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MAXIMAL FUNCTIONS AND RELATED WEIGHT CLASSES

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Abstract

The famous result of Muckenhoupt on the connection between weights ω in A_p-classes and the boundedness of the maximal operator in $L_p(\omega)$ is extended to the case $p = \infty$ by the introduction of the geometrical maximal operator. Estimates of the norm of the maximal operators are given in terms of the A_p-constants. The equality of two differently defined A_∞-constants is proved. Thereby an answer is given to a question posed by R. Johnson. For non-increasing functions on the positive real line a parallel theory to the A_p-theory is established for the connection between weights in B_p-classes and maximal functions, thereby extending and developing the recent results of Ariño and Muckenhoupt.

1. Introduction

Let f be a non-negative, locally integrable function defined on $(0, \infty)$. The well-known Carleman inequality (see [**4**, p. 250])

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx \leq e\int_0^\infty f(x)dx,$$

in which e is the best possible constant, can be considered as the limit case, as p tends to infinity, of the Hardy inequality for $f^{\frac{1}{p}}$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)^{\frac{1}{p}}dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)dx.$$

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In fact, the geometrical mean of f, $\exp(\frac{1}{x}\int_0^x \ln f(t)dt)$, satisfies (see [**4**, p. 139])

$$\lim_{p\to\infty}(\frac{1}{x}\int_0^x f(t)^{\frac{1}{p}}dt)^p = \exp(\frac{1}{x}\int_0^x \ln f(t)dt). \tag{1.1}$$

We recall that $(\frac{p}{p-1})^p$ is the best constant in Hardy's inequality and so we deduce

$$\lim_{p\to\infty}\sup_{\|g\|_p=1}\int_0^\infty(\frac{1}{x}\int_0^x g(t)dt)^p dx = \sup_{\|f\|_1=1}\int_0^\infty \exp(\frac{1}{x}\int_0^x \ln f(t)dt)dx. \tag{1.2}$$

In the first part of the paper we study analogues of these results in n dimensions for maximal functions and corresponding weights. To be more precise we need some notations.

We let Q stand for a cube with axes parallell to the coordinate axes and $|Q|$ its Lebesgue measure. It is convenient to use a special sign for the mean value over Q of a function f

$$⨍_Q f(x)dx = \frac{1}{|Q|}\int_Q f(x)dx.$$

First we define, for $g \in L^q_{loc}(\mathbb{R}^n),\ q > 0$, the *q-maximal function* of g by

$$M_q g(x) = \sup_{Q\ni x}(⨍_Q |g(t)|^q dt)^{\frac{1}{q}}, \tag{1.3}$$

where the supremum extends over all cubes $Q \subset \mathbb{R}^n$. For $q = 1$ we get the familiar Hardy-Littlewood maximal function $Mg = M_1 g$.

As a limit case as q tends to zero, we introduce the *geometrical maximal function*, $M_0 g$, by defining

$$M_0 g(x) = \sup_{Q\ni x}\exp(⨍_Q \ln|g(t)|dt).$$

For non-increasing, non-negative functions f on $(0,\infty)$ it is easy to show that

$$M_{\frac{1}{p}} f(x) = (⨍_0^x f^{\frac{1}{p}}(t)dt)^p \quad \text{and} \quad M_0 f(x) = \exp ⨍_0^x \ln f(t)dt.$$

The left and right hand sides of (1.1) therefore are $\lim_{p\to\infty} M_{\frac{1}{p}} f(x)$ and $M_0 f(x)$ respectively. We prove in Theorem 2 that

$$\lim_{p\to\infty} M_{\frac{1}{p}} f(x) = M_0 f(x) \quad \text{for} \quad x \in \mathbb{R}^n.$$

The corresponding limit relation to (1.2) will be proved as a corollary to this theorem, but in a much more general situation, where the Lebesgue measure is replaced by a measure $\omega(x)dx$, with ω a weight in the A_∞-class of Muckenhoupt. In section 2.2 we study the limit case as p tends to infinity of the A_p-constant of a weight function ω.

$$A_p(\omega) = \sup_Q \fint_Q \omega(x)dx \Big(\fint_Q \omega^{-\frac{1}{p-1}}(x)dx\Big)^{p-1} \tag{1.4}$$

and define

$$A_\infty(\omega) = \sup_Q \fint_Q \omega(x)dx \exp\Big(\fint_Q \ln \frac{1}{\omega(x)} dx\Big)\ ,\ \overline{A}_\infty(\omega) = \lim_{p\to\infty} A_p(\omega).$$

Jensen's inequality implies

$$A_\infty(\omega) \le \overline{A}_\infty(\omega)$$

Johnson in [**6**] left it as an open problem whether there exists a constant c such that $\overline{A}_\infty(\omega) \le cA_\infty(\omega)$. We settle that problem by showing in Theorem 1, that the two quantities are actually equal.

In Theorem 3 we prove that the geometrical maximal function M_0 gives a bounded mapping of $L^1(\omega)$ into $L^1(\omega)$ if and only if the weight function belongs to A_∞, thereby extrapolating from A_p the classical result of Muckenhoupt [**7**] on the Hardy-Littlewood maximal function.

In the second part of the paper we restrict our concern to the case of non-increasing, non-negative functions on $(0,\infty)$. Following a recent paper of Ariño and Muckenhoupt [**1**], we continue to study the classes of weights for which the maximal operator is bounded on non-increasing functions in $L^p(\omega)$. It turns out that we have here a more or less complete analogy with the A_p-classes. Also in this case we study the limit case as p tends to infinity. Our final specialization is to the case when ω is non-decreasing.

2. The general case

2.1. Notations and definitions.

For a non-negative, locally integrable (weight)-function ω we define $L^p(\omega)$ as the class of all measurable functions f such that

$$\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx < \infty, \text{ with } \| f \|_{L^p(\omega)} = (\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx)^{\frac{1}{p}}.$$

A_p is the class of all weight functions ω with finite $A_p(\omega)$. As p tends to infinity in (1.4) the second factor on the right hand side tends to $\exp(\fint_Q \ln \frac{1}{f(x)} dx)$. See [**4**, p. 71]. It is therefore natural to define the A_∞-constant of ω as

$$A_\infty(\omega) = \sup_Q \fint_Q \omega(x)dx \cdot \exp(\fint_Q \ln \frac{1}{\omega(x)} dx). \tag{2.1}$$

Usually A_∞ is not defined as the class of functions for which the right hand side of (2.1) is finite. However, it has been proved by S. V. Hruščev [**5**], and J. Garcia-Cuerva- R. de Francia [**3**, p. 405] that this is an alternative definition of A_∞.

When studying the boundedness of the maximal operators it is convenient to have the following notations for weight functions ω

$$m_p(\omega) = \sup_{\|f\|_{L(\omega)}=1} \int_{\mathbb{R}^n} M_{\frac{1}{p}} f(x)\omega(x)dx.$$

It is easy to see that

$$\sup_{\|f\|_{L(\omega)}=1} \int_{\mathbb{R}^n} M_{\frac{1}{p}} f(x)\omega(x)dx = \sup_{\|f\|_{L^p(\omega)}=1} \int_{\mathbb{R}^n} (Mf(x))^p \omega(x)dx). \tag{2.2}$$

We therefore put

$$m_\infty(\omega) = \sup_{\|f\|_{L(\omega)}=1} \int_{\mathbb{R}^n} M_0 f(x)\omega(x)dx. \tag{2.3}$$

The non-increasing and non-decreasing rearrangements of a function f will be denoted by f^* and f_* respectively and are defined by

$$f^*(t) = \sup_{|E|=t} \operatorname*{ess\,inf}_E f(x)\ , \quad f_*(t) = \inf_{|E|=t} \operatorname*{ess\,sup}_E f(x)$$

and then become continuous to the right and left respectively.

