

THE NON-ARCHIMEDEAN SPACE $BC(X)$ WITH THE STRICT TOPOLOGY

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Abstract

Let X be a zero-dimensional, Hausdorff topological space and K a field with a non-trivial, non-archimedean valuation under which it is complete. Then $BC(X)$ is the vector space of the bounded continuous functions from X to K . We obtain necessary and sufficient conditions for $BC(X)$, equipped with the strict topology, to be of countable type and to be nuclear in the non-archimedean sense.

Introduction

Throughout the paper K is a complete non-archimedean valued field with a valuation $|\cdot|$ which is not trivial, and X is a zero-dimensional Hausdorff topological space. We denote by $C(X)$ (resp. $BC(X)$) the space of the continuous (resp. bounded and continuous) functions from X to K . For $A \subset X$ and $f \in BC(A)$ we define $\|f\|_A = \sup_{x \in A} |f(x)|$.

All the specific definitions needed are included in the paper. For more general facts on locally convex spaces and Banach spaces we refer to [7] and [8].

We consider on $BC(X)$ the strict topology τ_β (definition below). Many properties of the locally convex space $BC(X)$, τ_β have already been investigated. See e.g. [4], [5] and [6]. But so far no attention was paid to the properties “being of countable type” and “being nuclear”. This paper fills that gap. The curious fact is that the conditions for the strict topology turn out to be the same as those obtained for the compact open topology τ_c , in [3].

1. The strict topology on $BC(X)$

1.1. Definitions and notations.

Denote by $B_0(X)$ the bounded functions $\varphi : X \rightarrow K$ which vanish at infinity. The strict topology τ_β on $BC(X)$ is then defined by the family of semi-norms $\{p_\varphi; \varphi \in B_0(X)\}$, where $p_\varphi(f) = \|\varphi \cdot f\|_X$, $f \in BC(X)$. The strict topology lies between the compact-open topology τ_c (i.e. the topology of uniform convergence on compact subsets of X) and the uniform topology τ_u . Thus $\tau_c \leq \tau_\beta \leq \tau_u$.

In [4, p. 193], Katsaras shows that a basis for the zero-neighbourhoods in $BC(X)$, τ_β consists of the sets of the form:

$$W(A, k) = \bigcap_n \{f \in BC(X); \|f\|_{A_n} \leq k_n\},$$

where $A = (A_n)$ is an increasing sequence of compact subsets of X and $k = (k_n)$ is a sequence of real numbers, increasing to infinity, with $k_1 > 1$.

The next lemma follows easily.

1.2. Lemma.

The strict topology τ_β on $BC(X)$ can be determined by the family of semi-norms $p_{A,k}$ with A and k as above and where

$$p_{A,k}(f) = \sup_n k_n^{-1} \cdot \|f\|_{A_n}, \quad f \in BC(X).$$

1.3. Remark.

Let $k = (k_n)$ be as above. Then, taking $l_n = k_n^{1/2}$, the sequence $l = (l_n)$ gives us a continuous semi-norm $p_{A,l}$ on $BC(X)$, τ_β such that $p_{A,l}(f) \geq p_{A,k}(f)$ for all $f \in BC(X)$, and $\lim_n k_n \cdot l_n^{-1} = \infty$. This semi-norm will be used in Section 3.

2. Conditions for $BC(X)$, τ_β to be of countable type

2.1. Definition. ([8, p. 66]).

A normed space E over K is said to be of countable type if it is the closed linear span of a countable set.

2.2. Example.

If X is compact then $BC(X) = C(X)$, τ_u is of countable type if and only if X is ultrametrizable. ([8, 3.T]).

2.3. Definition. ([8, p. 52]).

Let $(E_n, \|\cdot\|_n)$ be a sequence of Banach spaces and put $\oplus_n E_n = \{(x_n); x_n \in E_n \text{ for all } n \text{ and } \lim_n \|x_n\|_n = 0\}$. This is a Banach space for the norm $\|(x_n)\| = \max_n \|x_n\|_n$.

The following is easily proved.

2.4. Proposition.

The space $\oplus_n E_n, \|\cdot\|$ is of countable type if and only if $E_n, \|\cdot\|_n$ is of countable type for all n .

2.5. Notation.

Let E be a locally convex Hausdorff space over K , the topology of which is determined by a family of semi-norms P (E, P in short). For each $p \in P$ put $E_p = E/\text{Ker } p$ and denote by $\pi_p : E \rightarrow E_p$ the canonical surjection. The space E_p is then normed by $\|\pi_p(x)\|_p = p(x), x \in E$.

2.6. Example.

Let $p_{A,k}$ be one of the semi-norms determining the strict topology τ_β on $BC(X)$. (See 1.2).

