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ON BOTT-PERIODIC ALGEBRAIC K-THEORY

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Abstract

Let $K_*(A;\mathbf{Z}/\ell^n)$ denote the mod-ℓ^n algebraic K-theory of a $\mathbf{Z}[1/\ell]$-algebra A. Snaith [**14**], [**15**], [**16**], has studied Bott-periodic algebraic K-theory $K_i(A;\mathbf{Z}/\ell^n)[1/\beta_n]$, a localized version of $K_*(A;\mathbf{Z}/\ell^n)$ obtained by inverting a *Bott element* β_n. For ℓ an *odd prime*, Snaith has given a description of $K_*(A;\mathbf{Z}/\ell^n)[1/\beta_n]$ using Adams maps between Moore spectra. These constructions are interesting, in particular, for their connections with the Lichtenbaum-Quillen conjecture [**16**].

In this paper we obtain a description of $K_*(A;\mathbf{Z}/2^n)[1/\beta_n]$, $n \geq 2$, for an algebra A with $1/2 \in A$ and $\sqrt{-1} \in A$. We approach this problem using low dimensional computations of the stable homotopy groups of $B\mathbf{Z}/4$, and transfer arguments to show that a power of the mod-4 *Bott element* is induced by an Adams map.

Introduction. Let ℓ be a prime number and let A be a commutative ring containing $1/\ell$. For ℓ odd or $\ell = 2$ and $n \geq 2$ there exists (see [**16**]) a "Bott element" $\beta_n \in K_*(A;\mathbf{Z}/\ell^n)$ and Snaith [**16**] forms $K_*(A;\mathbf{Z}/\ell^n)[1/\beta_n]$, the localization of $K_*(A;\mathbf{Z}/\ell^n)$ obtained by inverting the Bott element. Thus, $K_*(A;\mathbf{Z}/\ell^n)[1/\beta_n]$ is the direct limit of iterated multiplications by β_n using the K-theory product. The Lichtenbaum-Quillen conjecture [**18**] has been reformulated as the assertion that for a suitable regular ring A, the canonical localization map

$$\rho : K_i(A;\mathbf{Z}/\ell^n) \to K_i(A;\mathbf{Z}/\ell^n)[1/\beta_n] \tag{1.1}$$

is injective for large i (see [**17**]).

For ℓ an odd prime Snaith [**16**] has obtained a description of $K_i(A;\mathbf{Z}/\ell^n)[1/\beta_n]$ in terms of Adams maps. Recall [**2**] that an Adams map between mod-ℓ^n Moore spectra is a map $A_n : \Sigma^d P(\ell^n) \to P(\ell^n)$ which induces isomorphisms on topological K-theory. In [**16**] Snaith

proved that $K_i(A;\mathbf{Z}/\ell^n)[1/\beta_n]$ is the direct limit of iterated precompositions of suspensions of mod-ℓ^n Adams maps, i.e.
(1.2)
$$K_i(A;\mathbf{Z}/\ell^n)[1/\beta_n] \approx \lim_{\rightarrow}\left(K_i(A;\mathbf{Z}/\ell^n) \xrightarrow{(\Sigma^d A_n)^*} K_{i+d}(A;\mathbf{Z}/\ell^n)\right)$$

then using (1.2) he obtains a factorization of the localization map (1.1) through the Hurewicz map $K_i(A;\mathbf{Z}/\ell^n) \to h_i(BGLA^+;\mathbf{Z}/\ell^n)$, where h_i denotes mod-ℓ^n topological complex K-theory $KU_i(-,\mathbf{Z}/\ell^n)$ or a suitable defined J-theory $J_i(-,\mathbf{Z}/\ell^n)$.

In this paper we extended these results to the case $\ell = 2$, $n \geq 2$ assuming that the ring A contains a fourth root of unity.

Bott elements and Adams maps. Let $A = \mathbf{Z}[1/2,\zeta_4]$ be the ring obtained by adjoining $\zeta_4 = \sqrt{-1}$ to the ring of integers localized away from 2. Snaith [**16**, Section 3] considers the following construction:

The inclusion $\mathbf{Z}/4 \to GL_1A$ given by sending a generator of $\mathbf{Z}/4$ to ζ_4, and inclusion of permutation matrices induces morphisms

$$\Sigma_r \int \mathbf{Z}/4 \to \Sigma_r \int GL_1A \to GL_rA \tag{2.1}$$

where $\Sigma_r \int G$ is the *wreath product* of the symmetric group Σ_r with the group G. These morphisms induce an infinite loop space map

$$d_1 : \left(B\textstyle\sum_\infty \int \mathbf{Z}/4\right)^+ \to BGLA^+ \tag{2.2}$$

2.3. The *Bott element* $\beta \in K_2(A;\mathbf{Z}/4)$ is defined as the image under the map induced by d_1 of a generator of order 4,

$$b \in \pi_2\left(\left(B\textstyle\sum_\infty \int \mathbf{Z}/4\right)^+;\mathbf{Z}/4\right) \approx \pi_2^s(B\mathbf{Z}/4;\mathbf{Z}/4)$$

obtained by stabilization of the generator of $\pi_2(B\mathbf{Z}/4;\mathbf{Z}/4) \approx \mathbf{Z}/4$ which maps under the Bockstein morphism $\pi_2(B\mathbf{Z}/4;\mathbf{Z}/4) \to \pi_1(B\mathbf{Z}/4)$ to the generator of $\mathbf{Z}/4$.