2.2. A_∞ as a limit case of A_p.

It is well known that A_∞ can be defined as $\bigcup_{p>1} A_p$. Let ω be a function in A_∞. Then there exists a number p_1, $1 \le p_1 < \infty$ such that $\omega \in A_p$ for $p \ge p_1$. Since, by Hölder's inequality, $A_p(\omega)$ is a decreasing function of p, we have two candidates for the A_∞-constant of ω, namely

$$\overline{A}_\infty(\omega) = \lim_{p\to\infty} A_p(\omega) \quad \text{and} \quad A_\infty(\omega) \quad \text{as defined by (2.1).}$$

By Jensen's inequality

$$\exp\,(\fint_Q \ln g(x)dx) \le \fint_Q g(x)dx.$$

We apply this with $g = \omega^{-\frac{1}{p}}$, raise both sides to the power p and obtain

$$\exp\,(\fint_Q \ln \frac{1}{\omega(x)} dx) \le (\fint_Q \frac{dx}{\omega(x)^{\frac{1}{p}}})^p,$$

which means that $A_\infty(\omega) \le A_{p+1}(\omega)$ and thus $A_\infty(\omega) \le \overline{A}_\infty(\omega)$. It has been an open question, [**6**, p. 98], whether there exists a constant c such that $\overline{A}_\infty(\omega) \le cA_\infty(\omega)$. Here is the answer.

Theorem 1. *If $\omega \in A_\infty$, then $A_\infty(\omega) = \overline{A}_\infty(\omega)$.*

In the proof of the theorem we will use the following lemma.

Lemma 1. *Let f be a non-negative integrable function on $(0,1)$ and p a real number, $p \ge 1$. Put*

$$g(x) = \begin{cases} f(x), & \text{if } f(x) < e \\ e, & \text{elsewhere.} \end{cases}$$

Then

$$(2.4) \qquad (\int_0^1 f(x)dx)^p - (\int_0^1 g(x)dx)^p \le p(\int_E f(x)dx) \cdot (\int_0^1 f(x)dx)^{p-1},$$

where $E = \{x \in (0,1)\ ;\ f(x) \ge e\}$.

Proof: This is an immediate consequence of the inequality

$$b^p - a^p \leq p(b-a)b^{p-1}, \quad \text{for} \quad 0 \leq a \leq b, \quad p \geq 1,$$

with

$$b = \int_0^1 f(x)dx \quad \text{and} \quad a = \int_0^1 g(x)dx. \quad \blacksquare$$

Proof of Theorem 1: Suppose ω is a function in $A_{p_1}(\mathbb{R}^n)$ with A_{p_1}-constant A. We will show that, for every $p \geq p_1 - 1$, we have

$$\sup_Q \fint_Q \omega(x)dx \cdot \Big(\fint_Q \frac{dx}{\omega(x)^{\frac{1}{p}}}\Big)^p \leq (1 + \delta(p, p_1, A)) \cdot A_\infty(\omega), \tag{2.5}$$

where $\lim_{p\to\infty} \delta(p, p_1, A) = 0$. This implies $\overline{A}_\infty(\omega) \leq A_\infty(\omega)$ which proves the theorem.

Except for the supremum the left hand side of (2.5) is invariant under changes of scale in $\mathbb{R}^n$ and also under multiplication of ω by a positive constant. Without loss of generality we may therefore assume that that $|Q| = 1$ and $\omega(Q) = \int_Q \omega(x)dx = 1$.

We denote here by $\omega_*(t)$, $0 \leq t \leq |Q| = 1$ the non-decreasing rearrangement of the restriction of ω to Q. Then, from [**10**, p. 250] e.g., and the definition of $A_\infty(\omega)$, we conclude that

$$\omega_*(t) \geq A^{-1}t^{p_1-1} \quad \text{and} \quad \exp\Big(\int_0^1 \ln \frac{1}{\omega_*(t)}dt\Big) \leq A_\infty(\omega). \tag{2.6}$$

Since

$$e^x \leq 1 + x + x^2, \quad \text{for} \quad x \leq 1,$$

we have

$$\int_0^1 \frac{dt}{\omega_*(t)^{\frac{1}{p}}} = \int_0^1 \exp\Big(\frac{1}{p}\ln\frac{1}{\omega_*(t)}\Big)dt \leq \int_0^1 \Big(1 + \frac{1}{p}\ln\frac{1}{\omega_*(t)} + \frac{1}{p^2}\Big(\ln\frac{1}{\omega_*(t)}\Big)^2\Big)dt,$$

if $\omega_*(t) \geq e^{-p}$. This means that

$$\Big(\int_0^1 \frac{dt}{\omega_*(t)^{\frac{1}{p}}}\Big)^p \leq \exp p \ln\Big(1 + \frac{1}{p}\int_0^1 \ln\frac{1}{\omega_*(t)}dt + \frac{1}{p^2}\int_0^1 \Big(\ln\frac{1}{\omega_*(t)}\Big)^2 dt\Big).$$

The inequality: $\ln(1+x) \leq x$, for $x > -1$, implies

$$(2.7) \qquad \left(\int_0^1 \frac{dt}{\omega_*(t)^{\frac{1}{p}}}\right)^p \leq \exp\left(\int_0^1 \ln\frac{1}{\omega_*(t)}dt\right)\cdot \exp\left(\frac{1}{p}\int_0^1 \left(\ln\frac{1}{\omega_*(t)}\right)^2 dt\right).$$

We now use (2.6) to find

$$\int_0^1 \left(\ln\frac{1}{\omega_*(t)}\right)^2 dt \leq \int_0^1 \left(\ln A + (p_1-1)\ln\frac{1}{t}\right)^2 dt = c(p_1, A).$$

The second factor on the right hand side of (2.7) therefore converges to 1 as p tends to infinity. This proves the theorem for functions ω that are bounded below by a positive constant a. (We just choose p so large that $e^{-p} < a$.) If that is not the case we construct a new function

$$\omega_p(x) = \begin{cases} \omega(x), & \text{if } \omega(x) > e^{-p} \\ e^{-p}, & \text{if } \omega(x) \leq e^{-p}. \end{cases}$$

It is easy to check that (2.7) is valid with ω_* replaced by $(\omega_p)_*$. After the replacement we increase the right hand side of (2.7) and use the second inequality of (2.6) to find

$$(2.8) \quad \left(\int_0^1 \frac{dt}{(\omega_p)_*(t)^{\frac{1}{p}}}\right)^p \leq \exp\left(\int_0^1 \ln\frac{1}{\omega_*(t)}dt\right)\cdot \exp\left(\frac{1}{p}\int_0^1 \left(\ln\frac{1}{\omega_*(t)}\right)^2 dt\right) \leq$$
$$\leq A_\infty(\omega)\cdot \exp\left(\frac{c(p_1,A)}{p}\right).$$

We now take a closer look at the left hand side of this inequality. We want to replace $(\omega_p)_*$ by ω_* and estimate the difference in a way that is independent of our particular choice of cube Q. For this we will use Lemma 1, applied with $f(x) = \omega_*(t)^{-\frac{1}{p}}$. From (2.6) we conclude that

$$\omega_*(t)^{-\frac{1}{p}} > e \implies A^{-1}t^{p_1-1} < e^{-p} \implies t < A^{\frac{1}{p_1-1}} e^{-\frac{p}{p_1-1}} = t_p.$$

Hence $E \subset (0, A^{\frac{1}{p_1-1}} e^{-\frac{p}{p_1-1}})$ in the lemma and we have the estimate

$$\int_E \frac{dt}{\omega_*(t)^{\frac{1}{p}}} \leq A^{\frac{1}{p}} \int_0^{t_p} \frac{dt}{t^{\frac{p_1-1}{p}}} = \frac{A^{\frac{1}{p}} t_p^{1-\frac{p_1-1}{p}}}{(1-\frac{p_1-1}{p})} = \frac{p}{p-p_1+1} A^{\frac{1}{p_1-1}} e^{-\frac{p}{p_1-1}+1} =$$
$$= c(p, p_1, A).$$

This estimate is used in (2.8) and, combined with Lemma 1, the conclusion is

$$(\int_0^1 \frac{dt}{\omega_*(t)^{\frac{1}{p}}})^p \le A_\infty(\omega) \cdot \exp\frac{c(p_1,A)}{p} + d(p,p_1,A)(\int_0^1 \frac{dt}{\omega_*(t)^{\frac{1}{p}}})^{p-1},$$

where $d(p,p_1,A) = p \cdot c(p,p_1,A)$. This quantity obviously tends to zero as p tends to infinity.

