V-ring is a right max ring since $\operatorname{rad} M \neq M$ for every $M \neq 0$.

Kaplansky's Theorem. [2] *A commutative ring R is a V-ring iff R is Von Neuman regular (= VNR).*

[2] According to my inquiry of Professor Kaplansky, "It worked its way into the public domain" (Letter of October 12, 1994).

Let $J = \operatorname{rad} R$. Then J is **left vanishing** (= T**-nilpotent in** [**B**], [**H**]) if for every sequence $\{a_n\}_{n=1}^{\infty}$ of elements of A, there is an $n \geq 1$ so that $a_n \cdots a_1 = 0$, that is the left-hand partial product $a_n \cdots a_1$ vanishes.

First Max Theorem ([**H**], [**K**]). *A commutative ring R is a max ring iff R/J is VNR and $J = \operatorname{rad} R$ is vanishing.*

Expressed otherwise: R is a max ring iff R/J is a V-ring, and J is vanishing. The **radical series** $\operatorname{rad}^{\alpha}(M)$ is defined inductively for each ordinal α in the usual way, where $\operatorname{rad}(M)$ is the intersection of all maximal submodules of M, $\operatorname{rad}^{\alpha+1}(M) = \operatorname{rad}(\operatorname{rad}^{\alpha}(M))$ for any ordinal, and

$$\operatorname{rad}^{\beta}(M) = \bigcap_{\alpha \in \beta} \operatorname{rad}^{\alpha}(M)$$

for each limit ordinal β.

Second Max Theorem ([**H**], [**K**]). *A ring R is right max iff R/J is right max and J is left vanishing.*

We next show that the modules in the radical series are test submodules for a Hamsher module.

2. Theorem. [3] *The f.a.e.c.'s on a right R-module M.*

(1) *M is Hamsher.*
(2) *$\operatorname{rad}^{\beta}(M)$ has a maximal submodule, or is 0, for every ordinal β.*
(3) *$\operatorname{rad}^{\alpha}(M) = 0$ for some α.*

Proof: (1) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (3) follows by cardinal number theory for any α of cardinal greater than that of R. (3) $\Rightarrow$ (1). If $S \neq 0$ is a submodule of M, then $S \nsubseteq \operatorname{rad}^{\lambda}(M)$ for least ordinal $\lambda < \alpha$, and obviously λ is not a limit ordinal, so $S \subseteq \operatorname{rad}^{\lambda-1}(M)$. If $S = \operatorname{rad}^{\lambda-1} M$, then S has a maximal submodule since $\operatorname{rad} S = \operatorname{rad}^{\lambda}(M) \neq S$. And if $S \neq \operatorname{rad}^{\lambda-1}(M)$, then S is not contained in a maximal submodule M' of $\operatorname{rad}^{\lambda-1}(M)$, hence $S \cap M'$ is a maximal submodule of S. This proves that M is a Hamsher module. ∎

3.1. Corollary. *Let E be a right cogenerator module for R. The R is right max iff E has transfinite nilpotent radical. A n.a.s.c. for*

[3]The equivalence (1) $\Leftrightarrow$ (3) is a theorem of Shock [**S**] who also proved that every semi-Artinian Hamsher module is Noetherian.

this is that $E(V)$ have transfinite nilpotent radical for each simple right R-module V.

3.2. Lemma. *If M is a quasi-injective right R-module, then so is every fully invariant submodule, in particular, so is $\operatorname{rad}^\alpha(M)$, for each ordinal α.*

Proof: A theorem of Wong-Johnson ([**W-J**]) characterizes a quasi-injective module as the fully invariant submodules of their injective hulls (see, e.g. [**F2**, p. 63, Prop. 19.2]). For example, if $E = E(M)$ has endomorphism Λ, then M is quasi-injective iff $\lambda(M) \subseteq M\ \forall \lambda \in \Lambda$. Now let M_0 be a fully invariant submodule of M. Since $E_0 = E(M) \subseteq E$, and since E is injective, then every element $\lambda_0 \in \Lambda_0 = \operatorname{End} E_0$ is induced by an element $\lambda \in \Lambda$. Since λ induces an endomorphism $\bar\lambda$ in M, and since $\bar\lambda(M_0) \subseteq M_0$ by the hypothesis that M_0 is fully invariant in M, then $\lambda_0(M_0) \subseteq M_0$ for each $\lambda_0 \in \Lambda_0$, that is, M_0 is fully invariant in $E(M_0)$, hence is quasi-injective.

