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EXISTENCE DOMAINS FOR HOLOMORPHIC L^p FUNCTIONS

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Abstract

If Ω is a domain of holomorphy in $\mathbb{C}^n$, having a compact topological closure into another domain of holomorphy $U \subset \mathbb{C}^n$ such that (Ω, U) is a Runge pair, we construct a function F holomorphic in Ω which is singular at every boundary point of Ω and such that F is in $L^p(\Omega)$, for any $p \in (0, +\infty)$.

1. Statement of the problem

The following notation and terminology will be used without further explanation. The open polydisc in $\mathbb{C}^n$ with center α and radious r is denoted by $\Delta^n(\alpha; r)$; if $n = 1$, then we use the notation $\Delta(\alpha; r)$. For every open set D in $\mathbb{C}^n$, $\theta(D)$ denotes the space of all holomorphic functions in D. If K is a compact subset of D, we define the $\theta(D)$-hull $\hat{K}_D$ of K by $\hat{K}_D := \{z \in D;\ |f(z)| \leqq \sup_{w \in K} |f(w)|, \text{ for all } f \in \theta(D)\}$. For $p \in (0, +\infty]$, we set $\theta L^p(D) := \theta(D) \cap L^p(D)$. Obviously, $\theta L^\infty(D)$ equals the algebra $H^\infty(D)$ of bounded holomorphic functions in D. If D carries a function $F \in \theta L^p(D)$, which cannot be holomorphically extended across the boundary of D, then D is said to be an existence domain for θL^p or of type θL^p.

Asking for the conditions under which a bounded domain of holomorphy Ω is of type θL^p, we recall the following result: *If $\Omega \subset\subset \mathbb{C}^n$ is a domain of holomorphy with C^∞ boundary and $(\alpha_\nu \in \Omega;\ \nu \in \mathbb{N})$ is a sequence such that $(\lim_{\nu \to \infty} \alpha_\nu) \in \partial\Omega$, then there exists a function $F \in \theta L^2(\Omega)$ satisfying $\overline{\lim}_{\nu \to \infty} |F(\alpha_\nu)| = +\infty$* ([**4**]). The question we are interested is the following: *Is any bounded domain of holomorphy in $\mathbb{C}^n$ existence domain for θL^p, for every $p \in (0, +\infty)$?* In [**1**] Catlin showed that *any smoothly bounded domain of holomorphy in $\mathbb{C}^n$ is of type θL^∞ (, and consequently of type θL^p, for every $p \in (0, +\infty)$)*. However in [**6**], Sibony showed that *there is a bounded Runge complete Hartogs domain of holomorphy $\Omega_S \subset \Delta^2(0; 1)$ ($\Omega_S \neq \Delta^2(0; 1)$) such that all bounded* *holomorphic functions in Ω_S extend holomorphically to the open unit bidisc, that is Ω_S is not of type θL^∞.*

The concern of this note is to give an answer to the above question. Our approach illustrates a partial extension of Catlin's improvement. More precisely, we shall prove that *any domain of holomorphy $\Omega \subset\subset \mathbb{C}^n$, having a compact topological closure into another domain of holomorphy U such that (Ω, U) is a Runge pair, is of type θL^p for any $p \in (0, +\infty)$.*

2. Unbounded holomorphic functions in Runge domains

Let $\Omega \subset\subset U$ be domains of holomorphy in $\mathbb{C}^n$. Assume that Ω is a bounded Runge domain relative to U.

Let $(z_m;\, m \in \mathbb{N})$ be a dense sequence in Ω, such that every point of the sequence is counted infinitely many times. Let r_m be the largest number with $\Delta^n(z_m; r_m) \subset \Omega$. Ω can be exhausted by compact sets E_j, so that $E_j \subset \mathring{E}_{j+1}$. Letting $K_1 := E_1$, we find a point $w_1 \in \Delta^n(z_1; r_1) - \hat{K}_{1,U}$. Obviously, there exists a $j_1 > 1$, with $w_1 \in E_{j_1}$. Put $K_2 := E_{j_1}$. Now, there is a point $w_2 \in \Delta^n(z_2; r_2) - \hat{K}_{2,U}$. If we set $K_3 := E_{j_2}$ (, $j_2 > j_1$), then $w_2 \in E_{j_2}$. Continuing like this, we find an exhaustive sequence $(K_m; m \in \mathbb{N})$ of compact subsets of Ω and a sequence $(w_m;\, m \in \mathbb{N})$ of points of Ω, with the following properties:

- $w_m \in K_{m+1} - \hat{K}_{m,U}$ (, $m \in \mathbb{N}$),
- whenever $w \in \partial\Omega \cap \Delta^n(\xi; \rho)$ for a polydisc $\Delta^n(\xi; \rho)$ and V is a connected component of $\Omega \cap \Delta^n(\xi; \rho)$ clustering at w, there exists a subsequence of $(w_m;\, m \in \mathbb{N})$ converging to w in V.