We now take an arbitrary ϵ, $0 < \epsilon < 1$, and choose p so large that $\exp\frac{c(p_1,A)}{p} \le (1+\epsilon)$ and $d(p,p_1,A) \le \epsilon$. Put $v = (\int_0^1 \frac{dt}{\omega_*(t)^{\frac{1}{p}}})^p$. Then $v \ge 1$ by Hölder's inequality and

$$v \le (1+\varepsilon)A_\infty(\omega) + \varepsilon v^{\frac{p-1}{p}} \le (1+\varepsilon)A_\infty(\omega) + \varepsilon v,$$

i.e.

$$v \le \frac{1+\varepsilon}{1-\varepsilon}A_\infty(\omega) < (1+3\varepsilon)A_\infty(\omega).$$

Now we take supremum over all cubes Q and get

$$A_{p+1}(\omega) = (1+\delta(p,p_1,A)) \cdot A_\infty(\omega) \le (1+3\varepsilon)A_\infty(\omega),$$

where $\delta(p,p_1,A)$ tends to zero as p tends to infinity. This means that

$$\lim_{p\to\infty} A_p(\omega) \le A_\infty(\omega),$$

which concludes the proof of the theorem. ■

It is also possible to have an estimate of the rate of convergence. A simple analysis of the various inequalities will give us the estimate

$$\delta(p,p_1,A) \le \frac{C(p_1,A)}{p},$$

where $C(p_1,A)$ is a constant depending only on p_1 and A.

2.3. M_0 as a limit case of $M_{\frac{1}{p}}$.

Corresponding to the preceding paragraph, we present here a result on the geometrical maximal function, $M_0 f$, the precise importance of which is demonstrated in Theorem 3 at the end of this paragraph.

Theorem 2. *Suppose that f lies in $L^{\alpha}_{loc}(\mathbb{R}^n)$, for some $\alpha > 0$. Then we have*

$$\lim_{p\to\infty} M_{\frac{1}{p}} f(x) = M_0 f(x), \quad \forall x.$$

Proof: By Jensen's inequality

$$\exp \fint_Q \ln f(x)dx \leq (\fint_Q f^{\frac{1}{p}}(x)dx)^p.$$

We take the supremum over all Q that contain x and obtain

$$M_0 f(x) \leq M_{\frac{1}{p}} f(x), \quad \forall x,$$

and letting p tend to infinity this gives

$$M_0 f(x) \leq \lim_{p\to\infty} M_{\frac{1}{p}} f(x), \quad \forall x. \tag{2.9}$$

It remains to prove the opposite inequality of (2.9). We assume first that the number α in the theorem equals one. Then we use Lemma 2 below, according to which we have, for every $\epsilon \in (0,1)$ and cube Q:
(2.10)

$$(\fint_Q f^{\frac{1}{p}}(x)dx)^p \leq \exp \fint_Q \ln f_\epsilon(x)dx \cdot \exp \frac{(\ln \epsilon)^2 + 1}{p} + \frac{p}{e^p - 1} \fint_Q f(x)dx,$$

where

$$f_\epsilon(x) = \begin{cases} f(x), & \text{if } f \geq \epsilon \fint_Q f(x)dx \\ \fint_Q f(x)dx & \text{elsewhere.} \end{cases}$$

From this we conclude

$$\sup_{Q\ni x}(\fint_Q f^{\frac{1}{p}}(t)dt)^p \leq \sup_{Q\ni x}(\exp \fint_Q \ln f_\epsilon(t)dt \cdot \exp \frac{(\ln \epsilon)^2 + 1}{p} + \frac{p}{e^p - 1}\fint_Q f(t)dt).$$

f_ϵ is independent of p. Letting p tend to infinity therefore gives us

$$\lim_{p\to\infty} M_{\frac{1}{p}} f(x) \leq \sup_{Q\ni x} \exp(\fint_Q \ln f_\epsilon(t)dt).$$

Now we let ϵ tend to zero. By monotone convergence

$$\lim_{p\to\infty} M_{\frac{1}{p}} f(x) \leq M_0 f(x), \quad \forall x. \tag{2.11}$$

This concludes the proof if $\alpha = 1$. For $\alpha \neq 1$ we put $g = f^\alpha$ and use (2.11) on g. This gives

$$\lim_{p\to\infty} \sup_{Q\ni x} (\fint_Q f^{\frac{\alpha}{p}}(t)dt)^p \leq \sup_{Q\ni x} \exp \fint_Q \ln f^\alpha(t)dt$$

or, with $q = p\alpha^{-1}$,

$$\lim_{q\to\infty} M_{\frac{1}{q}} f(x) \leq M_0 f(x), \quad \forall x.$$

This is (2.11), which thus is valid for all $\alpha > 0$. Combined with (2.9) this gives the desired equality.

What remains of the proof therefore is the main step, namely to prove the lemma. ■

Lemma 2. *Suppose that f is a locally integrable function, defined on $\mathbb{R}^n$. Then* (2.10) *is valid for every $\epsilon \in (0,1)$ and every cube Q in $\mathbb{R}^n$.*

Proof: The homogenity of (2.10) allows us to assume that $|Q| = 1$ and $\fint_Q f(x)dx = 1$. We may, by turning to the non-increasing rearrangement of the restriction of f to Q, even assume that we are dealing with a non-increasing function on $(0,1)$. This means that it is sufficient to prove that if $\epsilon \in (0,1)$ and $\int_0^1 f(x)dx = 1$ then

$$(\int_0^1 f^{\frac{1}{p}}(x)dx)^p \leq \exp \int_0^1 \ln f_\epsilon(x)dx \cdot \exp \frac{(\ln\epsilon)^2+1}{p} + \frac{p}{e^p - 1}, \tag{2.10'}$$

where

$$f_\epsilon(x) = \begin{cases} f(x), & \text{if } f \geq \epsilon \\ 1 & \text{elsewhere.} \end{cases}$$

Put

$$E_\epsilon = \{x \in (0,1); f(x) \geq \epsilon\} \quad \text{and} \quad |E_\epsilon| = 1 - l(\epsilon).$$

We first assume that $0 \leq f(x) \leq e^p$ on $(0,1)$. Since $f^{\frac{1}{p}}(x) = \exp\frac{\ln f(x)}{p}$ we can use the inequality $e^x \leq 1 + x + x^2$, for $x \leq 1$, to find

$$(\int_0^1 f_\epsilon^{\frac{1}{p}}(x)dx)^p \leq (\epsilon^{\frac{1}{p}} l(\epsilon) + \int_{E_\epsilon} f^{\frac{1}{p}}(x)dx)^p \leq ((\epsilon^{\frac{1}{p}} - 1)l(\epsilon) + 1 +$$

$$+ \int_{E_\epsilon} \frac{\ln f}{p} dx + \int_{E_\epsilon} \frac{(\ln f)^2}{p^2} dx)^p.$$

By assumption $\epsilon^{\frac{1}{p}} - 1 < 0$ and by definition $f_\epsilon \geq f$. Thus

$$(\int_0^1 f^{\frac{1}{p}}(x)dx)^p \leq (1 + \int_{E_\epsilon} \frac{\ln f}{p} dx + \int_{E_\epsilon} \frac{(\ln f)^2}{p^2} dx)^p =$$

$$= (1 + \int_0^1 \frac{\ln f_\epsilon}{p} dx + \int_0^1 \frac{(\ln f_\epsilon)^2}{p^2} dx)^p.$$