The element $\beta_1 = \beta^4 \in K_8(A;\mathbf{Z}/4)$ is also called a Bott element. The following characterization of the Bott elements is the mod-4 analogue of [**6**].

Lemma 2.4. *For $n > 1$, the 4^{n-1} cup power of β_1 in $K_*(A;\mathbf{Z}/4)$ is the reduction mod-4 of an element β_n in $K_{8\cdot 4^{n-1}}(A;\mathbf{Z}/4^n)$.*

Proof: As in [**6**, Lemma 2], the proof is by induction on $n > 1$ using the fact [**12**] that the differentials in the mod-4 stable homotopy Bockstein spectral sequence are derivations, and the definition $K_*(A;\mathbf{Z}/4^m) = \pi_*(\mathbf{K}A;\mathbf{Z}/4^m)$.

i) $n = 2$: Since $\beta_1^4 = \beta^{4\cdot 4}$ and since $d_1 : K_*(A;\mathbf{Z}/4) \to K_{*-1}(A;\mathbf{Z}/4)$ is a derivation, then

$$d_1(\beta_1^4) = d_1(\beta^{4\cdot 4}) = 4^2 \cdot \beta^{15} d_1(\beta) = 0$$

since $K_*(A;\mathbf{Z}/4)$ is a $\mathbf{Z}/4$-module. Thus, β_1^4 is a d_1-cycle and so it survives to E_2.

Now, by the description of E_r, see [**4**, Section 5], $\beta_1^4 = \beta^{16} \in E_2$ is represented by the class of a map $P^{8\cdot 4}(4) \to \mathbf{K}A$ such that there exists a factorization:

$$\begin{array}{ccc} P^{8\cdot 4}(4) & \longrightarrow & \mathbf{K}A \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \beta_2} & \\ & P^{8\cdot 4}(4^2) & \end{array}$$

i.e., $\beta_1^4 = \beta_2 \circ i = i^{\#}(\beta_2)$, i.e. β_1^4 is the mod-4 reduction of $\beta_2 \in K_{8\cdot 4}(A;\mathbf{Z}/4^2) = \pi_{8\cdot 4}(\mathbf{K}A;\mathbf{Z}/4^2)$ (the mod-4 reduction map is $r_{\#} = i^{\#}(-)$).

ii) Now, for $n > 2$, inductively we see that the cup powers:

$$\beta_1 = \beta^4, \quad \beta_1^4 = \beta^{4^2}, \dots, \beta_1^{4^{n-1}} = \beta^{4^n}$$

are d_r-cycles for $1 \leq r \leq n-1$, and so in particular

$$\beta_1^{4^{n-1}} = \beta^{4^n} \in E_n^{8\cdot 4^{n-1}}$$

can be represented as

$$P^{8\cdot 4^{n-1}}(4) \xrightarrow{\beta_1^{4^{n-1}}} \mathbf{K}A \xrightarrow{\beta_n} P^{8\cdot 4^{n-1}}(4^n)$$

by the description of $E_n^{8\cdot 4^{n-1}}$ [**4**, Section 5]. Thus, for $\beta_n \in K_{8\cdot 4^{n-1}}(A;\mathbf{Z}/4^n)$ we have: $\beta_1^{4^{n-1}} = i^{\#}(\beta_n))$ i.e., the mod-4 reduction of β_n is $\beta_1^{4^{n-1}}$. ■

2.5. Definition. Let X be an algebra over $A = \mathbf{Z}[1/2, \zeta_4]$, define [**16**]:

$$K_i(X;\mathbf{Z}/4^n)[1/\beta_n] := \varinjlim \left(K_i(X;\mathbf{Z}/4^n) \xrightarrow{\beta_n} K_{i+d}(X;\mathbf{Z}/4^n) \to \cdots \right)$$

where $d = \deg(\beta_n) = 8 \cdot 4^{n-1}$. Notice that $K_*(X;\mathbf{Z}/4^n)[1/\beta_n]$ is periodic of period d, i.e., $K_i(X;\mathbf{Z}/4^n)[1/\beta_n] \approx K_{i+d}(X;\mathbf{Z}/4^n)[1/\beta_n]$. These groups are called the mod-4^n *Bott-periodic algebraic K-theory groups of* X.

