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WEIGHTED NORM INEQUALITIES FOR MAXIMAL FUNCTIONS FROM THE MUCKENHOUPT CONDITIONS

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Abstract

For some pairs of weight functions u, v, which satisfy the well-known Muckenhoupt conditions, we derive the boundedness of the maximal fractional operator M_s $(0 \leq s < n)$ from L^p_v to L^q_u with $q < p$.

0. Introduction

Let u, v weight functions on $\mathbb{R}^n$, $n \geq 1$ (i.e. nonnegative locally integrable functions). The fractional maximal operator M_s $(0 \leq s < n)$ is given by

$$(M_s f)(x) = \sup \left\{ |Q|^{\frac{s}{n}-1} \int_Q |f|; \quad Q \text{ a cube with } Q \ni x \right\}$$

Throughout this paper Q will denote a cube with sides parallel to the co-ordinate planes.

Let $1 < p, q < \infty$, with $\frac{1}{p} - \frac{1}{q} \leq \frac{s}{n}$. It is fundamental in analysis to give a characterization of the pairs of weights (u, v) which satisfy

(0) $\|M_s f\|_{L^q_u} \leq C \|f\|_{L^p_v}$ for all functions f, $C = C(s,n,p,q,u,v) > 0$.

Here $\|g\|_{L^r_w}$ denotes $\left(\int_{\mathbb{R}^n} |g|^r w\, dx\right)^{\frac{1}{r}}$, with dx the Lebesgue measure on $\mathbb{R}^n$.

In the case of $1 < p \leq q < \infty$ Sawyer [**Sa2**] showed that the inequality (0) holds if and only if $(u, v) \in S(s,n,p,q)$ i.e

$$\|(M_s v^{-\frac{1}{p-1}} \mathbb{I}_Q) \mathbb{I}_Q\|_{L^q_u} \leq C \|v^{-\frac{1}{p-1}} \mathbb{I}_Q\|_{L^p_v} = \|\mathbb{I}_Q\|_{L^p_{v^{-\frac{1}{p-1}}}} < \infty$$

for all cubes Q, here $S = S(s,n,p,q,u,v) > 0$. A known necessary but not sufficient condition for (0) [**Mu**] is $(u,v) \in A(s,n,p,q)$ i.e

$$|Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q u\right)^{\frac{1}{q}} \left(\frac{1}{|Q|}\int_Q v^{-\frac{1}{p-1}}\right)^{1-\frac{1}{p}} \le A \text{ for all cubes } Q,$$

with $A = A(s,n,p,q,u,v) > 0$.

As we will recall in Section 2, this condition is verified more easily than the first one. Pérez [**Pe**] (see also [**Sa1**]) proved that $(u,v) \in A(s,n,p,q)$ implies the inequality (0) whenever $d\sigma = v^{-\frac{1}{p-1}}\,dx \in A_\infty$ i.e. for some $\delta > 0$:

$$\frac{|E|_\sigma}{|Q|_\sigma} \le \left(\frac{|E|}{|Q|}\right)^\delta \text{ for all cubes } Q \text{ and for all measurable sets } E \subset Q$$

here $|E|_\sigma$ denotes $\int_Q \sigma$. In fact the equivalence between (0) and $(u,v) \in A(s,n,p,q)$ is also valid with a weaker condition on $d\sigma$, for instance in [**Ra3**] it was proved that it is sufficient $d\sigma \in B_\delta$ i.e.

$$\frac{|Q'|_\sigma}{|Q|_\sigma} \le \left(\frac{|Q'|}{|Q|}\right)^\delta \text{ for all cubes } Q,\, Q' \text{ with } Q' \subset Q$$

with $\left[1-\frac{s}{n}\right] \le \delta$. As we will see in Section 2, measures $d\mu$ can be found such as $d\mu \in B_\delta$ but $d\mu \notin A_\infty$. The condition $\frac{1}{p}-\frac{1}{q} \le \frac{s}{n}$ can be derived from the inequality (0) by the Lebesgue differentiation theorem. Hence for $s = 0$ (M_0 is the Hardy-Littlewood maximal operator), the inequality (0) must only considered for $q \le p$. The case $p = q$ was studied by Muckenhoupt [**Mu**] for $u = v$ and by Sawyer [**Sa2**] for general weights u, v. For $q < p$, a characterization of the pairwise of weights (u,v) satisfying the inequality (0) was given by the author [**Ra1**]; but the condition used is difficult to check.