What is inside the last parenthesis obviously is positive and we can use the inequality: $\ln(1+x) \leq x$ for $x > -1$, to obtain

$$(2.12) \qquad (\int_0^1 f^{\frac{1}{p}}(x)dx)^p \leq \exp \int_0^1 \ln f_\epsilon(x)dx \cdot \exp \frac{1}{p} \int_0^1 (\ln f_\epsilon(x))^2 dx.$$

It is easy to see that $(\ln t)^2 < t$ if $t > 1$. Therefore

$$\int_0^1 (\ln f_\epsilon)^2 dx \leq (\ln \epsilon)^2 |E_\epsilon| + \int_0^1 f(x)dx \leq (\ln \epsilon)^2 + 1.$$

When we plug that into formula (2.12) we get something which is a little stronger than (2.10′). However, we have to get rid of our extra assumption that $f < e^p$ on $(0,1)$. We consider the truncated function

$$g_p(x) = \begin{cases} f(x), & \text{if } f(x) < e^p \\ e^p, & \text{elsewhere.} \end{cases}$$

We can apply exactly the same arguments as before to the function g_p and obtain

$$(2.13) \qquad (\int_0^1 g_p^{\frac{1}{p}}(x)dx)^p \leq (\exp(\int_{E_\epsilon} \ln g_p dx)) \cdot \exp \frac{(\ln \epsilon)^2 + 1}{p}.$$

Since $g_p(x) \leq f(x)$, we can replace g_p by f on the right hand side. To estimate the left hand side we use Lemma 1 with $f(x)$ replaced by $f^{\frac{1}{p}}(x)$. Then $g(x)$ of the lemma will be $g_p^{\frac{1}{p}}(x)$ and the result

$$(\int_0^1 f^{\frac{1}{p}}(x)dx)^p \leq (\int_0^1 g_p^{\frac{1}{p}}(x)dx)^p + p \int_0^{x_p} f^{\frac{1}{p}}(x)dx \cdot (\int_0^1 f^{\frac{1}{p}}(x)dx)^{p-1},$$

where $x_p = \sup\{x \in (0,1); f(x) > e^p\}$. For $x \in (0, x_p)$ we have

$$f^{\frac{1}{p}}(x) = \frac{f(x)}{f(x)^{1-\frac{1}{p}}} \leq \frac{f(x)}{e^{p-1}},$$

which, when integrated, gives

$$\int_0^{x_p} f^{\frac{1}{p}}(x)dx \leq \frac{1}{e^{p-1}}.$$

Also, by Hölder's inequality,

$$\int_0^1 f^{\frac{1}{p}}(x)dx \leq (\int_0^1 f(x)dx)^{\frac{1}{p}} = 1.$$

Therefore

$$(\int_0^1 f^{\frac{1}{p}}(x)dx)^p \leq (\int_0^1 g_p^{\frac{1}{p}}(x)dx)^p + \frac{p}{e^{p-1}},$$

which, combined with (2.13) gives

$$(\int_0^1 f^{\frac{1}{p}}(x)dx)^p \leq (\exp(\int_0^1 \ln f_\epsilon(x)dx)) \cdot \exp\frac{(\ln\epsilon)^2+1}{p} + \frac{p}{e^{p-1}}.$$

So we have proved (2.10′) and the proof is complete. ■

Corollary.

$$\lim_{p\to\infty} m_p(\omega) = m_\infty(\omega). \tag{2.14}$$

Proof: Choose an arbitrary $\epsilon > 0$. As an immediate consequence of Hölder's inequality and the monotone convergence theorem there exists, for every f, a number p_0, such that

$$|\int_{\mathbb{R}^n} M_{\frac{1}{p_0}} f(x)\omega(x)dx - \int_{\mathbb{R}^n} M_0 f(x)\omega(x)dx| < \epsilon.$$

In particular we can take an f with $\| f \|_{L(\omega)} = 1$ such that the second integral differs from $m_\infty(\omega)$ with at most ϵ. Since $M_{\frac{1}{p}} f \geq M_0 f$ we obviously have

$$m_\infty(\omega) \leq m_{p_0}(\omega) \leq m_\infty(\omega) + 2\epsilon.$$

However, $\epsilon > 0$ is arbitrary and we obtain (2.14). ■

Muckenhoupt [**7**, p. 222] has shown that the maximal operator M gives a bounded mapping from $L^p(\omega)$ to $L^p(\omega)$ if and only if $\omega \in A_p$. In other words :

$$\omega \in A_p \Longleftrightarrow \sup_{\|f\|_{L^p(\omega)}=1} \int_{\mathbb{R}^n} (Mf(x))^p \omega(x)dx) < \infty.$$

Put here $g = f^p$ and take into account that $M_{\frac{1}{p}} g = (Mg^{\frac{1}{p}})^p$. Then, using our terminology (1.4) and (2.2), Muckenhoupt's result can be rephrased as

Theorem M. *A weight function ω is in A_p if and only if*

$$m_p(\omega) = \sup_{\|f\|_{L(\omega)}=1} \int_{\mathbb{R}^n} M_{\frac{1}{p}} f(x)\omega(x)dx < \infty \tag{2.15}$$

and we have

$$A_p(\omega) \le m_p(\omega) \le g(A_p(\omega), p, n).$$

In the theorem below we will show that the limit case, $p = \infty$, ($M_{\frac{1}{p}}$ replaced by M_0), of this theorem is true. Furthermore, we will give an estimate of $m_\infty(\omega)$ in terms of the A_∞-constant of ω.

Theorem 3. *A weight function ω is in A_∞ if and only if*

$$m_\infty(\omega) = \sup_{\|f\|_{L(\omega)}=1} \int_{\mathbb{R}^n} M_0 f(x)\omega(x)dx < \infty. \tag{2.16}$$

and we have

$$A_\infty(\omega) \le m_\infty(\omega) \le C_1(n)(A_\infty(\omega))^{150n},$$

where $C_1(n)$ is a constant, depending only on n.

Proof: For the sufficiency we just note that, by the corollary above, $m_\infty(\omega) < \infty$ implies $m_p(\omega) < \infty$ for p large enough and by Theorem M it follows that $\omega \in A_p$ for p large enough and

$$A_p(\omega) \le m_p(\omega).$$

The sufficiency part and the first inequality of the theorem now follow from (2.14) and Theorem 1 by letting p tend to infinity in this formula.

For the necessity part we assume that ω is in A_∞ with $A_\infty(\omega) = A$. We use the result by Hruščev [**5**, p. 255], according to which, for a subset E of any cube Q, we have

$$\frac{|E|}{|Q|} \ge \frac{1}{2} \Longrightarrow \frac{\omega(E)}{\omega(Q)} \ge \frac{1}{1+4A^2} \Longrightarrow \frac{\omega(E)}{\omega(Q)} \ge \frac{1}{5A^2}.$$

Now we can use the estimate in theorem 3 of [**10**, p. 252] to deduce that for $\beta > (n+2)\log_2(5A^2) = \beta_0$ we have, for any $E \subset Q$

$$\frac{\omega(E)}{\omega(Q)} \ge \frac{1}{5A^2}(\frac{|E|}{|Q|})^\beta.$$

According to corollary 1, p. 250 of the same paper this implies that ω is in A_p for $p > \beta_0$ and with

$$A_p(\omega) \le (5A^2)(\frac{p-1}{p-\beta_0})^{p-1} \le 5A^2 e^{2\beta_0} \le (5A^2)^{3n+7} = B \tag{2.17}$$

for $p > 3\beta_0$.