In this section we prove that an appropriate choice for the 2-primary Bott elements is given by an Adams map between mod-4^n Moore spectra. First, we recall some properties of these 2-primary Adams maps, see [**5**] for details on these maps.

2.6. 2-Primary Adams maps. Let $u \in KU_0(S^0) = \pi_2(BU) = \mathbf{Z}$ be a (Bott) generator. Then, $\bar{u} = u^{2r} \in KU_{2r}(S^0) = \pi_{2r}(BU) = \mathbf{Z}$ is independent of the choice of u. This $\bar{u}$ will be called a *Bott class.*

Now, consider real K-theory KO_* and the complexification map

$$c : KO_*(\_) \to KU_*(\_)$$

Choose a generator $v \in KO_{8r}(S^0) = \pi_{8r}(BO) = \mathbf{Z}$ such that $c(v) = \bar{u}$ is the Bott class in $KU_{8r}(S^0)$.

Now, let $n \geq 1$ and consider the Moore spectrum $\mathbf{P}(2^n) = \mathbf{S}^0 \cup_{2^n} \mathbf{e}^1$.

Using this spectrum to introduce coefficients in KO-theory, write:

$$KO_*(X; \mathbf{Z}/2^n) = [\mathbf{P}(2^n), X \wedge \mathbf{KO}]_*$$

for X any spectrum and $\mathbf{KO}$ the spectrum representing KO_*-theory (see [**1**, Part 3]).

Now, for $v \in KO_{8r}(S^0) = [S^0; \mathbf{KO}]_{8r}$ we have that:

$$\begin{aligned} \bar{v} = 1 \wedge v &\in [\mathbf{P}(2^n) \wedge \mathbf{S}^0, \mathbf{P}(2^n) \wedge \mathbf{KO}]_{8r} \\ &= [\mathbf{P}(2^n), \mathbf{P}(2^n) \wedge \mathbf{KO}]_{8r} \\ &= KO_{8r}(\mathbf{P}(2^n); \mathbf{Z}/2^n) \end{aligned}$$

is a generator, called the mod-2^n *Bott class.*

Now, let $h_{KO} : \pi_*^s(X; \mathbf{Z}/2^n) \to KO_*(X; \mathbf{Z}/2^n)$ be the KO-Hurewicz map defined as follows: If $[f] \in \pi_r^s(X; \mathbf{Z}/2^n) = [\mathbf{P}(2^n), X]_r$ is represented by a map $f : \mathbf{P}(2^n) \to X$ of degree r, then f induces

$$f_* : KO_*(P(2^n); \mathbf{Z}/2^n) \to KO_*(X; \mathbf{Z}/2^n)$$

and we define: $h_{KO}[f] = f_*(e) \in KO_r(X; \mathbf{Z}/2^n)$ where $e \in KO_0(\mathbf{P}(2^n); \mathbf{Z}/2^n) \approx \mathbf{Z}/2^n$ is a generator.

2.7. Definition. A map $A_n : \Sigma^d \mathbf{P}(2^n) \to \mathbf{P}(2^n)$, representing an element $A_n \in \pi_d^s(\mathbf{P}(2^n); \mathbf{Z}/2^n)$, is called an *Adams map* iff

$$h_{KO}[A_n] = \bar{v} = \text{ Bott class } \in KO_d(P(2^n); \mathbf{Z}/2^n)$$

2.8. Remark. Observe that if A_n is an Adams map then $(A_n)_*$ is a KO_*-isomorphism. M. C. Crabb and K. Knapp, [**5**], have proved the following:

2.9. Proposition. [**5**, 3.2]. *Let $d = d(n) = \max(8, 2^{n-1}), n \geq 1$. Then, there exists a family of maps $A_n \in \pi_d^s(\mathbf{P}(2^n); \mathbf{Z}/2^n) = [\Sigma^d \mathbf{P}(2^n), \mathbf{P}(2^n)]_0$ such that:*

(1) *A_n is an Adams map.*

(2) *In the homotopy commutative diagram:*

$$\begin{array}{ccc} \Sigma^d \mathbf{P}(2^n) & \xrightarrow{\quad A_n \quad} & \mathbf{P}(2^n) \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle j} \\ \Sigma^d S^0 & \xrightarrow[\quad \alpha_n \quad]{} & \Sigma S^0 \end{array}$$

where i and j are inclusion into the bottom cell and projection onto the top cell respectively, and α_n is defined by the composite $\alpha_n \simeq j \circ A_n \circ i$, we have that $[\alpha_n] \in \pi_{d-1}^s(S^0)$ generates the 2-primary component of the image of J if $n \geq 4$ (a subgroup of order 2^n if $1 \leq n < 4$).