It follows that $\operatorname{rad}^{\alpha+1}(M)$ is quasi-injective for all α, since $\operatorname{rad}^{\alpha+1}(M)$ is fully invariant in $\operatorname{rad}^\alpha(M)$ which by an inductive hypothesis may be assumed to be quasi-injective. Furthermore, $\operatorname{rad}^\beta(M)$ is fully invariant hence quasi-injective for each limit ordinal β, since it is the intersection of fully invariant submodules of M. ■

3.3. Theorem. *For a ring R, the f.a.e.c.'s:*

(1) *R is right max.*
(2) *Every nonzero quasi-injective module has a maximal submodule.*
(3) *Every nonzero subdirectly irreducible quasi-injective module has a maximal submodule.*

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (1) is an immediate consequence of Theorem 2, Corollary 3.1 and Lemma 3.2. ■

4. Corollary. *If a right R module M is faithful and has transfinite nilpotent radical, then R has transfinite nilpotent radical J.*

Proof: One shows inductively that $\operatorname{rad}^\alpha(M) \supseteq MJ^\alpha$, where $J = \operatorname{rad} R$. ■

Note. Let R be a commutative Noetherian ring. Then $J^\omega = 0$ by the Krull intersection Theorem and if R is a domain, then $I^\omega = 0$ for any ideal $I \neq R$ ([**Z-S**, p. 216, Theorem 12 and Corollary]). Thus, J is transfinite but not T-nilpotent when R is e.g., a Noetherian local domain not a field.

LOEWY SERIES AND TRANSFINITE SEMISIMPLE MODULES

A **descending** or **dual Loewy series** for a module M is descending chain $\{M_\alpha\}_{\alpha\in\Lambda}$ of submodules indexed by an ordinal Λ such that $M_0 = M$, and $M_\alpha/M_{\alpha+1}$ is semisimple

$$M_\beta = \cap_{\alpha\in\beta} M_\alpha$$

for any limit ordinal $\beta \in \Lambda$. We say that M is **transfinitely semisimple** if there is a descending Loewy series $\{M_\alpha\}$ with $M_\alpha = 0$ for some $\alpha \in \Lambda$.

5. Theorem. *Any transfinitely semisimple module M is a Hamsher module.*

Proof: By transfinite induction,

$$M_\alpha \supseteq \operatorname{rad}^\alpha(M)$$

for each M_α as defined above, hence $\operatorname{rad}^\alpha(M) = 0$ for some ordinal α, and Theorem 1 applies: M is Hamsher module. ∎

By Theorem 1, we also have the following:

5.1. Corollary. *If $E(V)$ is transfinite semisimple for each simple right R-module V, then R is right max.*

BASS MODULES

Recall that a module M is a **Bass module** ([**F2**]) if every submodule $M' \neq M$ is contained in a maximal submodule of M.

6. Theorem. *Let E be an quasi-injective right R-module that contains a copy of each simple image of E and $\Lambda = \operatorname{End} E_R$. If E is a Bass module, then Λ has essential left socle,* $\operatorname{soc}_\ell \Lambda$.

Proof: By the Harada-Ishii ([**H-I**]) double annihilator condition (= *DAC*) for a quasi-injective modules,

$$\operatorname{ann}_\Lambda \operatorname{ann}_E I = I$$

for finitely generated left ideals of Λ, one can show that each such $I \neq 0$ contains a minimal left ideal L. For if E' is a maximal submodule, containing $\operatorname{ann}_E I$ the fact $V = E/E' \hookrightarrow E$ yields $\lambda \in \Lambda$ such that $\lambda E \approx V$, hence $L = \Lambda\lambda$ is a minimal left ideal contained in I. Thus, $\operatorname{soc}_\ell \Lambda$ is an essential left ideal of Λ. ∎

In the next corollary, we see what happens to Λ when E is Noetherian.

6.1. Corollary. *If E is a Noetherian quasi-injective right module over R, then $\Lambda = \operatorname{End} E_R$ is a right perfect ring, hence a right max ring.*

Proof: By the Harada-Ishii DAC cited in the proof of Theorem 6, E_R Noetherian implies that Λ satisfies the DAC on finitely generated left ideals, hence Λ is right perfect ([**B**]). ■

DOUBLE ANNIHILATOR CONDITIONS FOR COGENERATORS

It is known that any cogenerator F satisfies the double annihilator conditions (DAC)

$$I = \operatorname{ann}_R \operatorname{ann}_F I$$

(see, e.g. [**F1**]). We next prove another DAC for F.