Therefore $1 < q < p < \infty$ a natural question is: "does $(u,v) \in A(s,n,p,q)$ imply (0) whenever $d\sigma \in A_\infty$. In this paper we give a positive answer with the additional assumptions $u\,dx \in B_\nu$, $v^{-\frac{1}{p-1}}\,dx \in B_\rho$ with $0 < \nu$, ρ and $\left[1-\frac{s}{n}\right] < \rho\left(1-\frac{1}{p}\right) + \nu\frac{1}{p}$.

We state our main result in Section 1. In Section 2 we give some useful remarks and observations about the weight condition B_ρ. The proof of our main result is in Section 3. A paper of Verbitsky [**Ve**] concerning the characterization of the problem (0) with $q < p$ and for general weights u, v appeared when this manuscript was written.

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1. The main result

To include many classical maximal functions, we deal with the operator

$$(M_\Phi f)(x) = \sup\left\{\Phi(Q)|Q|^{-1}\int_Q |f|; \quad Q \text{ a cube with } Q \ni x\right\}$$

where Φ is a map defined on the set of cubes, taking its value in $]0,\infty[$ and satisfying the following growth conditions:

$\mathcal{H}_1$ there is $C > 0$ such as

$$\Phi(Q_1) \le C\Phi(Q_2) \text{ for all cubes } Q_1,\ Q_2 \text{ with } Q_1 \subset Q_2;$$

$\mathcal{H}_2$ there are $C_1,\ C_2 > 0,\ \lambda,\ \eta \ge 0$ such as

$$C_1 t^{n\lambda}\Phi(Q) \le \Phi(tQ) \le C_2 t^{n\eta}\Phi(Q) \text{ for all cubes } Q \text{ and all } t \ge 1.$$

When $\Phi(Q) = 1$ the Hardy-Littlewood maximal operator is obtained. The fractional maximal operator M_s $(0 < s < n)$ is given by $\Phi(Q) = |Q|^{\frac{s}{n}}$. Maximal operators connected to the Bessel potential operator [**Ke-Sa**] are defined by $\Phi(Q) = \int_0^{|Q|^{\frac{1}{n}}} \varphi(s)\,ds$; and generally M_Φ arises in studies of other potential operators [**Ch-St-Wh**].

Let $1 < p,\ q < \infty$. We say that the inequality $P(M_\Phi, p, q, u, v)$ holds for a constant $C > 0$ when

$$\|M_\Phi f\|_{L^q_u} \le C\|f\|_{L^p_v} \text{ for all functions } f$$

and we write $(u, v) \in A(\Phi, p, q)$ if for some constant $A > 0$

$$\Phi(Q)|Q|^{\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q u\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q v^{-\frac{1}{p-1}}\right)^{1-\frac{1}{p}} \le A \text{ for all cubes } Q.$$

In this paper we always adopt the convention $0.\infty = 0$. By $P(M_\Phi, p, q, u, v)$ and the Lebesgue theorem, we see that if $u \ne 0$ it is necessary to suppose

$$(\mathcal{H}_3) \qquad \lim_{|Q|\to 0}\left(\Phi(Q)|Q|^{\frac{1}{q}-\frac{1}{p}}\right) < \infty.$$

For instance $\mathcal{H}_3$ is satisfied if $\frac{1}{p} - \frac{1}{q} \le \lambda$. For $\Phi(Q) = 1$, the hypothesis $\mathcal{H}_3$ implies $q \le p$, and for $\Phi(Q) = |Q|^{\frac{s}{n}}$ it means $\frac{1}{p} - \frac{1}{q} \le \frac{s}{n}$.