To each w_m there corresponds a holomorphic function $f_m \in \theta(U)$, such that $|f_m(w_m)| > \sup_{z \in K_m} |f_m(z)| = 1$. If we let $0 < \varepsilon_m < |f_m(w_m)| - 1$, then $|f_m(z)| < |f_m(w_m) - \varepsilon_m|$, whenever $z \in K_m$. Hence, for suitably choosen numbers $v_m > 0$, the series

$$F(z) = \sum_{m=1}^{\infty} ([f_m(z)]^{v_m} / [f_m(w_m) - \varepsilon_m]^{v_m})$$

converges absolutely and compactly on Ω and $|F(w_m)| > m$, for any m. It follows that whenever $w \in \partial\Omega$, $\Delta^n(\xi; \rho)$ is a polydisc containing w and V is a connected component of $\Omega \cap \Delta^n(\xi; \rho)$ clustering at w, F is unbounded in V. So, F is a function holomorphic on Ω, which is singular (unbounded) at every boundary point of Ω ([**3**]).

3. Runge domains of type θL^p

Let the notations and assumptions be as in Section 2. The principal purpose of this paragraph is to announce the following:

Theorem 1. *Let $\Omega \subset\subset U$ be domains of holomorphy in $\mathbb{C}^n$ such that (Ω, U) is a Runge pair. Then, $F \in \theta L^p(\Omega)$, for any $p \in (0, +\infty)$.*

Proof: The evaluation of more useful choice of v_m is our first aim. Let $\delta > 2$. For each $m \in \mathbb{N}$, choose v_m so that $|f_m(w_m| - \varepsilon_m|^{v_m} \geqq \delta^m$. It is easily seen that the power series

$$h(\zeta) = \sum_{m=1}^{\infty} [f_m(w_m) - \varepsilon_m]^{-v_m} \cdot \zeta^m$$

converges into the disc $\Delta(0;\delta)$. Define a linear functional

$$\Lambda_h : \mathbb{P}(\mathbb{C}) \to \mathbb{C}; \quad x^m \to \Lambda_h(x^m) := [f_m(w_m) - \varepsilon_m]^{-v_m},$$

where $\mathbb{P}(\mathbb{C})$ is the vector space of complex polynomials in $\mathbb{C}$. In order to prove the theorem two lemmas play crucial role:

Lemma 1. ([**2**]) *The functional Λ_h is continuous and there is a continuous extension of Λ_h into $\theta(\overline{\Delta(0;\delta^{-1})})$. Further, for each $\zeta \in \Delta(0;\delta)$ there holds $\Lambda_h((1 - x\zeta)^{-1}) = h(\zeta)$ $(, x \in \Delta(0;\delta^{-1}))$.*

Proof of Lemma 1: Let $r < \delta$. If $p(x)$ is a polynomial in $x \in \mathbb{C}$, then by Cauchy's integral formula we have

$$|\Lambda_h(p)| \leqq M(r) \cdot \sup_{|x| \leqq r} |h(x)| \cdot \sup_{|x| \leqq r^{-1}} |p(x)|,$$

where the constant $M(r)$ depends only on r. Hence, by density, there is a continuous extension of Λ_h on $\theta(\overline{\Delta(0;\delta^{-1})})$. If now $\zeta \in \Delta(0;\delta)$ and if ζ is fixed, then the number $\Lambda_h((1 - x\zeta)^{-1})$ is well defined (: Λ_h acts on the variable $x \in \overline{\Delta(0;\delta^{-1})}$ and ζ is regarded as a parameter). By the continuity of Λ_h, we obtain $\Lambda_h((1 - x\zeta)^{-1}) = h(\zeta)$. ■

The next lemma is a consequence of Lemma 1, but is much more useful since the choice of the functional Λ_h is eliminated.