Buckley, [**2**, p. 9], has shown that the maximal operator is of weak type (p,p) on $L^p(\omega)$ with weak-norm $(C(n)A_p(\omega))^{\frac{1}{p}}$. We use this result and Marcinkiewicz interpolation theorem (see Torchinsky [**9**, p. 87]) to interpolate in the interval $(p_0 =)3\beta_0 < 2p_0 < \infty$ and find that

$$m_{2p_0} \le (8e^{\frac{1}{e}})^{2p_0} C(n)^2 B p_0^{-1}$$

Taking into account $p_0 = 3\beta_0$ and the definition (2.17) of B, this implies

$$m_p(\omega) \le C_1(n) A^{50n+100} \quad \text{for} \quad p \ge 3\beta_0.$$

Hence

$$m_\infty(\omega) \le C_1(n)(A_\infty(\omega))^{150n}. \quad \blacksquare$$

3. The case of non-increasing functions on $(0,\infty)$.

3.1. Notations and definitions.

For non-negative, non-increasing functions on $(0,\infty)$ the maximal functions $M_q f$ and $M_0 f$ satisfy

$$M_q f(x) = \Big(\fint_0^x f^q(t)dt\Big)^{\frac{1}{q}} \qquad \text{and} \qquad M_0 f(x) = \exp \fint_0^x \ln f(t)dt.$$

Ariño and Muckenhoupt [**1**, p. 727–734] have shown that in this case and for $1 \le p < \infty$ a necessary and sufficient condition on ω to secure that there exists a constant C, such that

$$\int_0^\infty (Mf(x))^p \omega(x)dx \le C \int_0^\infty f^p(x)\omega(x)dx \tag{3.1}$$

is valid, for all non-negative, non-increasing functions in $L^p(\omega)$ on $(0,\infty)$, is the existence of a constant B, such that

$$\int_x^\infty \frac{\omega(t)}{t^p}dt \le \frac{B}{x^p}\int_0^x \omega(t)dt\ , \quad \forall x > 0. \tag{3.2}$$

They also proved that a sufficient condition on ω is

$$\sup_{x>0}\Big[\fint_0^x \omega(t)dt\Big]\ \Big[\fint_0^x (\omega(t))^{-\frac{1}{p-1}}dt\Big]^{p-1} = A'_p(\omega) < \infty, \tag{3.3}$$

and that this condition is also necessary if the weight function ω is non decreasing.

We will denote by $B_p, 0 < p < \infty$ and $A'_p, 1 \le p < \infty$ the class of all functions ω satisfying (3.2) and (3.3) respectively. (For $p = 1$ the second factor to the left in (3.3) should be interpreted as $\operatorname{ess\,sup}_{0<t<x} \frac{1}{\omega(t)}$.) We also say that ω lies in B_p with constant $B_p(\omega)$ if $B_p(\omega)$ is the minimal constant for which (3.2) is valid. Let p tend to infinity in (3.3). This natural way leads us to the definition of A'_∞ as those non-negative, measurable functions ω that satisfy

$$\sup_{x>0}\Big[\fint_0^x \omega(t)dt\Big]\ \Big[\exp \fint_0^x \ln \frac{1}{(\omega(t)}dt\Big] = A'_\infty(\omega) < \infty.$$

In analogy with the n-dimensional case we define, for $p > 0$

$$m'_p(\omega) = \sup \int_0^\infty M_{\frac{1}{p}} f(x)\omega(x)dx,$$

but now the supremum is taken over all non-increasing f on $(0, \infty)$ with $\| f \|_{L(\omega)} = 1$. We note that $m'_p(\omega)$ is the infimum of all C such that (3.1) holds. Correspondingly we define

$$m'_\infty(\omega) = \sup_f \int_0^\infty M_0 f(x)\omega(x)dx,$$

where the supremum is taken over the same class.

3.2. The analogy between A_p and B_p.

In Lemma (2.1) of [**1**] there is a proof, of the fact that $\omega \in B_p$ implies that $\omega \in B_{p-\epsilon}$ for some $\epsilon > 0$ (a similar result is in Strömberg-Torchinsky [**8**, p. 12]). We give here a short and sharp proof of that lemma.

Lemma 3. *Suppose that* $0 < p < \infty$ *and* ω *is a function in* B_p *such that*

$$\int_x^\infty \frac{\omega(t)}{t^p} dt \le \frac{B}{x^p} \int_0^x \omega(t)dt, \qquad \forall x > 0. \tag{3.4}$$

Then $\omega \in B_{p-\epsilon}$ *for* $\epsilon < \frac{p}{B+1}$ *i.e.* $\omega \in B_{p_1}$ *for* $p_1 < \frac{B}{B+1}p$ *and* $B_p(\omega) \le \frac{Bp}{p-\epsilon(B+1)}$. *The upper bound of* ϵ *is best possible.*

Proof: Choose $\epsilon < \frac{p}{B+1}$, multiply (3.4) by $x^{\epsilon-1}$ and integrate from r to infinity. A change of the order of integration on both sides then results in

$$\frac{1}{\epsilon} \int_r^\infty \frac{\omega(t)}{t^p}(t^\epsilon - r^\epsilon)dt \le \frac{B}{p-\epsilon}\left(\int_0^r \frac{\omega(t)}{r^{p-\epsilon}}dt + \int_r^\infty \frac{\omega(t)}{t^{p-\epsilon}}dt\right),$$

which gives us, after once more using (3.4)

$$\left(\frac{1}{\epsilon} - \frac{B}{p-\epsilon}\right) \int_r^\infty \frac{\omega(t)}{t^{p-\epsilon}}dt \le \frac{r^\epsilon}{\epsilon} \int_r^\infty \frac{\omega(t)}{t^p}dt + \frac{B}{(p-\epsilon)r^{p-\epsilon}} \int_0^r \omega(t)dt \le$$

$$\le \left(\frac{1}{\epsilon} + \frac{1}{p-\epsilon}\right)\frac{B}{r^{p-\epsilon}} \int_0^r \omega(t)dt.$$

This is to say that $\omega \in B_{p-\epsilon}$ for $\epsilon < \frac{p}{B+1}$ and $B_p(\omega) \leq \frac{Bp}{p-\epsilon(B+1)}$.

To show that the limit is best possible we just take $\omega(x) = x^\alpha$, $\alpha > -1$, and $p > \alpha + 1$. Then the B_p-constant of ω is $\frac{\alpha+1}{p-\alpha-1}$. By the result above we see that $\omega \in B_{p_1}$ for

$$p_1 > p - \frac{p}{\frac{\alpha+1}{p-\alpha-1} + 1} = \alpha + 1.$$

Of course no smaller p's are possible, if the left side of (3.6) is to converge. ■

We will extend the results of [**1**] to the geometrical maximal function $M_0 f$ (and also in some cases to $0 < p \leq 1$. To make apparent the parallellity with the ordinary A_p-classes, we introduce a class B_∞. It will soon become evident that the corresponding to the definition of A_∞ would be to define B_∞ as the class of weight functions, for which there exist two constants $r < 1$ and $k > 0$ such that

$$1 > \frac{t}{x} \geq r \Longrightarrow \frac{\int_0^t \omega(u)du}{\int_0^x \omega(u)du} \geq k.$$

This is equivalent to the following definition, which is more easy to grasp.

Definition. B_∞ is the class of non-negative, locally integrable functions ω on $(0, \infty)$ with the property that there exist two constants $r, 0 < r < 1$ and $C > 0$ such that

$$C \int_0^{rx} \omega(t)dt \geq \int_0^x \omega(t)dt, \quad \forall x > 0. \tag{3.5}$$

Remark. We could equally well have made the definition with $r = \frac{1}{2}$ instead of being arbitrary. This would seemingly be more restrictive for $r > \frac{1}{2}$. However if ω satisfies our definition with an $r > \frac{1}{2}$ we can iterate the inequality approximately $(-\log_2 r)^{-1}$ of times to see that it is satisfied for $r = \frac{1}{2}$, but with a larger C.

Definition. The doubling constant, $d(\omega)$ is the minimum of all C such that (3.5) is valid with $r = \frac{1}{2}$. If $d(\omega)$ is finite we will say that ω has the doubling property.