Let $\varrho > 0$ and w be a weight function. As in Section 0, we write $w\,dx \in B_\varrho$ if there is $C > 0$ such as

$$\frac{|Q'|w}{|Q|w} \le \left(\frac{|Q'|}{|Q|}\right)^\varrho \text{ for all cubes } Q,\ Q' \text{ with } Q' \subset Q.$$

Now our main result can be stated:

Theorem.

Let $1 < p,\ q < \infty$ and Φ be a function which satisfies $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$.

A) *If the inequality $P(M_\Phi, p, q, u, v)$ holds for a constant $C > 0$ then $(u, v) \in A(\Phi, p, q)$ with the constant $A = C$.*

B) *Let $1 < q < p < \infty$ and $d\sigma = v^{-\frac{1}{p-1}}\,dx \in A_\infty$. Moreover assume $u\,dx \in B_\nu$, $v^{-\frac{1}{p-1}}\,dx \in B_\varrho$ with $0 < \nu$, ϱ and $(1-\lambda) < \varrho\left(1 - \frac{1}{p}\right) + \nu\left(\frac{1}{p}\right)$. If $(u, v) \in A(\Phi, p, q)$ then the inequality $P(M_\Phi, p, q, u, v)$ holds for a constant Ac, $c = c(\Phi, n, p, q, u, v) > 0$. The part B) is also valid when $d\sigma \in B_\varrho$ with $1 - \lambda \le \varrho$.*

Actually the constant c depends on the fact that $u\,dx \in B_\nu$, $v^{-\frac{1}{p-1}}\,dx \in B_\varrho$ but not directly on u and v. The result stated in the introduction is now easily derived from the theorem by taking $\Phi(Q) = |Q|^{\frac{s}{n}}$.

Let $0 < s < n$ and I_s the fractional integral operator defined by

$$I_s = \int_{\mathbb{R}^n} |x - y|^{s-n} f(y)\,dy.$$

In the case of $1 < p \le q < \infty$ it is known [**Pe**] that the inequality $P(I_s, p, q, u, v)$, i.e.

$$\|I_s f\|_{L^q_u} \le C\|f\|_{L^p_v} \text{ for all nonnegative functions } f$$

holds if and only if $(u, v) \in A(s, n, p, q)$ whenever $u\,dx$, $v^{-\frac{1}{p-1}}\,dx \in A_\infty$. By the results in [**Ra2**] and [**Ra3**] this equivalence also holds if $u\,dx \in B_\nu \cap D_\infty$, $v^{-\frac{1}{p-1}}\,dx \in B_\varrho$ with $1 - \frac{s}{n} < \nu$ and $1 - \frac{s}{n} \le \varrho$ (see also [**Pe**] for such a result). The condition $w\,dx \in D_\infty$ means:

$$|2Q|_w \le C|Q|_w \text{ for all cubes } Q.$$

$2Q$ is the cube with the same center as Q but the edge lenght expanded twice. As a consequence of our theorem, for $1 < q < p$ we have

Corollary.

Let $1 < q < p < \infty$, $0 < s < n$ and $u\,dx$, $v^{-\frac{1}{p-1}}\,dx \in A_\infty$. Moreover assume $u\,dx \in B_\nu$, $v^{-\frac{1}{p-1}}\,dx \in B_\varrho$ with $0 < \nu, \varrho$ and $\left(1 - \frac{s}{n}\right) < \varrho\left(1 - \frac{1}{p}\right) + \nu\left(\frac{1}{p}\right)$. Then the inequality $P(I_s, n, p, q, u, v)$ holds if and only if $(u, v) \in A(s, n, p, q)$. This equivalence also holds when $u\,dx \in B_\nu \cap D_\infty$, $v^{-\frac{1}{p-1}} \in B_\varrho$ with $1 - \frac{s}{n} < \nu$ and $1 - \frac{s}{n} \le \varrho$.

For seeing this, it is sufficient to remind that the Muckenhoupt-Wheeden inequality [**Mu-Wh**]

$$\|I_s f\|_{L^q_u} \le C\|M_s f\|_{L^q_u}$$

holds whenever $u\,dx \in A_\infty$. This is also the case when $u\,dx \in B_\nu \cap D_\infty$ with $1 - \frac{s}{n} < \nu$ (see [**Pe**] or [**Ra2**]).