7. Dac Theorem. [4] *If F is any right cogenerator of R, and I and M are submodules of R_R and F_R respectively, then they satisfy the DAC's:*

(a) $$I = \operatorname{ann}_R \operatorname{ann}_F I$$

(b) $$M = \operatorname{ann}_F \operatorname{ann}_\Omega M$$

where $\Omega = \operatorname{End} F_R$.

Proof:

(1) Since F is a cogenerator then $R/I \hookrightarrow F^\alpha$ for some cardinal α, and if (x_i) is the image in F of the coset $1 + I$ in R/I, one sees that

$$I = \operatorname{ann}_R\{x_i\},$$

so (a) follows.

(2) F/M embeds in a direct product F^α of copies of F, and hence there is a map $h : F \to F^\alpha$ that has $\ker h = M$. Then, if $p_\alpha : F^\alpha \to F$ is the α-th projection, it follows that $\omega_\alpha = p_\alpha \circ h \in \Omega$ and that

(3) $$M = \cap_\alpha \ker \omega_\alpha.$$

Then,

(4) $$M = \operatorname{ann}_F L,$$

where $L = \Sigma_\alpha \Omega \omega_\alpha$.

Since (4) $\Longrightarrow$ (b), the proof is complete. ■

[4] After this was written, I found Kurata's report [**Ku**] where (b) is stated without proof in greater generality.

INJECTIVE COGENERATORS

If any cogenerator of mod-R is a Hamsher module, then R is a right max ring. In this section we list two conditions on a minimal injective cogenerator E that are each necessary and sufficient in order that R be a right V-ring: (1) $\operatorname{rad} E = 0$. (Theorem 8.1) and (2) E_R is a Bass module, and $\Lambda = \operatorname{End} E_R$ has zero Jacobson radical (Theorem 8.2).

8.1. Theorem. *Let E be a minimal injective cogenerator of R, and W the direct sum of a complete set of non-isomorphic simple right R-modules. (Thus, E is the injective hull of W, and W is the socle of E.) Then, the f.a.e.c.'s:*

(1) *R is a right V-ring.*
(2) $\operatorname{rad} E = 0$.

Proof: (1) $\Rightarrow$ (2). As stated, (1) $\Leftrightarrow$ $\operatorname{rad} M = 0$ for every right R-module M.

(2) $\Rightarrow$ (1). If V is a simple submodule of E, then (2) implies that there exists a maximal submodule M of E not containing V. Then since $V \cap M = 0$, and $V + M \supset M$, we see that $E = V \oplus M$, so V is injective. Since every simple right R-module embeds in E, then R is a right V-ring. ■

8.2. Theorem. *If the right minimal injective cogenerator E of a ring R is a Bass Module, and if $\Lambda = \operatorname{End} E_R$ has zero Jacobson radical, then R is a right V-ring (and E is semisimple).*

Proof: Let $W = \operatorname{soc} E$, the sum of all simple module, one for each isomorphy class. If $W = E$, then every submodule of E is a direct summand, hence is injective, so R is right V-ring. We may therefore assume that $E \neq W$, and hence by our Bass module assumption that there is a maximal submodule M of E that contains W. Since $V = E/M \hookrightarrow W$, there is an endomorphism λ of E such that $\ker \lambda = M$. Since M is an essential submodule of E, then $\lambda \in J = J(\Lambda)$ by a theorem of Utumi (e.g. [**F2**, p. 76, Theorem 19.27(a)]) contradicting the $J = 0$ assumption, and completing the proof. ■

8.3. Proposition. *If S is any semisimple right R-module with injective hull $E = E(S)$, then the endomorphism ring Λ has radical*

$$J(\Lambda) = \{\lambda \in \wedge \ker \lambda \supseteq S\}, \qquad (1)$$

and moreover,

(2) $$J(\Lambda) = \mathrm{ann}_\Lambda S.$$

Furthermore,

(3) $$\overline{\Lambda} = \Lambda/J(\Lambda) = \mathrm{End}\, S_R$$

is a full product $= \Pi_{i\in A} L_i$ *of full linear rings, where* $L_i = \mathrm{End}\, W_{Di}$*, and* W_i *is a vector space over a sfield* D_i, $\forall i \in A$.