Lemma 2. *If $z \in \Omega$, then there holds*

$$|F(z)| \leqq f\left(\frac{1}{\tau}\right) \cdot \sum_{m=1}^{\infty} \frac{|[f_m(z)]^{v_m}|}{\tau^m},$$

for any $\tau \in (2, \delta)$ and where the constant $\mathcal{L}\left(\frac{1}{\tau}\right)$ depends only on τ but is independent of z.

Proof of Lemma 2: Assuming that $z \in \Omega$, $x \in \Delta(0; \delta^{-1})$ and $\tau \in (2, \delta)$, we have by Cauchy's integral formula and by Lemma 1:

$$\begin{aligned}|F(z)| &= \left|\sum_{m=1}^{\infty} \Lambda_h(x^m) \cdot [f_m(z)]^{v_m}\right| = \\ &= \left|\frac{1}{2\pi i} \cdot \int_{|\zeta|=\frac{1}{\tau}} \Lambda_h(1/(\zeta - x)) \cdot \left(\sum_{m=1}^{\infty} [f_m(z)]^{v_m} \cdot \zeta^m\right) d\zeta\right| \leqq \\ &\leqq L\left(\frac{1}{\tau}\right) \cdot \left(\sup_{|\zeta|=\frac{1}{\tau}} |\Lambda_h(1/(\zeta - x))|\right) \left(\sup_{|\zeta|=\tau} \left\{\sum_{m=1}^{\infty} \frac{|[f_m(z)]^{v_m}|}{\zeta^m}\right\}\right),\end{aligned}$$

that is

$$|F(z)| \leqq \mathcal{L}\left(\frac{1}{r}\right) \cdot \sum_{m=1}^{\infty} \frac{|[f_m(z)]^{v_m}|}{\tau^m}. \quad \blacksquare$$

End of Proof of Theorem 1: Let $0 < p < +\infty$. By Lemma 2 and by Fatou's Theorem, it is enough to show that

$$\sup\left\{\int_{\Omega} \left(\sum_{m=1}^{v} \frac{|[f_m(z)]^{v_m}|}{\tau^m}\right)^p d\lambda(z); \quad v \in \mathbb{N}\right\} < +\infty,$$

for some $\tau \in (2, \delta)$. ($d\lambda(\cdot)$ is the Lebesgue measure in $\mathbb{C}^n$).

Suppose $\tau \in (2, \delta)$. For any $v \in \mathbb{N}$, choose a positive number $\frac{2k_v - 1}{k_v}$ (, $k_v \in \mathbb{N}$), such that

$$\int_{\Omega} \left(\sum_{m=1}^{v} |[f_m(z)]^{v_m}|\right)^p d\lambda(z) \leqq \left(\frac{2k_v - 1}{k_v}\right)^p.$$

This choice permits us to obtain the following inequalities

$$\int_{\Omega} \left(\sum_{m=1}^{v} \frac{|[f_m(z)]^{v_m}|}{2}\right)^p d\lambda(z) \leqq \left(\frac{2k_v - 1}{2k_v}\right)^p \leqq 1,$$

for any $v \in \mathbb{N}$. Therefore,

$$\int_\Omega \left(\sum_{m=1}^{v} \frac{|[f_m(z)]^{v_m}|}{\tau} \right)^p d\lambda(z) \leqq 1,$$

for any $v \in \mathbb{N}$ and consequently,

$$\int_\Omega \left(\sum_{m=1}^{v} \frac{|[f_m(z)]^{v_m}|}{\tau^m} \right)^p d\lambda(z) \leqq 1,$$

for any $v \in \mathbb{N}$, which completes the proof. ∎

We are now in position to formulate the main result of this note, which is an immediate consequence of Theorem 1:

Theorem 2. *Let $\Omega \subset\subset U$ be domains of holomorphy in $\mathbb{C}^n$. Assume that Ω is a bounded Runge domain relative to U. Then, Ω is an existence domain for θL^p, for any $p \in (0, +\infty)$. In particular, any bounded Runge domain of holomorphy is of type θL^p, for any $p \in (0, +\infty)$.*

We finally turn to the question whether Sibony's example Ω_S in [**6**] is an existence domain of L^p holomorphic functions. The answer is a direct consequence of Theorem 2: Since Sibony's example is a bounded Runge domain of holomorphy, it is an existence domain for θL^p, for any $p \in (0, +\infty)$.

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