2. On $A(\Phi, p, q)$ and B_ϱ conditions

Now we also assume the functions Φ defined on the set of balls by $\Phi(B) = \Phi(Q)$ whenever Q is the smallest cube which contains the ball B. A weight function w satisfies the condition $\mathcal{C}$ when there are constants c, $C > 0$ so that

$$\sup_{\frac{1}{4}R < |x| \le 4R} w(x) \le \frac{C}{R^n} \int_{|y| \le cR} w(y)\,dy.$$

Many of usual weight functions w satisfy this growth condition, since nonincreasing and nondecreasing radial functions are included. Condition $(u, v) \in A(\Phi, p, q)$ for u and v satisfying $\mathcal{C}$ can be easily realized, mainly for radial weights. Indeed we have

Proposition 2.1.

Let $1 < p$, $q < \infty$ and $\frac{1}{p} - \frac{1}{q} \le \lambda$. Assume u, v satisfying the growth condition $\mathcal{C}$. Then $(u, v) \in A(\Phi, p, q)$ for a constant $A > 0$ if and only if $(u, v) \in A_0(\Phi, p, q)$; i.e

$$\Phi(B(0, R)) R^{n(\frac{1}{q} - \frac{1}{p})} \left(\frac{1}{R^n} \int_{|y|<R} u \right)^{\frac{1}{q}} \left(\frac{1}{R^n} \int_{|y|<R} v^{-\frac{1}{p-1}} \right)^{1-\frac{1}{p}} \le A_0$$

for all $R > 0$, where $A_0 = A \times c(\Phi, n, p, q, u, v)$.

As an example for $0 \le s < n$, $\frac{1}{p} - \frac{1}{q} \le \frac{s}{n}$, $-n < \alpha < n(p-1)$, $ps - n < \alpha$, $\beta = \frac{q}{p}(n + \alpha) - qs - n$, $u(x) = |x|^\beta$, $v(x) = |x|^\alpha$ then $(u, v) \in A(s, n, p, q)$.

Now let us discuss how we can verify in practise, for usual weights the condition $w\,dx \in B_\varrho$, $\varrho > 0$. To do this, we first recall some known classes of weights.

The Muckenhoup class A_p.

Let us recall that $w\,dx \in A_p$ $(1 < p < \infty)$ if and only if $(w, w) \in A(0, n, p, p)$. It is known [**Ga-Rb**] that $A_\infty = \cup_{r>1} A_r$.

The reverse Holder class RH_r.

We write $w\,dx \in RH_r$ $(1 < r < \infty)$ if and only if

$$\left(\frac{1}{|Q|}\int_Q w^r\right)^{\frac{1}{r}} \leq R\left(\frac{1}{|Q|}\int_Q w\right) \text{ for all cubes } Q \quad C = C(w) > 0.$$

The classes RH_r and A_p are related; for instance $w\,dx \in RH_r$ if and only if $w^{-1}\,dx \in A_{\frac{r}{r-1}}$. If $w\,dx \in A_p$ then it is known [**Ga-Rb**] that $w\,dx \in RH_{1+\rho}$ for some $\rho > 0$; the converse is also true.

The reverse doubling class RD_ϱ.

We write $w\,dx \in D_\varrho$ $(\varrho > 0)$ if and only if

$$Ct^{n\varrho}|Q|_w \leq |tQ|_w \text{ for all cubes } Q \text{ and all } t \geq 1, \quad C = C(w) > 0.$$

If $w\,dx \in RH_{\frac{r}{r-1}}$ then, by the Holder inequality, $w\,dx \in RD_{\frac{1}{r}}$. Suppose $w\,dx \in D_\infty$ with the doubling constant D, i.e

$$|2Q|_w \leq D|Q|_w \text{ for all cubes } Q \quad D = D(w) > 1,$$

then [**St-To**] $w\,dx \in RD_\varrho$ for some $\varrho > 0$. Precisely [**Ra3**] we can take $\varrho = \frac{1}{\ln 2^n}\ln\frac{D^c}{D^c-1}$ where $c = 4 + \frac{\ln 3}{\ln 2}$. But the reverse doubling condition RD_r is weaker than the doubling condition D_∞ (take for instance $w(x) = e^{|x|}$).