2.10. Remark. Recall, see e.g. [**19**], that ${}_2\pi_7^s(S^0) = \mathbf{Z}/16$ generated by the Hopf map σ. From the coefficient sequence

$$\cdots {}_2\pi_8^s(S^0) \xrightarrow{4} {}_2\pi_8^s(S^0) \xrightarrow{r} {}_2\pi_8^s(S^0; \mathbf{Z}/4) \xrightarrow{\partial} {}_2\pi_7^s(S^0) \xrightarrow{4} {}_2\pi_7^s(S^0) \rightarrow \cdots$$

and since ${}_2\pi_8^s(S^0) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$, it follows that r is injective.

Also, from the Universal Coefficient Sequence [**11**] we see that

$${}_2\pi_8^s(S^0; \mathbf{Z}/4) = \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

Let $\hat{\sigma}$ a generator of order 4 in ${}_2\pi_8^s(S^0; \mathbf{Z}/4)$. Observe that $\partial(\hat{\sigma}) = 4\sigma$. Consider now the following diagram for q sufficiently large:

$$\begin{array}{ccc} P^{q+8}(4) & \overset{A_1}{\dashrightarrow} & P^q(4) \\ {\scriptstyle i}\uparrow & \searrow^{a_1} & \downarrow{\scriptstyle j} \\ S^{q+7} & \xrightarrow[\quad \alpha_1 \quad]{} & S^0 \end{array}$$

where a_1 is a map that represents $\hat{\sigma}$, i and j are inclusion into the bottom cell and projection onto the top cell respectively, and $\alpha_1 \simeq a_1 \circ i = i^{\#}(a_1)$ represents $\partial(\hat{\sigma}) = 4\sigma$ (recall that the Bockstein morphism ∂ is given by $i^{\#}$). Then, $\alpha_1 \in \pi_7^s(S^0)$ has order 4 since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and σ is of order 16.

The Toda bracket $\{4, \alpha_1, 4\} = 0$ by [**19**, 3.7].

The Adams e-invariant [**2**, Section 3] of α_1 is: $e(\alpha_1) = 1/4(\bmod 1)$ since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and $e(\sigma) = 1/16 \pmod 1$ by [**1**].

It follows, from [**2**, 12.5] that there exists a map A_1 making the previous diagram homotopy commutative, and moreover A_1 is an Adams map.

2.11. Transfer maps. Let $H \subseteq G$ be finite groups, and let $n = [G : H]$ be the index of H in G. As usual, let Σ_r denote the r-th symmetric group for $1 \leq r \leq \infty$. The natural morphisms:

$$G \longrightarrow \Sigma_n \int H \longrightarrow \Sigma_\infty \int H$$

induce, upon applying the classifying space functor $B(-)$ and the plus construction $(-)^+$ (Section 1.1.), maps:

$$BG \longrightarrow B(\Sigma_n \int H) \longrightarrow (B \textstyle\sum_\infty \int H)^+ \simeq Q_0(BH_+)$$

where the equivalence is that of [**7**]. The natural extension of the map $BG \longrightarrow Q_0(BH_+)$ to $Q_0(BG_+)$ is called the (stable) *transfer map*, and we will denote it by:

$$t : Q_0(BG_+) \longrightarrow Q_0(BH_+)$$

2.12. Theorem. *Let $b \in \pi_2^s(B\mathbf{Z}/4; \mathbf{Z}/4) = \mathbf{Z}/4 \oplus \mathbf{Z}/2$ be the generator of order 4. Let $t : Q_0(B\mathbf{Z}/4)_+ \to Q_0(S^0)$ be the transfer map associated to the inclusion $1 \hookrightarrow \mathbf{Z}/4$. Consider $b^4 \in \pi_8^s(B\mathbf{Z}/4; \mathbf{Z}/4)$ and let $\hat{\sigma} \in \pi_8^s(S^0; \mathbf{Z}/4)$ be as in (2.10). Then*

$$t_\# : \pi_8^s(B\mathbf{Z}/4; \mathbf{Z}/4) \to \pi_8^s(S^0; \mathbf{Z}/4)$$

sends b^4 to $\hat{\sigma}$.

Proof: The transfer map $t_\#$ can be factored as:

$$t_\# : \pi_8^s(B\mathbf{Z}/4; \mathbf{Z}/4) \xrightarrow{t_2} \pi_8^s(RP^\infty; \mathbf{Z}/4) \xrightarrow{t_1} \pi_8^s(S^0; \mathbf{Z}/4)$$

where $\mathbf{R}P^\infty = B\mathbf{Z}/2$, t_1 is the transfer map associated to $1 \hookrightarrow \mathbf{Z}/2$ and t_2 is the transfer associated to $\mathbf{Z}/2 \hookrightarrow \mathbf{Z}/4$.

Consider now the following commutative diagram:

$$\begin{array}{ccccc}
\pi_8^s(BZ/4; Z/4) & \underset{f_2}{\overset{t_2}{\rightleftarrows}} & \pi_8^s(RP^\infty; Z/4) & \underset{f_1}{\overset{t_1}{\rightleftarrows}} & \pi_8^s(S^0; Z/4) \\
\Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\partial} \\
\pi_7^s(BZ/4) & \underset{f_2}{\overset{t_2}{\rightleftarrows}} & \pi_7^s(RP^\infty) & \underset{f_1}{\overset{t_1}{\rightleftarrows}} & \pi_7^s(S^0)
\end{array}$$

where f_i, $i = 1, 2$ are the morphisms induced by the group inclusions, t_i are the corresponding transfer maps, and ∂ the Bockstein morphisms.