Then $\text{Ker } p_{A,k} = \{f \in BC(X); f(x) = 0 \text{ for } x \in \bigcup_n A_n\}$. Denote by $BC(A, k)$ the space $BC(\bigcup_n A_n)$, normed by $p_{A,k}$. Then the normed space $BC(X)/\text{Ker } p_{A,k}$ is linearly isometric with a subspace of $BC(A, k)$. Indeed we have a commutative diagram

$$\begin{array}{ccc}
 & BC(X) & \\
 \pi_{A,k} \swarrow & & \searrow R \\
 BC(X)/\text{Ker } p_{A,k} & \xrightarrow{S} & BC(A, k)
 \end{array}$$

where $\pi_{A,k}$ is the canonical surjection and R is the restriction map which sends $f \in BC(X)$ onto its restriction to $\bigcup_n A_n$. Then S is the desired isometry.

For later use we prove:

2.7. Lemma.

$BC(A, k)$ is linearly isometric to a subspace of the space $\oplus_n C(A_n, k_n)$, where, for all $n, C(A_n, k_n)$ is the space $C(A_n)$, normed by

$$\|g\|_n = \|g\|_{A_n, k_n^{-1}}, \quad g \in C(A_n).$$

Note that each of the spaces $C(A_n, k_n)$ is a Banach space, linearly homeomorphic to $C(A_n)$, τ_u .

Proof:

For $f \in BC(A, k)$ let f_n stand for the restriction of f to A_n .

Then $f_n \in C(A_n, k_n)$, and $\|f_n\|_n = \|f_n\|_{A_n} \cdot k_n^{-1} \leq \|f\|_{\bigcup_n A_n} \cdot k_n^{-1}$. Hence $\lim_n \|f_n\|_n = 0$, and we can define $T : BC(A, k) \rightarrow \bigoplus_n C(A_n, k_n) : f \rightarrow (f_n)$. This T is the desired linear isometry. ■

2.8. Definition. ([7, 4.3]).

The locally convex space E, P is said to be of countable type if each of the normed spaces $E_p, \|\cdot\|, p \in P$, is of countable type.

2.9. Theorem.

The following are equivalent:

- i) $BC(X), \tau_\beta$ is of countable type.
- ii) $BC(X), \tau_c$ is of countable type.
- iii) Every compact subset of X is ultramerizable.

Proof:

i) \Rightarrow ii) follows directly from the fact that $\tau_c \leq \tau_\beta$. (See 1.1).

ii) \Leftrightarrow iii): see [3, Prop. 3.2].

iii) \Rightarrow i): Let $p_{A,k}$ be one of the seminorms determining τ_β (see 1.2). We have to prove that the normed space $BC(X)/\text{Ker } p_{A,k}$ is of countable type. Now every subspace of a space of countable type, is of countable type ([8, 3.16]). Hence, by 2.4, 2.6, and 2.7 it suffices to show that each of the normed spaces $C(A_n, k_n)$ is of countable type. Now make use of the remark made in 2.7 and apply 2.2. ■

2.10. Remark.

The conditions in Theorem 2.9 are not equivalent to " $BC(X), \tau_u$ is of countable type". Indeed, take $X =$ the natural numbers with the discrete topology. Then $BC(X), \tau_u = l^\infty, \|\cdot\|$ which is not of countable type. Also note that the strict topology on l^∞ coincides with the natural topology $n(l^\infty, c_0)$ in the sense of perfect sequence spaces ([1, p. 473]) and that the compact open topology on l^∞ is the weak topology $\sigma(l^\infty, c_0)$. Hence the inequalities in 1.1 may be strict.

3. The nuclearity of $BC(X)$, τ_β

3.1. Definitions.

Let E be a locally convex space over K . A subset B of E is called compactoid if for every zero-neighbourhood U in E there exists a finite subset S of E such that $B \subset U + Co(S)$, where $Co(S)$ is the absolutely convex hull of S .

A linear map T from a normed space E to a normed space F is called compact if it maps the unit ball of E into a compactoid subset of F .

The following is easily seen:

3.2. Lemma.

Let E, F and G be normed spaces over K , $T : E \rightarrow F$ a linear map, and $S : F \rightarrow G$ a linear isometry. If SoT is compact, then so is T .

3.3. Definition.

Let E, P be a locally convex space over K . If $p \in P$ and q is a continuous seminorm on E with $p \leq q$. Then there exists a unique continuous linear map $\varphi_{pq} : E_q \rightarrow E_p$ which makes the diagram

$$\begin{array}{ccc}
 & E & \\
 \pi_q \swarrow & & \searrow \pi_p \\
 E_q & \longrightarrow & E_p
 \end{array}$$

The space E, P is called nuclear if for every $p \in P$ there exists a continuous seminorm q on E with $p \leq q$ such that the map φ_{pq} is compact.

3.4. Theorem.

The following are equivalent:

- i) $BC(X)$, τ_β is nuclear.
- ii) Every τ_β -bounded subset of $BC(X)$ is τ_β -compactoid.
- iii) $BC(X)$, τ_c is nuclear.
- iv) Every τ_c -bounded subset of $BC(X)$ is τ_c -compactoid.
- v) $C(X)$, τ_c is nuclear.
- vi) Every τ_c -bounded subset of $C(X)$ is τ_c -compactoid.
- vii) Every compact subset of X is finite.