Proof: By Utumi's theorem cited above (proof of 8.2), (2) has the description (1) above. Since a submodule M of $E = E(S)$ is essential iff $M \supseteq S$, this shows that (2) holds. Furthermore since E is injective, any element of End S_R is induced by some $\lambda \in \Lambda$, so (2) $\Rightarrow$ (3). Finally, $\bar{\Lambda}$ is a product as described by classical ring theory. ■

8.4. Corollary. *If* E *is a minimal injective cogenerator of* mod-R*, and* $\Lambda = \mathrm{End}\, E_R$*, then* $\overline{\Lambda} = \Lambda/J(\Lambda)$ *is product* $\Pi_{i\in A} D_i$ *of sfields* $D_i = \mathrm{End}(V_i)_R$*, one for each isomorphy class* $[V_i]$ *of simple modules. Consequently,* $\overline{\Lambda}$ *is a* V*-ring.*

Proof: Follows from 8.3. $\bar{\Lambda}$ is thus abelian VNR (=strongly regular), hence is a right and left V-ring. ■

8.5. Corollary. *If (in Theorem 8.3)* E *is a minimal injective cogenerator, then* $E = E(S)$*, where* $S = \oplus V_i$*, exactly one simple module* V_i *of each isomorphy class, and*

$$\bar{\Lambda} = \Lambda/J(\Lambda) = \Pi_{i\in A} D_i$$

where $D_i = \mathrm{End}\, V_i$*, one for each* V_i*.*

Furthermore, $\bar{\Lambda}$ *is a right and left* V*-ring. Finally,* Λ *is a right (left) max ring iff* $J(\Lambda)$ *is left (right) vanishing. Moreover,* Λ *is right max iff* E_R *satisfies the acc on kernels of finite products* $\{j_n \cdots j_2 j_1\}$ *of elements of* $J(\Lambda)$*.*

Proof: Follows from Corollary 8.4, the Harada-Ishii theorem, and the Second Max Theorem. ■

8.6. Corollary. *If the minimal injective cogenerator E of* mod-R *satisfies the acc on essential submodules (equivalently, $E/\operatorname{soc} E$ is Noetherian), then $\Lambda = \operatorname{End} E_R$ is a right max ring.*

Proof: Since $\Lambda/J(\Lambda)$ is a V-ring (both sides) hence a max ring, then by Hamsher's theorem, Λ is right max iff $J(\Lambda)$ is left vanishing. But this follows from Corollary 8.5 and the Harada-Ishi Theorem as in the proof of Theorem 6. (Since $\operatorname{soc} E$ is the intersection of all essential submodule by a theorem of Kasch-Sandomierski, the parenthetical equivalence holds.) ■

Remark 8.6A. The condition of Corollary 8.6 implies that $E(V)$ is Noetherian for any simple module V, and by Corollary 1.1, this is also a sufficient condition for R to be right max.

8.7. Theorem (Partial Converse of Theorem 6)**.** *If E is an injective cogenerator for* mod-R*, and if $\Lambda = \operatorname{End} E_R$ has essential left socle then E is a Bass module.*

Proof: The proof is a straightforward application of the Harada-Ishii theorem. For if M is a proper submodule of E, the fact that E is an injective cogenerator yields $\hom(E/M, E) \neq 0$, hence some $\lambda \in \Lambda$ with $\ker \lambda \supseteq M$. Then, if $\Lambda\lambda_0$ is a minimal left ideal of Λ contained in $\Lambda\lambda$, by the Harada-Ishii theorem, $E_0 = \ker \lambda_0$ is a maximal submodule containing $\ker \lambda$, hence M. ■

In the proof of the next theorem, we let $\ker L = \cap_{\lambda \in L} \ker \lambda$.