We know that $\partial(\hat{\sigma}) = 4 \cdot \sigma$.

Similarly, if $\tilde{a}$ is a generator of order 4 of $\pi_8^s(RP^\infty; \mathbf{Z}/4) = \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$, then: $\partial(\tilde{a}) = 4 \cdot a$.

Also, if $\tilde{b}$ is a generator of order 8 of $\pi_7^s(B\mathbf{Z}/4) = \mathbf{Z}/8 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$, then: $\partial(b^4) = 2 \cdot \tilde{b}$.

By Kahn-Priddy [**8**], see also [**7**, Remark 4, p. 26], t_1 is split surjective on the 2-primary components. Thus: $t_1(a) = \sigma \in {}_2\pi_7^s(S^0)$, and so, by the commutativity of the right-hand side square of the diagram above we have: $t_1(\tilde{a}) = \hat{\sigma}$.

Now, observe that $f_2 \cdot t_2 =$ multiplication by 2 on $\pi_*^s(B\mathbf{Z}/4)$ so that $t_2(\tilde{b}) = 2 \cdot a \in \pi_7^s(RP^\infty)$. Therefore $t_2 \cdot \partial(b^4) = 4 \cdot a = \partial(\tilde{a})$. Hence $t_2(b^4) = \tilde{a}$. ■

2.13.

i) Recall that for $A = \mathbf{Z}[1/2, \zeta_4]$ we defined (2.3)

$$\beta = (d_1)_\#(b) \in K_2(A; \mathbf{Z}/4)$$

where $d_1 : (B \sum_\infty \int \mathbf{Z}/4)^+ \to BGLA^+$.

ii) We also defined (2.3)

$$\beta_1 = \beta^4 = \big((d_1)_\#(b)\big)^4 = (d_1)_\#(b^4) \in K_8(A; \mathbf{Z}/4)$$

iii) Now, in order to have a similar description for the higher Bott elements $\beta_n \in K_*(A; \mathbf{Z}/4^n)$ of (2.4) for $n > 1$, we proceed as in [**16**, Section 3], but first we point out, as communicated to me by V. Snaith in a corrigendum of his paper [**16**], that diagram (3.7) of [**16**] should be replaced by the following commutative (up to an inner automorphism) diagram, where in the case considered by Snaith we replace 4 by and *odd* prime ℓ:

$$\begin{array}{ccccc}
\Sigma_n \int Z/4 & \xrightarrow{\quad d_1 \times d_2 \eta \quad} & & & GL_nZ[1/2,\zeta_4] \times GL_nZ[1/2,\zeta_4] \\
\Big\downarrow t & & & & \Big\downarrow \oplus \\
\Sigma_{4n} & \xrightarrow{d_2} & GL_{4n}Z[1/2,\zeta_4] & \xleftarrow{s} & GL_{2n}Z[1/2,\zeta_4]
\end{array}$$

where $s : GL_r \to GL_m, m \geq r$, is the stabilization map,

$$d_1 : \Sigma_r \wr \mathbf{Z}/4 \to \Sigma_r \wr GL_1\mathbf{Z}[1/2, \zeta_4] \to GL_r\mathbf{Z}[1/2, \zeta_4]$$

is induced by the inclusion $\mathbf{Z}/4 \approx \mu_4 \to GL_1\mathbf{Z}[1/2, \zeta_4]$,

$$d_2 : \Sigma_m \to GL_m\mathbf{Z} \to GL_m\mathbf{Z}[1/2, \zeta_4]$$

is induced by inclusion of permutation matrices,

$$\eta : \Sigma_n \wr \mathbf{Z}/4 \to \Sigma_n$$

is induced by the morphism $\mathbf{Z}/4 \to 1$,

$$t : \Sigma_m \wr \mathbf{Z}/4 \to \Sigma_{4m}$$

is the transfer morphism induced by sending a generator of $\mathbf{Z}/4$ to the cycle $(1, 2, 3, 4) \in \Sigma_4$.