It is immediately evident from the definition that $B_{p_1} \subset B_p$ and also that $B_p(\omega) \leq B_{p_1}(\omega)$ if $p_1 < p$.

A function in B_p obviously has the doubling property. (Just relax in the definition (3.2) by reducing the interval of integration on the left in (3.2) to become $(x, 2x)$. However, we can do much better and obtain an estimate of C in (3.5), an estimate that depends on r and also can be used as an alternative characterization of B_p. (Compare corollary 1, p. 250 of [**10**].)

Theorem 4. *A weight function ω is in B_p, if and only if there exist constants p_1, $0 < p_1 < p$, and C such that*

$$(3.6) \qquad \int_0^t \omega(u)du \geq C\left(\frac{t}{x}\right)^{p_1} \int_o^x \omega(u)du\ , \qquad \textit{for} \quad x \geq t.$$

If $C_{p_1}(\omega)$ is the maximal C for which (3.6) holds, then

$$C_{p_1}(\omega) \geq \frac{1}{2B_p(\omega)+1} \textit{ for } p_1 > \frac{2B+1}{2B+2}p \textit{ and } B_p(\omega) \leq \frac{p}{C_{p_1}(\omega)(p-p_1)}.$$

Proof: Suppose first that $\omega \in B_p$ and put $B_p(\omega) = B$. By the preceding lemma we know that, for $p_1 = \frac{2B+1}{2B+2}p < p$, $\omega \in B_{p_1}$ with constant $2B$. Thus

$$2B \int_0^x \omega(u)du \geq x^{p_1} \int_x^\infty \frac{\omega(u)}{u^{p_1}}du \geq \sum_{k=0}^{\infty} 2^{-(k+1)p_1} \int_{x2^k}^{x2^{k+1}} \omega(u)du \geq$$

$$\geq \sum_{k=0}^{N-1} 2^{-(k+1)p_1} \int_{x2^k}^{x2^{k+1}} \omega(u)du.$$

This gives, for every $x > 0$,

$$(1-2^{-p_1}) \sum_{k=1}^{N-1} 2^{-kp_1} \int_0^{x2^k} \omega(u)du + 2^{-Np_1} \int_0^{x2^N} \omega(u)du \leq$$

$$\leq (2B + 2^{-p_1}) \int_0^x \omega(u)du.$$

Therefore, taking only the last term on the left into account and replacing x by $x2^{-N}$, we find

$$\int_0^{x2^{-N}} \omega(u)du \geq \frac{1}{2^{Np_1}(2B+1)} \int_0^x \omega(u)du.$$

For $x2^{-(N+1)} \leq t \leq x2^{-N}$ we have

$$\int_0^t \omega(u)du \geq \int_0^{x2^{-(N+1)}} \omega(u)du \geq \frac{1}{2^{Np_1}(2B+1)} \int_0^x \omega(u)du \geq$$

$$\geq \frac{1}{2B+1}(\frac{t}{x})^{p_1} \int_0^x \omega(u)du.$$

Thereby we have proved the necessity of the condition and the first inequality between the constants.

To prove the suffiency we assume $p_1 < p$ and

$$\int_0^t \omega(u)du \geq C(\frac{t}{x})^{p_1} \int_0^x \omega(u)du \ , \quad \text{for} \quad 0 \leq t \leq x.$$

Multiply this inequality by $t^{-p_1}x^{p_1-1-p}$. We get

$$\frac{1}{Ct^{p_1} \cdot x^{1+p-p_1}} \int_0^t \omega(u)du \geq \frac{1}{x^{p+1}} \int_0^x \omega(u)du.$$

This inequality is valid for $0 \leq t \leq x$. We integrate with respect to x over the interval (t, ∞) and change the order of integration in the right member. The result is

$$\frac{1}{C \cdot (p-p_1) \cdot t^p} \int_0^t \omega(u)du \geq \frac{1}{p} \int_0^t \frac{\omega(u)}{t^p} du + \frac{1}{p} \int_t^\infty \frac{\omega(u)}{u^p} du.$$

Hence

$$\int_t^\infty \frac{\omega(u)}{u^p} du \leq \frac{p}{C \cdot (p-p_1)} \cdot \frac{1}{t^p} \int_0^t \omega(u)du.$$

This completes the proof of the necessity and the second inequality between the constants. ■

We complete the analogy by

Theorem 5.

$$B_\infty = \bigcup_{p>0} B_p.$$

Proof: Suppose $\omega \in B_p$ for some $p > 0$. It is immediate from Theorem 4 that ω satisfies the requirements for being in B_∞. Thus

$$B_\infty \supset \bigcup_{p>0} B_p.$$

Suppose on the other hand that $\omega \in B_\infty$ with $d(\omega) = C$, i.e.

$$\int_0^{2x} \omega(t)dt \leq C \int_0^x \omega(t)dt.$$

This means that

$$\int_x^{2x} \omega(t)dt \leq (C-1) \int_0^x \omega(t)dt.$$

Thus

$$\int_x^\infty \frac{\omega(t)}{t^p} dt = \sum_{k=0}^\infty \int_{2^k x}^{2^{k+1}x} \frac{\omega(t)}{t^p} dt \leq \sum_{k=0}^\infty \frac{1}{2^{kp}x^p} \int_{2^k x}^{2^{k+1}x} \omega(t)dt \leq (C-1)$$

$$\sum_{k=0}^\infty \frac{1}{2^{kp}x^p} \int_0^{2^k x} \omega(t)dt \leq (C-1) \sum_{k=0}^\infty \frac{C^k}{2^{kp}x^p} \int_0^x \omega(t)dt = \frac{2^p(C-1)}{(2^p - C)x^p} \int_0^x \omega(t)dt$$

for $p > \log_2 C$. So $\omega \in B_p$ for $p > \log_2 C$ and

$$B_\infty \subset \bigcup_{p>0} B_p.$$

and the proof is complete. ■

3.3. m'_∞ as limit case of m'_p.

In this section we will for convenience use a special notation, $L_d(\omega)$, for the set of all non-negative, non-increasing functions in $L(\omega)$.

Theorem 6. *M_0 is a bounded operator on $L_d(\omega)$, (i.e. $m'_\infty < \infty$), if and only if $\omega \in B_\infty$ and*

$$d(\omega) \leq (2m'_\infty(\omega) - 1)^2 \leq C_0(d(\omega))^{9.4},$$

where C_0 is an absolute constant.

Proof of part 1: We give first a short proof of the first part of the theorem without estimates of the constants. Suppose therefore that M_0 is a bounded operator on $L_d(\omega)$. Since $M_{\frac{1}{p}}f(x)$ tends monotonically to $M_0f(x)$ as p tends to infinity, it is an immediate consequence of the monotone convergence theorem that

$$\lim_{p\to\infty} m_p'(\omega) = m_\infty'(\omega) < \infty.$$

Thus $m_p'(\omega) < \infty$ for p large enough. By [**1**] this implies that $\omega \in B_p$ for p large enough and then, by Theorem 5, $\omega \in B_\infty$.

If on the other hand $\omega \in B_\infty$, then, by Theorem 5 again, $\omega \in B_p$ for p large enough and the result in [**1**] implies $m_p'(\omega) < \infty$. Hence $m_\infty(\omega) < \infty$, which means that M_0 is bounded on $L_d(\omega)$. ■

We will now present a complete proof of Theorem 6 that does not rely on the results of Arino and Muckenhoupt, but is based on another technique. It has the advantage that it gives estimates of $m_\infty'(\omega)$ in terms of $d(\omega)$. To complete the proof we need the following lemma.

