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LOCALLY SOLUBLE GROUPS WITH ALL NONTRIVIAL NORMAL SUBGROUPS ISOMORPHIC

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Abstract

Let G be an infinite, locally soluble group which is isomorphic to all its nontrivial normal subgroups. If G/G' has finite p-rank for $p = 0$ and for all primes p, then G is cyclic.

In paper [**2**] we considered groups which are isomorphic to all of their nontrivial normal subgroups. The question as to which infinite groups have this property P, say, was first raised by Philip Hall. It was shown in [**2**] that, if G is a finitely generated infinite P-group which contains a proper normal subgroup of finite index, then G is cyclic, and our conjecture is that $\mathbb{Z}$ is the only finitely generated infinite P-group which is not simple. It was further remarked in [**2**] that is should perhaps be possible to deal with the locally soluble case, and this note represents a step in this direction. The following is proved.

Theorem. *Let G be an infinite, locally soluble group which is isomorphic to all of its nontrivial normal subgroups. If G/G' has finite p-rank for all $p = 0$ or a prime then G is cyclic.*

We recall that an abelian group A has p-rank r if the cardinality of a maximal independent subset of elements of A of order p is equal to r. In particular, if A has finite (Prüfer) rank then the p-ranks of A are boundedly finite.

There is one aspect of the proof of our theorem which recalls part of the proof from [**2**], namely the exploitation of "linearity conditions" which are forced by the rank restrictions (in conjunction with property P). In the case where G has normal abelian p-sections of possibly infinite rank, such a technique is bound to fail, and it is not clear how one might approach the case where, for example, G is an arbitrary locally nilpotent group with P. Clearly such a group is either torsionfree or a p-group, but beyond that there is little that we can say at the moment.

Proof of the theorem:

Suppose first that G has a nontrivial, torsionfree soluble image S and let r be the 0-rank of G/G'. Then, because of property P, S has finite Hirsch length (that is, the sum of the 0-ranks of the derived factors of G is finite). Let F denote the Fitting subgroup of S. Then F is locally nilpotent and its abelian subgroups have finite 0-rank. Since F is torsionfree, it is nilpotent (see Lemma 6.37 of [**3**]). Let A be a maximal normal abelian subgroup of F. Then A is self-centralizing in F and of rank at most r (again by P), and so F/A embeds in the group of (upper) unitriangular $r \times r$ matrices over $\mathbb{Q}$. It follows that F/A and hence F has bounded rank and bounded nilpotency class c, say. For each $i = 1, \ldots, c$, let Z_i denote the i-th term of the upper central series of F and let D_i be the centralizer in S of Z_i/Z_{i-1} (where $Z_0 = 1$). Then S/D_i is a soluble group of automorphisms of Z_i/Z_{i-1}, which is torsionfree abelian of rank at most r, and so S/D_i embeds in $GL(r, \mathbb{Q})$. By the result of Zassenhaus ([**3**, Theorem 3.23]), S/D_i has bounded derived length. Let $D = \bigcap_{i=1}^{c} D_i$. Then S/D has bounded derived length. Further, D stabilizes a series of length c in F and so, writing C for the centralizer of F in S, we see that D/C is nilpotent ([**1**, Lemma 3.5]). But $C = Z(F)$ (e.g. Lemma 2.17 of [**3**]) and so $[C, D] = 1$ and D is nilpotent and hence in F. It follows that S has bounded derived length and we can choose N minimal subject to $N \lhd G$ and G/N torsionfree soluble. If $N \neq 1$ then, by property P, N has a nontrivial, torsionfree soluble image, contradicting the definition of N. Thus $N = 1$ and G is soluble. Clearly $G \cong \mathbb{Z}$ in this case. From now on, we may assume that all soluble images of G are periodic. (If G were to have a nonperiodic soluble image then some abelian normal factor of G would be nontrivial and torsionfree and so, again by P, G/G' would have a nontrivial torsionfree image.) Let H/K be an arbitrary chief factor of G - such exists in every nontrivial group. Then H/K is an elementary abelian p-group, for some prime p, and we see that G therefore has a nontrivial finite p-image. Let $P_1 = G'G^p$ and, for $i \geq 1$, let $P_{i+1} = P_i'P_i^p$. By property P, the subgroups P_i form a strictly descending chain of normal subgroups of G. Also, each G/P_i is a finite p-group. Let $R = \bigcap_{i=1}^{\infty} P_i$ and write $\overline{G} = G/R$, $\overline{P}_i = P_i/R$, $i = 1, 2, \ldots$. Let s be the rank of G/P_1 and let $\overline{A}$ be an arbitrary finitely generated abelian subgroup of $\overline{G}$. The subgroups $\overline{A} \cap \overline{P}_i$ form a descending chain, with trivial intersection, such that each $\overline{A}/\overline{A} \cap \overline{P}_i$ is a finite (abelian) p-group of rank at most s (since $\overline{A}\,\overline{P}_i/\overline{P}_i$ is subnormal in $\overline{G}/\overline{P}_i$). It follows that $\overline{A}$ has rank at most s, and so $\overline{G}$ is a locally soluble group whose abelian subgroups have bounded rank. By a result of Merzljakov (see p. 89, vol. 2 of [**3**] for a reference), $\overline{G}$ has finite rank.

Now by Lemma 10.39 of [**3**], $\overline{G}$ is periodic-by-soluble and hence periodic. Clearly, therefore, $\overline{G}$ is a locally nilpotent p-group and hence a Černikov group (Corollary 1 to Theorem 6.36 of [**3**]). Since $\overline{G}$ is residually finite, it must be finite, contradicting the choice of the subgroups P_i. This completes the proof of the theorem. ∎

References

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