Snaith's proof of the commutativity (up to an inner automorphism) of this diagram runs as follows:

The right hand route corresponds to the module

$$(\mathbf{Z}[1/4]^n \otimes_{\mathbf{Z}[1/4]} \mathbf{Z}[1/4]^3) \oplus (\mathbf{Z}[1/4]^n)$$

where $\Sigma_n \wr \mathbf{Z}/4$ acts on the second summand by the permutation representation η, and on the first summand by the tensor product action on the first factor of η with the translation action of $\mathbf{Z}/4$ on $\mathbf{Z}[\zeta_4][1/4]$ as $\mathbf{Z}[1/4]$-module. As in the proof of Lemma (3.8) of [**16**], this is conjugate to the tensor product action of $\eta \otimes \iota$ where

$$\iota : \mathbf{Z}/4 \to \Sigma_4$$

is the natural inclusion.

iv) From this (modified) diagram, applying the classifying space functor, the plus construction and taking $n \to \infty$, we obtain a commutative diagram, which replaces Corollary (3.10) of [**16**],

$$\begin{array}{ccc}
\pi_*(B\Sigma_\infty \wr Z/4^+; Z/4) & \xrightarrow{(d_1)_\# \times (d_2\eta)_\#} & K_*(Z[1/2, \zeta_4]; Z/4) \times K_*(Z[1/2, \zeta_4]; Z/4) \\
\Big\downarrow{\scriptstyle t_\#} & & \Big\downarrow{\scriptstyle +} \\
\pi_*(B\Sigma_\infty^+; Z/4) & \xrightarrow[(d_2)_\#]{} & K_*(Z[1/2, \zeta_4]; Z/4)
\end{array}$$

and since $(B\sum_\infty \int \mathbf{Z}/4)^+ \simeq Q_0(B\mathbf{Z}/4_+)$ and $b \in \pi_2^s(B\mathbf{Z}/4_+;\mathbf{Z}/4)$ originates in $\pi_2(B\mathbf{Z}/4;\mathbf{Z}/4)$ then $\eta_\#(b) = 0$ and hence $\eta_\#(b^4) = 0$ also.

Therefore, we have the formula:

$$(d_2)_\# t_\#(b^4) = (d_1)_\#(b^4),$$

which is essentiallly (3.12) in [**16**] and is used to derive Lemma (3.13) of [**16**] and its consequences.

I thank professor Snaith for communicating the above results to me.

v) Now, using the formula in (iv) we have:

$$\beta_1 = (d_1)_\#(b^4) = (d_2)_\# t_\#(b^4) = (d_2)_\#(\hat{\sigma})$$

by (2.12) where $\hat{\sigma} = j \circ A_1$, with A_1 an Adams map (2.10). vi) Now, $d_2 : B\sum_\infty^+ \to BGL\mathbf{Z}[1/2,\zeta_4]^+$ is the base-point component of the 0-th spaces of the unit

$$D : \mathbf{S}^0 \to \mathbf{KZ}[1/2,\zeta_4]$$

of the algebraic K-theory ring spectrum of $A = \mathbf{Z}[1/2,\zeta_4]$. Therefore,

$$\beta_1 = (d_2)_\#(\hat{\sigma}) = D_\#(\hat{\sigma})$$

2.14. Now, to have the desired description for the higher Bott elements $\beta_n \in K_*(A;\mathbf{Z}/4^n)$ of (2.4) for $n > 1$, we proceed as in [**16**, Section 3] as follows: We want $\beta_n \in D_\#(\pi^s_{8\cdot 4^n-1}(S^0;\mathbf{Z}/4^n))$ where

$$D_\# : \pi_*^s(S^0;\mathbf{Z}/4^n) \to K_*(A;\mathbf{Z}/4^n).$$

By induction on n suposse $\beta_n \in D_\#(\pi^s_{8\cdot 4^n-1}(S^0;\mathbf{Z}/4^n))$ and consider $\beta_{n+1} \in K_{8\cdot 4^n}(A;\mathbf{Z}/4^{n+1})$.

Let $r_\# : \pi_*(-;\mathbf{Z}/4^{n+1}) \to \pi_*(-;\mathbf{Z}/4^n)$ be the reduction map.

Let $x_n \in \pi^s_{8\cdot 4^n-1}(S^0;\mathbf{Z}/4^n)$ be such that $D_\#(x_n) = \beta_n$, and consider $x_n^4 \in \pi^s_{8\cdot 4^n}(S^0;\mathbf{Z}/4^n)$. Since the differentials in the homotopy Bockstein spectral sequence are derivations [**12**] then $\partial_n(x_n^4) = 0$ since $\pi_*^s(S^0;\mathbf{Z}/4)$ is a $\mathbf{Z}/4$ module. Thus, there exists $x_{n+1} \in \pi^s_{8\cdot 4^n}(S^0;\mathbf{Z}/4^{n+1})$ such that $r_\#(x_{n+1}) = x_n^4$. Now, since $D_\#$ is a ring map we have $D_\#(x_n^4) = \beta_n^4$.