Thus it is clear that $w\,dx \in A_\infty$ implies $w\,dx \in B_\varrho$ for some ϱ. On the otherhand we can state

Proposition 2.2.

If $w\,dx \in B_\varrho$ for some $\varrho > 0$ then $w\,dx \in RD_\varrho$. Conversely if $w\,dx \in RD_\varrho \cap D_\infty$ then $w\,dx \in B_\varrho$.

So in practice to obtain $w\,dx \in B_\varrho$ it is sufficient to get $w\,dx \in RD_\varrho \cap D_\infty$. By the above condition $w\,dx \in RD_\varrho$ $(0 < \varrho \leq 1$, with the precise value of $\varrho)$ can be realized from $w\,dx \in RH_{\frac{1}{1-\varrho}}$ or $w\,dx \in D_\infty$. Consequently, it is interesting to know when we have $w\,dx \in D_\infty$. It is well known [**Ga-Rb**] that $w\,dx \in D_\infty$ when $w\,dx \in A_\infty$. But we can find $w\,dx \in D_\infty$ with $w\,dx \notin A_\infty$ [**Wi**]. As a tool for $w\,dx \in D_\infty$ Stromberg and Wheeden [**St-Wh**] proved that $|x|^\alpha u(x)$, $\left(\frac{|x|}{1+|x|}\right)^\alpha u(x) \in D_\infty$ when $u\,dx \in RD_\varrho \cap D_\infty$ and $\alpha > -np$. By adapting an argument in [**St-Wh**], this result can be extended for weights $w(x) = \theta(|x|)u(x)$ where $u\,dx \in RD_\varrho \cap D_\infty$ $(\varrho > 0)$, and θ essentially constant on annuli and satisfying a condition like: $\sum_{k\geq 0} 2^{-kn_\varrho}\theta(2^{-k}L) \leq \theta(L)$ for all $L > 0$.

3. Proofs of results

Our main theorem is a direct consequence of the inequalities (3.1), (3.2), (3.3) in the following propositions.

Proposition 3.1.

Let $1 < q < p < \infty$. Assuume Φ be a function which satisfies hypotheses $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$. Let define

$$\Theta(x) = \sup\{\Phi(Q)|Q|^{-1}|Q|_{\sigma}^{1-\frac{1}{p}}|Q|_{u}^{\frac{1}{p}}; \quad Q \text{ a cube with } Q \ni x\},$$

$d\sigma = v^{-\frac{1}{p-1}}\,dx$ and $\tilde{u}(x) = \Theta^{-p}(x)u(x)$. Then $\tilde{u} \in L^1_{\text{loc}}(\mathbb{R}^n, dx)$ and

$$\|M_{\Phi}f\|_{L^q_u} \leq \|M_{\Phi}f\|_{L^p_{\tilde{u}}}\|\Theta\|_{L^r_u} \text{ for all functions } f, \tag{3.1}$$

where $r = \frac{qp}{p-q}$.

Proposition 3.2.

Let $1 < p < \infty$ and $\tilde{u}$ defined as above. Assume $d\sigma \in A_{\infty}$ or $d\sigma \in B_{\varrho}$ with $1 - \lambda \leq \varrho$ (λ is the exponent in the hypothesis $\mathcal{H}_2$). Then there is $c = c(\Phi, n, p, q, u, v) > 0$ such that

$$\|M_{\Phi}f\|_{L^p_{\tilde{u}}} \leq c\|f\|_{L^p_v} \text{ for all functions } f. \tag{3.2}$$

Proposition 3.3.