8.8. Theorem. *For a ring R, right injective cogenerator E, and $\Lambda = \operatorname{End} E_R$ the f.a.e.c.'s:*

(1) *R is right max.*
(2) *E is a Hamsher module.*
(3) *Λ/L has nonzero socle for any left ideal $L = \operatorname{ann}_\Lambda M$, where M is a nonzero submodule of E.*

Proof: (1) $\Leftrightarrow$ (2) by Theorem 1. (2) $\Rightarrow$ (3). By the *DAC* Theorem 6.2, if $L = \operatorname{ann}_\Lambda M$, then $M = \ker L$, hence, since E is Hamsher module, M has a maximal submodule M_0. Since $\hom_R(M/M_0, E) \neq 0$ and E is injective, then there exists $\lambda_0 \in \Lambda$ such that $\lambda_0 M_0 = 0$ and $\lambda_0 M \neq 0$. Moreover, if $L_0 = \operatorname{ann}_\Lambda M_0$, then by the *DAC* Theorem 7, $\operatorname{ann}_E L_0 = M_0$, and since $M \cap (\ker \lambda_0) = M_0$, then:

$$\operatorname{ann}_E(L+\Lambda\lambda_0) = (\ker L)\cap(\ker \lambda_0) = M\cap(\ker \lambda_0) = M_0 = \operatorname{ann}_E L_0. \quad ■$$

By the Harada-Ishii theorem, $L + \Lambda\lambda_0$ satisfies the DAC, hence

$$L + \Lambda\lambda_0 = \operatorname{ann}_\Lambda \operatorname{ann}_E(L + \Lambda\lambda_0) = \operatorname{ann}_\Lambda M_0 = L_0.$$

Moreover, the same argument shows that

$$L_0 = L + \Lambda\lambda' \quad \text{for all} \quad \lambda' \in L_0 \backslash (L)$$

that is, necessarily $\operatorname{ann}_E(L + \Lambda\lambda') = M_0$ so $L + \Lambda\lambda' = \operatorname{ann}_\Lambda M_0 = L$. Thus $L_0 \backslash L$ is a minimal submodule of $\Lambda \backslash L$, so (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2). Let $L = \operatorname{ann}_\Lambda M$. Then, by the DAC Theorem 6.2, $M = \operatorname{ann}_E L$. Let L_0/L be a minimal submodule of Λ/L, and let $M_0 = \operatorname{ann}_E L_0$. Since $L_0 = L + \Lambda\lambda$ for any $\lambda \in L_0 \backslash L$, then by the Harada-Ishii DAC, necessarily $L_0 = \operatorname{ann}_\Lambda M_0$. If $M' \neq M$ is a submodule of M containing M_0, then by simplicity of L_0/L, necessarily $\operatorname{ann}_\Lambda M' = L_0$ whence by the DAC Theorem 6.2,

$$M' = \operatorname{ann}_E \operatorname{ann}_\Lambda M' = \operatorname{ann}_E L_0 = M_0$$

so M_0 is a maximal submodule of M. Thus, (3) $\Rightarrow$ (2). ■

8.9A. Corollary. *If R is right max, E an injective cogenerator, and $\Lambda = \operatorname{End} E_R$, then Λ/I has nonzero socle for each proper left ideal I of the (3) types:*

(0) *L_0 finitely generated left ideal of Λ.*
(1) *L_1 an annihilator left ideal of Λ.*
(2) *$L_2 = L + L_0$, where L_0 is finitely generated and $L = \operatorname{ann}_\Lambda M$ for a submodule M of E.*

In particular, $L_1 = \operatorname{ann}_\Lambda M_1$, where $M_1 = L_1^\perp E$, so L can have the form L_1 in (2).

Proof: By the Harada-Ishii DAC, any left ideal L_2 of the form (2) satisfies the DAC, hence $L_2 = \operatorname{ann}_\Lambda M_2$, where $M_2 = \operatorname{ann}_\Lambda L_2 = \ker L_2$, so Theorem 8.8 applies.

Furthermore, if L_1 is the left annihilator ${}^\perp X$ in Λ of a subset X of Λ, then $L_1 =^\perp (L_1^\perp)$ so

$$L_1 = \operatorname{ann}_\Lambda({}^\perp L_1 E)$$

is the annihilator of an R-submodule of E. ■

8.9B. Corollary. *If E is an injective cogenerator of* mod-R *with left Loewy (equivalently, left semiartinian) endomorphism ring Λ, then R is right max and Λ is right perfect. Moreover, R has just finitely many simple right modules.*