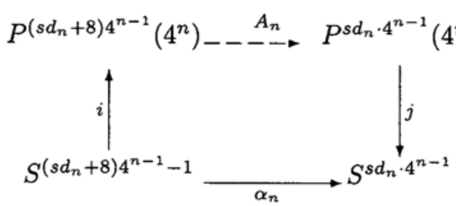
Therefore, by naturality we have:

$$\begin{array}{ccc} x_{n+1} & \xmapsto{\ r_\#\ } & x_n^4 \\ \Big\downarrow{\scriptstyle D_\#} & & \Big\downarrow{\scriptstyle D_\#} \\ D_\#(x_{n+1}) & \overset{r_\#}{\dashrightarrow} & \beta_n^4 \end{array}$$

i.e., $D_{\#}(x_{n+1})$ is an element of $K_{8\cdot 4^n}(A;\mathbf{Z}/4^{n+1})$ that reduces mod-4 to β_n^4.

Therefore, we may choose $\beta_{n+1} = D_{\#}(x_{n+1})$ since this element reduces to β_n^4 which itself reduces to $(\beta_1^{4^{n-1}})^4 = \beta_1^{4^n}$ by (2.4).

2.15. Remark. Analogously to [**16**, Section 3], we can see that for $n \geq 1$, a suitable choice for $x_n \in \pi_*^s(S^0;\mathbf{Z}/4^n)$ is given by an Adams map, i.e. by $a_n = j \circ A_n$ where j and A_n are maps in the diagram:

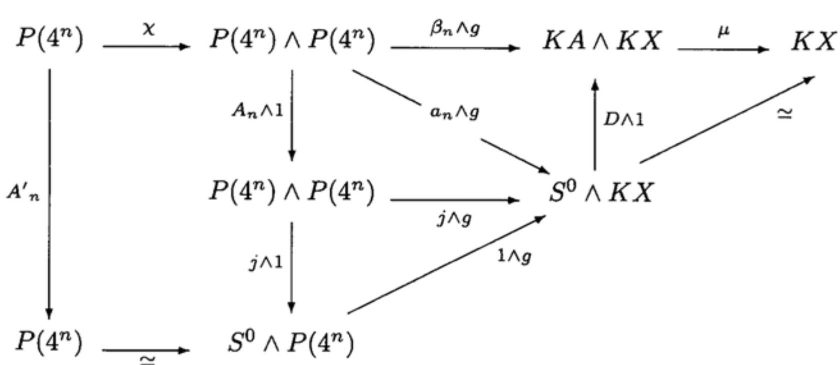


where $d_n = \max(8, 4^{n-1}) = \deg(A_n)$, and A_n an Adams map.

2.16. Now, let X be a commutative A-algebra, $A = \mathbf{Z}[1/2, \zeta_4]$. Then $\mathbf{K}X$ is a $\mathbf{K}A$-module. We denote this action by

$$\mu : \mathbf{K}A \wedge \mathbf{K}X \to \mathbf{K}X.$$

Let $[g] \in K_i(X;\mathbf{Z}/4^n) = \pi_i(\mathbf{K}X;\mathbf{Z}/4^n)$ be represented by a map of spectra $g : \mathbf{P}(4^n) \to \mathbf{K}X$ of degree i. Consider a representative $\beta_n : \mathbf{P}(4^n) \to \mathbf{K}A$ of the Bott element $\beta_n \in K_{8\cdot 4^{n-1}}(A;\mathbf{Z}/4^n)$ of (2.4). We have a commutative diagram of spectra:



where the composite of the top row represents the product

$$\beta_n \cdot [g] \in K_{i+d}(X;\mathbf{Z}/4^n),$$

$\mathbf{S}^0$ is the sphere spectrum, χ is the copairing of Moore spectra of [**4**], μ is the multiplication induced by the action of A on X, A_n and j are the maps of spectra of (2.15) and $a_n \simeq j \cdot A_n$ in (2.15), and D is the unit of $\mathbf{K}A$.

It follows that A'_n is also an Adams map between Moore spectra.

From the commutativity of this diagram it follows that:

$$\beta_n \cdot [g] = [g \cdot A'_n] = A'^*_n[g] \in K_{i+d}(X; \mathbf{Z}/4^n)$$

i.e., multiplication by β_n is precomposition with an Adams map A'_n.

From this remark, we obtain the analogue of Snaith's theorem [**16**, 3.22]:

2.17. Theorem. *Let X be a commutative $\mathbf{Z}[1/2,\zeta]$-algebra. Suppose that there exists a map of Moore spaces $A_n : P^{q+d}(4^n) \to P^q(4^n)$ with $d = 8 \cdot 4^{n-1}$, such that its stable homotopy class is $A'_n : \mathbf{P}(4^n) \to \mathbf{P}(4^n)$ an Adams map of Moore spectra as in (2.11). Suppose $i \geq q$. Then:*

$$K_i(X; \mathbf{Z}/4^n)[1/\beta_n] \approx \varinjlim_k (K_{i+kd}(X; \mathbf{Z}/4^n) \xrightarrow{(\Sigma^{i+kd-q} A_n)^*} K_{i+(k+1)d}(X; \mathbf{Z}/4^n))$$

Proof: First, recall that there exist Adams maps

$$A_n : P^{q+d}(4^n) \to P^q(4^n)$$

for $d = \max(8, 2^{2n-1})$ and q large enough.