Proof:

i) \Rightarrow ii) and iii) \Rightarrow iv) are general properties of nuclear spaces (see [2, 5.1]).

ii) \Leftrightarrow vi) is proved in [6, 2.2].

v) \Leftrightarrow vi) \Leftrightarrow vii) is proved in [3, 3.3].

We prove here vii) \Rightarrow i), v) \Rightarrow iii) and iv) \Rightarrow ii).

vii) \Rightarrow i):

Let $p_{A,k}$ be one of the semi-norms determining τ_β on $BC(X)$ (see 1.2) and let $p_{A,l}$ be the corresponding semi-norm given in 1.3. We prove that the canonical map

$$\varphi_{k,l} : BC(X)/\text{Ker } p_{A,l} \rightarrow BC(X)/\text{Ker } p_{A,k}$$

is compact.

Denote by $B(A, l)$ (resp. $B(A, k)$) the space of the bounded continuous functions from $\bigcup_n A_n$ to K , normed by $p_{A,l}$ (resp. $p_{A,k}$). We first show that the identity map $I : B(A, l) \rightarrow B(A, k)$ is compact.

Let B_l (resp. B_k) denote the unit ball of $B(A, l)$ (resp. $B(A, k)$). We have to prove that B_l is compactoid in $B(A, k)$. Choose $\alpha \in K, \alpha \neq 0$. We need to find a finite subset S of $B(A, k)$ such that

$$(*) \quad B_l \subset \alpha B_k + Co(S).$$

We fix an index n_0 such that $l_n k_n^{-1} < |\alpha|$ for $n \geq n_0$, and consider $A_1 \subset A_2 \subset \dots \subset A_{n_0}$. Note that A_{n_0} is finite by vii).

Let $A_1 = \{x_1^1, \dots, x_{k_1}^1\}$, $A_2 \setminus A_1 = \{x_1^2, \dots, x_{k_2}^2\}, \dots, A_{n_0} \setminus A_{n_0-1} = \{x_1^{n_0}, \dots, x_{k_{n_0}}^{n_0}\}$. Further denote by $\xi(x_i^j)$ the characteristic function of $\{x_i^j\}$ and take

$$S = \{l_1 \xi(x_1^1), \dots, l_1 \xi(x_{k_1}^1), l_2 \xi(x_1^2), \dots, \dots, l_{n_0} \xi(x_{k_{n_0}}^{n_0})\}.$$

Then $S \subset B(A, k)$ and it is easy to calculate that for this S condition (*) is satisfied.

Now consider the diagram

$$\begin{array}{ccc} BC(X)/\text{ker } p_{A,l} & \xrightarrow{\varphi_{k,l}} & BC(X)/\text{Ker } p_{A,k} \\ s_l \downarrow & & \downarrow s_k \\ BC(A, l) & & BC(A, k) \\ i_l \downarrow & & \downarrow i_k \\ B(A, l) & \xrightarrow{I} & B(A, k) \end{array}$$

where S_l (resp. S_k) is the isometry described in 2.6 and i_l (resp. i_k) is the canonical injection. One sees easily that this diagram is commutative. Now, since I is compact, the map $I \circ i_l \circ S_l$ is compact as well. Hence $i_k \circ S_k \circ \varphi_{kl}$ is compact. Since $i_k \circ S_k$ is an isometry we finally obtain from Lemma 3.2 that $\varphi_{l,k}$ is compact.

v) \rightarrow iii):

Follows from the fact that every subspace of a nuclear space is nuclear ([2, 5.7]).

iv) \rightarrow ii):

Let B be a τ_β -bounded subset of $BC(X)$. Then B is τ_u -bounded ([4, 2.11]) and therefore the topologies τ_β and τ_c coincide on B ([4, 2.9]). On the other hand, since τ_c is coarser than τ_β (see 1.1) B is τ_c -bounded. So B is τ_c -compactoid by iv). It then follows from [7, 10.5] that B is τ_β -compactoid. ■

3.4. Remarks.

3.4.1. If the space $BC(X)$, τ_β is complete the conditions in Theorem 3.3 are equivalent to

viii) X is a discrete topological space.

Indeed, $BC(X)$, τ_β is complete if and only if X is an ultra k -space ([5, Prop. 9]). Then from vii) it follows, as in [3, 3.3], that X is discrete. On the other hand, if X is discrete, then vii) follows trivially.

3.4.2. From vii) it does not follow that τ_β coincides with the weak topology $\sigma(BC(X), BC(X)')$ where $BC(X)'$ is the topological dual of $BC(X)$, τ_β (compare with [3, 3.3]). To see this consider the example in 2.10. There $(l^\infty, \tau_\beta)' = c_0$ and $\sigma(l^\infty, c_0)$ is strictly weaker than $n(1^\infty, c_0) = \tau_\beta$ ([1, Prop. 6]).

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Primera versió rebuda el 8 de Setembre de 1993,
darrera versió rebuda el 15 de Desembre de 1993