Let $1 < q < p < \infty$. Assume

i) *Φ be a function which satisfies $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$;*

ii) *$u\,dx \in B_{\nu}$, $v^{-\frac{1}{p-1}}\,dx \in B_{\varrho}$ with $0 < \nu$, ϱ and $(1-\lambda) < \varrho\left(1 - \frac{1}{p}\right) + \nu\left(\frac{1}{p}\right)$;*

iii) *$(u, v) \in A(\Phi, p, q)$ for a constant $A > 0$. Then there is $C(\Phi, n, p, q, u, v) > 0$ so that*

$$\|\Theta\|_{L^r_u} \leq CA \quad r = \frac{qp}{p-q}. \tag{3.3}$$

Proof of Proposition 3.1:

Let us first observe the locally integrability of the function $\tilde{u}$. Indeed for each cube Q with $\left(\Phi(Q)|Q|^{-1}|Q|_{\sigma}^{1-\frac{1}{p}}|Q|_{u}^{\frac{1}{p}}\right) > 0$ and for each $x \in Q$:

$$\Theta^{-1}(x) \leq \left(\Phi(Q)|Q|^{-1}|Q|_{\sigma}^{1-\frac{1}{p}}|Q|_{u}^{\frac{1}{p}}\right)^{-1} > 0$$

and so

$$(3.4)\qquad |Q|_{\tilde{u}} = \int_Q \Theta^{-p}(x)u\,dx \leq \left(\Phi(Q)|Q|^{-1}|Q|_\sigma^{1-\frac{1}{p}}|Q|_u^{\frac{1}{p}}\right)^{-p} < \infty.$$

Note that for $|Q|_u = 0$, by the convention $0.\infty = 0$, we immediatly have $|Q|_{\tilde{u}} = 0$.

Inequality (3.1) comes from the Holder inequality, indeed for $1 < q < p < \infty$ and $r = \frac{qp}{p-q}$ we get

$$\begin{aligned}\|M_\Phi f\|_{L^q_u}^q &= \int_{\mathbb{R}^n} \left[(M_\Phi f)\tilde{u}^{\frac{1}{p}}\Theta u^{\frac{1}{q}-\frac{1}{p}}\right]^q dx \leq \\ &\leq \|(M_\Phi f)\tilde{u}^{\frac{1}{p}}\|_{L^p}^q \|\Theta u^{\frac{1}{q}-\frac{1}{p}}\|_{L^r}^q = \\ &= \|M_\Phi f\|_{L^p_{\tilde{u}}}^q \|\Theta\|_{L^r_u}^q. \blacksquare\end{aligned}$$

Proof of Proposition 3.2:

First let us note that by (3.4), $(\tilde{u}, v) \in A(\Phi, p, p)$ i.e.

$$\left(\Phi(Q)|Q|^{-1}|Q|_\sigma^{1-\frac{1}{p}}|Q|_{\tilde{u}}^{\frac{1}{p}}\right) > 0 \text{ for all cubes } Q.$$

For $d\sigma \in A_\infty$, an easy modification of the proof in [**Pe**] yields to the conclusion (3.2). For $d\sigma \in B_\varrho$ with $1 - \lambda \leq \varrho$, we get $(\tilde{u}, v) \in S(\Phi, p, p)$ [**Ra3**] and then by a similar argument as in [**Sa2**] the inequality (3.2) holds for a constant $c = c(\Phi, n, p, \tilde{u}) > 0$. ■

Proof of Proposition 3.3:

For each $R > 0$, let us define

$$\Theta_R(x) = \sup\{\Phi(Q)|Q|^{-1}|Q|_\sigma^{1-\frac{1}{p}}|Q|_u^{\frac{1}{p}};\ Q \text{ a cube with } Q \ni x,\ |Q|^{\frac{1}{n}} \leq R\}.$$

The conclusion appears once we obtain

$$(3.3')\qquad \|\Theta_R\|_{L^r_u} \leq cA \quad c = c(\Phi, n, p, q, u, \sigma) > 0, \quad r = \frac{qp}{p-q}.$$

Then in order to prove (3.3'), we take a cube Q_0 with $|Q_0|^{\frac{1}{n}} = R$. Then

$$\|\Theta\|_{L^r_u}^r = \theta_{1,R} + \Theta_{2,R}$$

where $\Theta_{1,R} = \int_{Q_0} \Theta_R^r u\,dx$, $\Theta_{2,R} = \int_{\mathbb{R}^n \setminus Q_0} Q_R^r u\,dx$. ■

Estimate of $\Theta_{1,R}$.