Now, by choosing appropriate compositions of suspensions of these Adams maps we get maps

$$A'_n : P^{q+8\cdot 4^{n-1}}(4^n) \to P^q(4^n)$$

that still induce isomorphisms in K-theory, i.e. they are Adams maps.

Now, by the remark (2.16)

$$\beta_n \cdot [g] = A'^*_n[g] = [g \cdot A'_n]$$

and since the isomorphisms

$$K_i(X; \mathbf{Z}/4^n) = [P^i(4^n), BGLX^+] \approx [\Sigma^i \mathbf{P}(4^n), \mathbf{K}X]$$

are such that the following diagram commutes

$$\begin{array}{ccc} [P^i(4^n), BGLX^+] & \xrightarrow{\approx} & [\Sigma^i P(4^n), KX] \\ \downarrow{\scriptstyle (\Sigma^{i-q}A_n)^*} & & \downarrow{\scriptstyle (A'_n)^*} \\ [P^i(4^n), BGLX^+] & \xrightarrow[\approx]{} & [\Sigma^{i+d} P(4^n), KX] \end{array}$$

provided $i \geq q$, since we are assuming that the stable homotopy class of the map A_n is A'_n. Therefore the result follows. ■

References

1. J. F. ADAMS, "*Stable Homotopy and Generalized Homology,*" The University of Chicago Press, Chicago, 1974.
2. J. F. ADAMS, On the group $J(X)$ IV, *Topology* **5** (1966), 21–71.
3. S. ARAKI AND H. TODA, Multiplicative structures in mod-q cohomology theories I, *Osaka Journal of Math.* **2** (1965), 71–115; II *Osaka Journal of Math.* **3** (1966), 870–120.
4. W. BROWDER, Algebraic K-Theory with coefficients $\mathbf{Z}/p$, in "*Geometric applications of homotopy theory,*" Lecture Notes in Mathematics **657**, Springer Verlag, Berlin-Heidelberg-New York, 1978, pp. 40–84.
5. M. C. CRABB AND K. KNAPP, Adams periodicity in stable homotopy, *Topology* **24** (1985), 475–486.
6. W. G. DWYER, E. M. FRIEDLANDER, V. P. SNAITH AND R. W. THOMASON, Algebraic K-theory eventually surjects onto topological K-theory, *Invent. Math.* **66** (1982), 481–491.
7. B. HARRIS AND G. SEGAL, K_i groups of rings of algebraic integers, *Ann. of Math.* **101** (1975), 20–33.
8. D. KAHN AND S. PRIDDY, The transfer and stable homotopy theory, *Math. Proc. Camb. Phil. Soc.* **83** (1978), 103–111.
9. A. LIULEVICIUS, A theorem in homological algebra and stable homotopy groups of projective spaces, *Trans. Am. Math. Soc.* **109** (1963), 540–552.
10. J. L. LODAY, K-théorie algébrique et représentations de groupes, *Ann. Sci. École Norm. Sup.* **9** (1976), 309–377.
11. J. NEISENDORFER, Primary homotopy theory, *Mem. Am. Math. Soc.* **232** (1980).
12. S. OKA, Multiplications in the Moore spectrum, *Mem. of the Fac. of Sci. Kyushu Univ. Ser. A.* **38** (1984), 257–276.

13. D. QUILLEN, On the cohomology and K-theory of the general linear group over a finite field, *Ann. of Math.* **96** (1972), 552–586.
14. V. P. SNAITH, Algebraic cobordism and K-theory, *Mem. Am. Math. Soc.* **221** (1979).
15. V. P. SNAITH, Algebraic K-theory and localized stable homotopy theory, *Mem. Am. Math. Soc.* **280** (1983).
16. V. P. SNAITH, Unitary K-homology and the Lichtenbaum-Quillen conjecture on the algebraic K-theory of schemes, in "*Proc. Algebraic Topology,*" Lecture Notes in Mathematics **1051**, Springer-Verlag, Berlin-Heidelberg-New York, 1984, pp. 128–155.
17. R. W. THOMASON, The Lichtenbaum-Quillen conjecture for $K/\ell_*[\beta^{-1}]$, in "*Current Trends in Algebraic Topology,*" Can. Math. Soc. Conference Proceedings Vol. 2 Part 1, 1984, pp. 117–140.
18. R. W. THOMASON, Algebraic K-theory and étale cohomology, *Ann. Sci. École Norm. Sup.* **18** (1985), 437–452.
19. H. TODA, "*Composition methods in homotopy groups of spheres,*" Annals of Mathematics Studies **49**, Princeton University Press, 1962.
20. F. ZALDÍVAR, The stable Homotopy of $B\mathbf{Z}/4$, *Aportaciones Matemáticas. Comunicaciones* **5** (1988), 95–108.

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