Lemma 4. *Suppose $\sum_{-\infty}^{\infty} a_k$ is a positive series with sum A. Form a new series with the convoluted terms*

$$b_k = \sum_{m=-\infty}^{\infty} \frac{a_m}{2^{\epsilon|k-m|}}.$$

Then

$$b_k \ge a_k, \qquad 2^{-\epsilon} \le \frac{b_{k+1}}{b_k} \le 2^{\epsilon} \quad \textit{and} \quad \sum_{k=-\infty}^{\infty} b_k \le \frac{2^\epsilon+1}{2^\epsilon-1}A.$$

Proof:

$$b_k = \cdots + a_{k-2}2^{-2\epsilon} + a_{k-1}2^{-\epsilon} + a_k + a_{k+1}2^{-2\epsilon} + a_{k+2}2^{-2\epsilon}\cdots$$

Now the two first properties are trivial and the third follows from a change of order of summation. ■

Proof of Theorem 6: Suppose first that $m_\infty'(\omega) = K < \infty$. Then

$$\int_0^\infty M_0f(x)\omega(x)dx \le K\int_0^\infty f(x)\omega(x)dx, \ \forall f \in L_d(\omega). \tag{3.7}$$

Choose a in $0 < a < 1$ and put

$$f(x) = \begin{cases} 1, & 0 < x \le r \\ a, & r < x \le 2r \\ 0, & x > 2r. \end{cases}$$

Then $f \in L_d(\omega)$ and

$$M_0 f(x) = \begin{cases} 1, & 0 < x \le r \\ a^{1-\frac{r}{x}}, & r < x \le 2r \\ 0, & x > 2r. \end{cases}$$

We apply formula (3.7) and obtain

$$\int_0^r \omega(x)dx + \int_r^{2r} a^{1-\frac{r}{x}}\omega(x)dx \le K(\int_0^r \omega(x)dx + \int_r^{2r} a\omega(x)dx).$$

Thus

$$a\int_r^{2r} (a^{-\frac{r}{x}} - K)\omega(x)dx \le (K-1)\int_0^r \omega(x)dx.$$

We choose $a = (2K)^{-2}$. Since $\frac{r}{x} \ge \frac{1}{2}$ we obtain

$$\int_r^{2r} \omega(x)dx \le 4K(K-1)\int_0^r \omega(x)dx,$$

which means that $\omega \in B_\infty$ with doubling constant at most $(2K-1)^2$. It also follows from this inequality that K has to be strictly greater than 1, otherwise ω has to be identically zero

Suppose on the other hand that $\omega \in B_\infty$ with doubling constant C. Choose the sequence $\{\alpha_k\}_{-\infty}^{\infty}$ such that

$$\int_0^{\alpha_k} \omega(x)dx = C^{-k}.$$

Using the doubling property we see

$$C^{-k} = \int_0^{\alpha_k} \omega(x)dx = C\int_0^{\alpha_{k+1}} \omega(x)dx \ge \int_0^{2\alpha_{k+1}} \omega(x)dx.$$

Therefore

$$\alpha_k \geq 2\alpha_{k+1}. \tag{3.8}$$

Take an arbitrary $f \in L_d(\omega)$ and put

$$\int_0^\infty f(x)\omega(x)dx = K.$$

Since f is non-increasing this means that

$$K \geq \sum_{k=-\infty}^{\infty} f(\alpha_k) \int_{\alpha_{k+1}}^{\alpha_k} \omega(x)dx = \frac{C-1}{C} \sum_{k=-\infty}^{\infty} f(\alpha_k)C^{-k}.$$

Now we can use Lemma 4 with $a_k = f(\alpha_k)C^{-k}$ and obtain $b_k \geq a_k$ with

$$\sum_{-\infty}^{\infty} b_k \leq \frac{KC}{C-1}\frac{2^\epsilon+1}{2^\epsilon-1}.$$

We can define a new non-increasing function g with $g(x) \geq f(x)$ and $g(\alpha_k) = C^k b_k$. Obviously $M_0 g \geq M_0 f$. Jensen's inequality gives

$$M_0 g(\alpha_k) = \exp \fint_0^{\alpha_k} \ln g(x)dx \leq (\fint_0^{\alpha_k} g^{\frac{1}{p}}(x)dx)^p \leq$$

$$\leq (\frac{1}{\alpha_k} \sum_{m=k}^{\infty} g^{\frac{1}{p}}(\alpha_{m+1})(\alpha_m - \alpha_{m+1}))^p \leq (\frac{1}{\alpha_k} \sum_{m=k}^{\infty} C^{\frac{m+1}{p}} b_{m+1}^{\frac{1}{p}} \alpha_m)^p.$$

By (3.8), the terms in the last series of this estimate decrease geometrically with a quotient that is at most $C^{\frac{1}{p}} 2^{\frac{\epsilon}{p}} 2^{-1}$. Thus

$$M_0 f(\alpha_k) \leq C^{k+1} b_{k+1} (\frac{1}{1 - C^{\frac{1}{p}} 2^{\frac{\epsilon}{p}} 2^{-1}})^p,$$

if p is large enough. We are still free to choose ϵ and p. We can for example choose $\epsilon = \frac{1}{2}$ and $p = 3 \ln C$ if $C \geq e^8$. If $C < e^8$ we take $p = 10$. Some elementary calculations then show that

$$M_0 f(\alpha_k) \leq DC^{k+1} b_{k+1} C^{3.7},$$

where D is an absolute constant. Therefore

$$\int_0^\infty M_0 f(x)\omega(x)dx \le D\frac{C-1}{C}\sum_{k=-\infty}^{\infty} M_0 f(\alpha_{k+1})C^{-k}C^{k+1}b_{k+1}C^{3.7} =$$

$$= DC^{3.7}(C-1)\sum_{-\infty}^{\infty} b_k \le EC^{4.7}K,$$

where E is an absolute constant. We deduce

$$m'_\infty(\omega) \le EC^{4.7}$$

and the theorem is proved. ■

Now that we have the tools, it is tempting to prove theorem (1.7) in [**1**], for $0 < p < \infty$. We will use Theorem 4 and the technique of Theorem 6.

Theorem 7. *For $0 < p \le \infty$, $M_{\frac{1}{p}}$ is a bounded operator on $L_d(\omega)$ if and only if $\omega \in B_p$.*

Proof: $p = \infty$ is already treated in Theorem 6.

In the easy necessity part, we have nothing new to offer. It follows directly by chosing $f = \chi_{(0,x)}$ in (3.1).

For the sufficiency part we suppose that $\omega \in B_p$ with $B_p(\omega) = B$. In Theorem 4 we take $\epsilon = \frac{p}{4(B+1)}$ and put $p_1 = p - 2\epsilon$ and $p_2 = p - \epsilon$. The conclusion is that $B_{p_1}(\omega) \le 2B$ and

$$\int_0^{rx} \omega(u)du \ge \frac{r^{p_1}}{2B+1}\int_0^x \omega(u)du = r^{p_2}\frac{r^{p_1-p_2}}{2B+1}\int_0^x \omega(u)du \quad \text{for} \quad r \le 1.$$

We now choose $r_0 < 1$ so small that $r_0^{p_2-p_1}(2B+1) = 1$. This gives

$$\int_0^{r_0 x} \omega(u)du \ge r_0^{p_2}\int_0^x \omega(u)du, \quad \forall x > 0.$$

Put $r_0^{p_2} = C_0^{-1}$ and choose $\{\alpha_k\}_{-\infty}^{\infty}$ so that $\int_0^{\alpha_k} \omega(u)du = C_0^{-k}$. Then we have

$$\int_0^{\alpha_k} \omega(x)dx = C_0\int_0^{\alpha_{k+1}} \omega(x)dx \ge \int_0^{\frac{\alpha_{k+1}}{r_0}} \omega(x)dx$$

and therefore

$$\alpha_{k+1} \leq r_0 \alpha_k. \tag{3.9}$$

Now we can proceed as in the proof of Theorem 6 (with C replaced by C_0) to find

$$M_{\frac{1}{p}} f(\alpha_k) \leq M_{\frac{1}{p}} g(\alpha_k) \leq \Big(\frac{1}{\alpha_k} \sum_{m=k}^{\infty} C_0^{\frac{m+1}{p}} b_{m+1}^{\frac{1}{p}} \alpha_m\Big)^p.$$