Let $x \in Q_0$, Q a cube with $Q \ni x$ and $|Q|^{\frac{1}{n}} \leq R$. Note that $Q \subset (3Q_0)$. Now using i), ii), iii) we get

$$\begin{aligned}\Lambda(Q) &= \Phi(Q)|Q|^{-1}|Q|_\sigma^{1-\frac{1}{p}}|Q|_u^{\frac{1}{p}} \leq \\ &\leq c(\Phi, n, \sigma, u)\left(\frac{|Q|}{|Q_0|}\right)^{[\lambda-1+\varrho(1-\frac{1}{p})+\nu\frac{1}{p}]}\Lambda(3Q_0) \leq \\ &\leq c(\Phi, n, \sigma, u)\Lambda(3Q_0).\end{aligned}$$

Thus $\Theta_R(x) \leq c(\Phi, n, \sigma, u)\Lambda(3Q_0)$, and consequently

$$\begin{aligned}\Theta_{1,R} &\leq c'(\Phi, n, \sigma, u)\left(\Lambda(3Q_0)|3Q_0|_u^{\frac{1}{q}}\right)^r = \\ &= c'(\Phi, n, \sigma, u)\left(\Phi(3Q_0)|3Q_0|^{-1}|3Q_0|_\sigma^{1-\frac{1}{p}}|3Q_0|_u^{\frac{1}{q}}\right)^r \leq \\ &\leq c'(\Phi, n, \sigma, u)A^r.\end{aligned} \tag{3.5}$$

Estimate of $\Theta_{2,R}$.

First we can write

$$\Theta_{2,R} = \sum_{k\geq 0}\int_{(2^{k+1}Q_0)\setminus(2^kQ_0)} \Theta_R^r u\,dx.$$

Let $k \in \mathbb{N}$, $x \in (2^{k+1}Q_0)\setminus(2^kQ_0)$ and $Q \ni x$ with $|Q|^{\frac{1}{n}} \leq R$. Then $Q \subset (32^{k+1}Q_0) = (6Q_0)$. As the above computation we have

$$\Lambda(Q) \leq c'(\Phi, n, \sigma, u)2^{-kn[\lambda+\varrho(1-\frac{1}{p})+\nu\frac{1}{p}]}\Lambda(62^kQ_0).$$

Next, since $1-\lambda < \varrho\left(1-\frac{1}{p}\right)+\nu\frac{1}{p}$, then

$$\begin{aligned}\Theta_{2,R} &\leq c'(\Phi, n, \sigma, u)\sum_{k\geq 0}2^{-kn[\lambda+\varrho(1-\frac{1}{p})+\nu\frac{1}{p}]}\left(\Lambda(62^kQ_0)|(62^kQ_0)|_u^{\frac{1}{r}}\right)^r \leq \\ &\leq c'(\Phi, n, \sigma, u)A^r\sum_{k\geq 0}2^{-kn[\lambda+\varrho(1-\frac{1}{p})+\nu\frac{1}{p}]} \leq \\ &\leq c'(\Phi, n, \sigma, u)A^r.\end{aligned} \tag{3.6}$$

Inequalities (3.5) and (3.6) yield (3.3'), and consequently by a limiting argument we get (3.3).

Proof of Proposition 2.1:

Let us assume the condition $(u,v) \in A(\Phi,p,q)$ holds for a constant $A > 0$. It is also equivalent to ask
(∗)

$$\Phi(B)|B|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|B|}\int_B u\right)^{\frac{1}{q}} \left(\frac{1}{|B|}\int_B v^{-\frac{1}{p-1}}\right)^{1-\frac{1}{p}} \leq A' \text{ for all balls } B,$$

here $A' = Ac(\Phi,n,p,q)$.