Proof: If Λ is left Loewy, then Λ/L has nonzero socle for all left ideals $L \neq \Lambda$, so Theorem 8.8 applies to establish that R is right max. Since $\overline{\Lambda} = \Lambda/J(\Lambda)$ is also left Loewy and right self-injective (see, e.g. (3) of Prop. 8.3), then $\overline{\Lambda}$ is semisimple Artinian and $J = J(\Lambda)$ is left vanishing, hence Λ is right perfect. (See, for example, the discussion in [**C-P**, esp. Lemma 1 and the proof of Proposition 2].) Furthermore, since $\overline{\Lambda}$ is semisimple and isomorphic to the endomorphism ring of the socle S of E (see the proof of 8.3), then S has finite length. This shows that the isomorphy set of simple right R-modules is finite. ■

8.10. Corollary. *If E is an injective cogenerator of* mod-R, *and $\Lambda = \operatorname{End} E_R$, then R is right max iff $J = \operatorname{rad} R$ left vanishing, and Λ/L has nonzero left socle for any left ideal $L = \operatorname{ann}_\Lambda M$, where M is a nonzero R-submodule of E annihilated by J.*

Proof: One knows that $F = \operatorname{ann}_E J$ is an injective cogenerator of mod-R/J (F is injective as an R/J-module and contains a copy of each simple R-module). Moreover, F is a fully invariant R-submodule of E, hence, by injectivity of E,

$$\bar{\Lambda} = \Lambda/\operatorname{ann}_\Lambda F \approx \operatorname{End} F_R.$$

The corollary now follows from Hamsher's Second Theorem and Theorem 8.8. ■

8.11. Example. Let M be any bimodule over a right max ring A. Then the split-null or trivial extension $R = (A, M)$ is a right max ring.

Proof: Let $J(A)$ be the (left vanishing) radical of A. Then $J(R) = (J(A), M)$ and

$$R/J(R) \approx A/J(A)$$

is a right max ring, so R is right max iff $J(R)$ is left vanishing. But

$$J(R)/(0, M) \approx J(A)$$

is left vanishing and $(0, M)^2 = 0$, and then an easy computation shows that $J(R)$ is left vanishing. ■

REMARKS ON THE LITERATURE

A module M is quotient finite dimensional (= $q.f.d.$) provided that all factor modules have finite Goldie dimension, i.e., contain no infinite direct sums. Generalizing a theorem of Shock [**S**], Camillo [**C1**] proved that an R-module M is q.f.d. iff every submodule N contains a finitely generated submodule K with N/K having no maximal submodules. This implies that a q.f.d. module M is Noetherian iff every factor module M/K is Hamsher. Since linearly compact modules are q.f.d., then by duality theory [**M**] one shows that a Morita ring R(= R has a Morita duality) is right max iff left Loewy (= semi-Artinian and iff R is right and left Artinian.

Results of Camillo and Fuller [**C-F1**], [**C-F2**] and Nastasescu and Popescu [**N-P**] are germane here: A left Loewy ring R of finite Loewy length is right max ([**C-F1**], [**N-P**]). More generally, any left Loewy ring with acc on primitive ideals is right max ([**C-F2**]). The example of a right but not left V-ring R of the author's in [**F4**] is a VNR of left Loewy length 2 hence left max.

As an application of Theorem 1, we prove in [**F3**] that for a commutative ring R that the f.e.c.'s : (1) R is locally a perfect ring (= R_m is perfect at each maximal ideal m); (2) R_m is a max ring for each maximal ideal m; (3) R is a max ring.

QUESTIONS

(1) If $\Lambda = \operatorname{End} E_R$ is a right max ring, for a minimal injective cogenerator E of mod-R, is R right max?

(2) If R is right max, is Λ?

In [**C2**], Camillo proves that a right max right and left $PID\,R$ is simple, and that given two maximal right ideals, pR and qR, either R/pqR or R/qpR is semisimple.

(3) Characterize when a PID ring R is right (or left) max. It is of course if R/aR (or R/Ra) is semisimple for any $0 \neq a \in R$. (See [**C2**].)

(4) (Hamsher [**H**]) When is a full linear ring right or left max? (Regarding the corresponding question for V-rings, see Osofsky [**0**].)

Acknowledgements. The author wishes to acknowledge support for this research from the Spanish Ministry of Science and Education, and the Rutgers University Academic Study Program (FASP).

The author also wishes to acknowledge the terrific work of Barbara Miller in rendering his scribbles into TEX: and Happy Birthday, Barbara!

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Rebut el 19 de Gener de 1995