By the definition of C_0 and (3.9) we deduce that the terms of this series decrease geometrically with a quotient that is at most $r_0^{1-\frac{p_2}{p}} 2^{\frac{\epsilon}{p}}$. We have not yet decided what $\epsilon > 0$ (in Lemma 4) should be. We just have to take $\epsilon < (p_2 - p)\frac{\ln r_0}{\ln 2}$ to be sure of obtaining geometrical decreasing. Take for instance ϵ equals half that quantity. Then we have

$$M_{\frac{1}{p}} f(\alpha_k) \leq C(B,p) C_0^{k+1} b_{k+1},$$

where $C(B,p)$ is a constant, depending only on the indicated quantities. This gives

$$\int_0^\infty M_{\frac{1}{p}} f(x)\omega(x)dx \leq C(B,p) \sum_{-\infty}^{\infty} C_0^{k+2} b_{k+2} C_0^{-k} \leq C_1(B,p)K =$$

$$= C_1(B,p) \int_0^\infty f(x)\omega(x)dx,$$

by which we have proved the sufficiency part of the theorem. ■

3.4. A'_∞ and non-decreasing weights.

We end this paper by proving two theorems, the first of which is an extension to $q = \infty$ of Theorem (1.10) in [**1**]. The second is an analogy with Theorem 5 for non-decreasing weights ω.

Theorem 8. *If $\omega \in A'_\infty$, then $m'_p(\omega) < \infty$ for p large enough.*

A non-decreasing ω lies in A'_∞ if and only if $m'_\infty(\omega) < \infty$ and then $m'_\infty(\omega) \geq A'_\infty(\omega)$.

Theorem 9. *For ω non-decreasing we have*

$$\omega \in A'_\infty \Leftrightarrow \omega \in \bigcup_{p>1} A'_p.$$

The proofs of these two theorems are based on the following lemma:

Lemma 5. *Suppose that $\omega \in A'_\infty$ with constant K. Then, for every $r > 1$ there is a constant C, depending on K and r, such that*

$$\int_0^{rx} \omega(t)dt \le C \int_0^x \omega(t)dt.$$

For $r = 2$, $\quad C = 4K^3$ will do.

Proof: Choose an arbitrary $r > 1$. For every $x > 0$, the assumption and Jensen's inequality give

$$\text{(3.10)} \qquad \fint_0^{rx} \omega(t)dt \exp \fint_0^{rx} \ln \frac{1}{\omega(t)} dt \le K \le K \fint_0^{x} \omega(t)dt \exp \fint_0^{x} \ln \frac{1}{\omega(t)} dt.$$

Put

$$\int_0^{rx} \omega(t)dt = c\alpha \quad \text{and} \quad \int_0^x \omega(t)dt = \alpha.$$

Then

$$\int_x^{rx} \omega(t)dt = (c-1)\alpha \quad \text{and} \quad \fint_x^{rx} \omega(t)dt = \frac{(c-1)\alpha}{(r-1)x}.$$

What we want to estimate is the exponential of

$$\fint_0^{rx} \ln \frac{1}{\omega(t)} dt - \fint_0^{x} \ln \frac{1}{\omega(t)} dt = \frac{1}{rx} \int_x^{rx} \ln \frac{1}{\omega(t)} dt - \frac{r-1}{rx} \int_0^{x} \ln \frac{1}{\omega(t)} dt =$$

$$= \frac{r-1}{r} \Big(\fint_x^{rx} \ln \frac{1}{\omega(t)} dt - \fint_0^{x} \ln \frac{1}{\omega(t)} dt \Big).$$

We now treat the two members on the left, the first by Jensen's inequality

$$\exp \fint_x^{rx} \ln \frac{1}{\omega(t)} dt \ge \Big(\fint_x^{rx} \omega(t)dt \Big)^{-1} = \frac{(r-1)x}{(c-1)\alpha} \ \textit{i.e.} \ \fint_x^{rx} \ln \frac{1}{\omega(t)} dt \ge \ln \frac{(r-1)x}{(c-1)\alpha}.$$

The second satisfies by assumption

$$\exp \fint_0^{x} \ln \frac{1}{\omega(t)} dt \le \frac{K}{\alpha} x \ \ \textit{i.e.} \ \ \fint_0^{x} \ln \frac{1}{\omega(t)} dt \le \ln K + \ln \frac{x}{\alpha}.$$

We therefore obtain

$$\frac{r-1}{r}\left(\fint_x^{rx}\ln\frac{1}{\omega(t)}dt-\fint_0^x\ln\frac{1}{\omega(t)}dt\right)\geq\frac{r-1}{r}\ln\frac{r-1}{K(c-1)}$$

and inequality (3.10) gives

$$cr\left(\frac{r-1}{K(c-1)}\right)^{\frac{r-1}{r}}\leq K,$$

or

$$cr\left(\frac{r-1}{c-1}\right)^{\frac{r-1}{r}}\leq K^{2-\frac{1}{r}}.$$

For any $r>1$ we see that c cannot be arbitrarily large, but has to be smaller than some number, which depends on r and K. $r=2$, for example, gives the doubling constant $d(\omega)<\frac{K^3}{4}$. This proves the lemma. ■

Proof of Theorem 8: Suppose that $\omega\in A'_\infty$ with constant K. By Lemma 5 , $\omega\in B_\infty$ with $d(\omega)\leq\frac{K^3}{4}$. By Theorem 6, $m'_\infty(\omega)<\infty$.

Suppose now that ω is a non-decreasing function with finite $m'_\infty(\omega)$. Then we use the inequality

$$\int_0^\infty M_0f(t)\omega(t)dt\leq m'_\infty(\omega)\int_0^\infty f(t)\omega(t)dt$$

with the non-increasing function $f=\frac{1}{\omega}\chi(0,x)$ to obtain

$$\int_0^x\exp\left(\fint_0^t\ln\frac{1}{\omega(s)}ds\right)\omega(t)dt\leq m'_\infty(\omega)x.$$

Since ω is non-decreasing and $t\leq x$ in the integration

$$\fint_0^t\ln\frac{1}{\omega(s)}ds\geq\fint_0^x\ln\frac{1}{\omega(s)}ds.$$

This gives

$$\fint_0^x\omega(t)dt\exp\fint_0^x\ln\frac{1}{\omega(t)}dt\leq m'_\infty(\omega),$$

and thus

$$A'_\infty(\omega) \le m'_\infty(\omega).$$

Thereby we have proved Theorem 8. ■

Proof of Theorem 9: This is now more or less a corollary. By Jensen's inequality $A'_\infty(\omega) \le A'_p(\omega)$ and therefore $A'_p \subset A'_\infty$, $\forall p > 1$. On the other hand, by Lemma 5 and Ariño-Muckenhoupt's result

$$\omega \in A'_\infty \Rightarrow \{\omega \in B_p \quad \text{for some } p > 1\} \Rightarrow \omega \in A'_p.$$

Therefore, for non-decreasing ω , $A'_\infty \subset \bigcup A'_p$ and the proof is complete. ■

It is natural to ask whether $\omega \in A'_\infty$ implies $\omega \in A'_p$ for some $p > 1$, i.e. if Theorem 8 could be strengthened to comprise also the case of weight functions that are not non-decreasing. This, however, is not true. We can for example take

$$\omega(x) = \begin{cases} \exp - \frac{1}{\sqrt{(1-x)}}, & 0 < x < 1, \\ 1, & x \ge 1. \end{cases}$$

This function clearly lies in A'_∞ but not in A'_p for any $p > 1$, but it is easy to see that $B_p(\omega)$ is finite for every $p > 1$ and therefore $m'_p(\omega) < \infty$ for every $p > 1$. This example also shows that $A'_\infty(\omega) \ne \lim_{p\to\infty} A'_p(\omega)$.

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