If $|x_0| \leq 2R$ then $B \subset B(0,3R)$ and hence the first member of (∗) is majorized by

$$c(\Phi,n,p,q)\times$$
$$\times\Phi(B(0,3R))(3R)^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{1}{(3R)^n}\int_{|y|<3R} u\right)^{\frac{1}{q}}\left(\frac{1}{(3R)^n}\int_{|y|<3R} v^{1-\frac{1}{p}}\right).$$

If $2R < |x_0|$ then $2^jR < |x_0| \leq 22^jR$ for some $j \in N^*$ and hence for each $y \in B : \frac{1}{4}2^jR < |x| \leq 42^jR$. Using the growth condition $\mathcal{C}$ for u and $v^{-\frac{1}{p-1}}$ it is found $c = c(u,v) > 0$, $C = C(u,v) > 0$ such as

$$\left(\frac{1}{|B|}\int_B v^{-\frac{1}{p-1}}\right) \leq C\left(\frac{1}{(c2^jR)^n}\int_{|y|<(c2^jR)} v^{-\frac{1}{p-1}}\right)$$

and

$$\left(\frac{1}{|B|}\int_B u\right) \leq C\left(\frac{1}{(c2^jR)^n}\int_{|y|<(c2^jR)} u\right).$$

Note that c, C depend on the constants on the growth condition $\mathcal{C}$ for u and $v^{-\frac{1}{p-1}}$ but not directly on these weights. Consequently in the case $2R < |x_0|$, the first number of (∗) is now majorized by

$$C(\Phi,n,p,q,c,C)2^{-jn[\lambda+\frac{1}{q}-\frac{1}{p}]}\Phi(B(0,c2^jR))\times$$
$$\times(c2^jR)^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{1}{(c2^jR)^n}\int_{|y|<(c2^jR)} u\right)^{\frac{1}{q}}\left(\frac{1}{(c2^jR)^n}\int_{|y|<(c2^jR)} v^{-\frac{1}{p-1}}\right)^{1-\frac{1}{p}}$$

Since $\frac{1}{q}-\frac{1}{p} \leq \lambda$, we can see that inequality (∗) is satisfied once $(u,v) \in A_0(\Phi,p,q)$ i.e.

$$\Phi(B(0,R))R^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\frac{1}{R^n}\int_{|y|<R} u\right)^{\frac{1}{q}}\left(\frac{1}{R^n}\int_{|y|<R} v^{-\frac{1}{p-1}}\right)^{1-\frac{1}{p}} \leq A_0$$

for all $R > 0$, here $A_0 = Ac'(\Phi, n, p, q, u, v)$. ■

Proof of Proposition 2.2:

Let $w\,dx \in B_\varrho$ for some $\varrho > 0$ i.e.

$$\frac{|Q_1|_w}{|Q_0|_w} \leq B\left(\frac{|Q_1|}{|Q_0|}\right)^\varrho \text{ for all cubes } Q_0,\, Q_1 \text{ with } Q_1 \subset Q_0.$$

Let Q be a cube and $t \geq 1$. Taking $Q_1 = Q$ and $Q_0 = tQ$ we obtain

$$t^{n\varrho}|Q|_w \leq R|Q|_w$$

with $R = B$, hence $w\,dx \in RD_\nu$.

Conversely let $w\,dx \in RD_\varrho$ for a constant $R > 0$. Also if $w\,dx \in D_\infty$ then for $Q_1 \subset Q_0$ and for all cubes Q_2 having the same center as Q_1 and with $|Q_2| = |Q_0|$

$$\begin{aligned} |Q_1|_w &\leq R\left(\frac{|Q_1|}{|Q_0|}\right)^\varrho |Q_2|_w \leq \\ &\leq R\left(\frac{|Q_1|}{|Q_0|}\right)^\varrho |3Q_0|_w \leq \\ &\leq RD|Q_0|_w. \end{aligned}$$

Here D depends on the constant which is in the doubling condition for $w\,dx$. Consequently $w\,dx \in B_\varrho$ with the constant $B = RD$. ■

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