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GABRIEL FILTERS IN GROTHENDIECK CATEGORIES

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Dedicated to the memory of Pere Menal

Abstract

In [1] it is proved that one must take care trying to copy results from the case of modules to an arbitrary Grothendieck category in order to describe a hereditary torsion theory in terms of filters of a generator. By the other side, we usually have for a Grothendieck category an infinite family of generators $\{G_i;\, i \in I\}$ and, although each G_i has good properties the generator $G = \bigoplus_{i \in I} G_i$ is not easy to handle (for instance in categories like graded modules, presheaves or sheaves of modules). In this paper the authors obtain a bijective correspondence between hereditary torsion theories in a Grothendieck category C and a appropriately defined family of Gabriel filters of subobjets of the generators of C. This has been possible by using the natural conditions of local projectiveness and local smallness for families of generators in a Grothendieck category, that the embedding theorem of Gabriel-Popescu provided us.

Introduction

As it is well known, Grothendieck categories provide a good setting to study torsion theories, for they can be applied in several different contexts [4], [5]. Nevertheless, up to now there was not a suitable characterization of hereditary torsion theories in terms of Gabriel filters in the generators, like the usual one in the category R-mod. This paper is

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devoted to obtain the bijective correspondence between hereditary torsion theories in a Grothendieck category $\mathcal{C}$ and the appropriately defined Gabriel filters of subobjects of a family of generators of $\mathcal{C}$. This has been possible by introducing the natural notions of local projectiveness and local smallness, both defined here, for families of objects in $\mathcal{C}$. With respect to this, it is convenient to emphasize the fact that properties 1) to 4) of (3.2) caracterizing Gabriel filters have been abstracted from the parallel ones in R-mod, with the idempotent condition 4) restricted to coverings of the elements of the filter, instead of the totality of morphisms between generators and these objects as in (3.5) 4'). We set the comparation between the two posibilities in (3.5) proving that they are equivalent (in presence of the other conditions 1) to 3), of course) in the case that the generators are projective and small. We take from [1] an illustrative counterexample (3.6) showing that if the generators are not small the conditions 1) to 4') are unable to characterize the Gabriel filters corresponding to idempotent kernel functors.

Let us finally say that we have learnt about the complete generality of our result (3.4), looking at the example (2.5) in the light of the Gabriel-Popescu Imbedding theorem, in discussions with Professor A. Verschoren.

1. Preliminaries

Let us first take a quick look at some of the basic topics in torsion theories [2], [4].

(1.1) If $\mathcal{C}$ is a category with zero, the relation

$$(A, B) \in \Phi \Longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B) = 0$$

defines a Galois connection in the class $|\mathcal{C}|$ of the objects of $\mathcal{C}$

$$\begin{aligned} \mathcal{P}|\mathcal{C}| \underset{f}{\overset{t}{\rightleftarrows}} \mathcal{P}|\mathcal{C}| &\quad \text{by means of} \\ B \in f(X) \Longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B) = 0 &\quad \text{for every } A \in X \\ A \in t(Y) \Longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B) = 0 &\quad \text{for every } B \in Y, \end{aligned}$$

X and Y being classes of objects in $\mathcal{C}$.

(1.2) A class $X \in \mathcal{P}|\mathcal{C}|$ (resp. $Y \in \mathcal{P}|\mathcal{C}|$) is a *torsion class* (resp. a *torsion-free class*) if, and only if, $X = t(Y)$ (resp. $Y = f(X)$) for any $Y \in \mathcal{P}|\mathcal{C}|$ (resp. $X \in \mathcal{P}|\mathcal{C}|$).

As in every Galois connection there is an inclusion-reversing bijection between tosion classes and free torsion classes defined by f, with inverse t.

A *polar pair* is a pair of classes (T, F), such that $T = t(F)$ and $F = f(T)$. It constitutes a *torsion theory.*

(1.3) If (T, F) is a torsion theory in an abelian category $\mathcal{C}$ we have:

a) $F \cap T = \{0\}$.

b) T is closed under quotient objects, extensions and coproducts.

c) F is closed under products, subobjects and extensions.

If T is also closed under subobjects, we say that T is a *hereditary torsion class* and (T, F) is a *hereditary torsion theory.* In this case, F is closed under taking essential extensions.

(1.4) If $\mathcal{C}$ is an abelian $AB5$ category (i.e. $\mathcal{C}$ has exact inductive limits and coproducts) there is a bijection between hereditary torsion theories and *idempotent kernel functors* in $\mathcal{C}$ [4].

If $\sigma : \mathcal{C} \to \mathcal{C}$ is an idempotent kernel functor in $\mathcal{C}$, the corresponding torsion theory (T_σ, F_σ) is given by taking for T_σ the class of objects M in $\mathcal{C}$, such that $\sigma(M) = M$, and for F_σ those M with $\sigma(M) = 0$.

Conversely, if (T, F) is a hereditary torsion theory, the kernel functor which corresponds to (T, F) is defined, for $M \in |\mathcal{C}|$, by $\sigma(M) = \sum_{M \supseteq N \in T} N$; i.e. $\sigma(M)$ is the bigger torsion subobject of M.

(1.5) In an abelian $AB5$ category $\mathcal{C}$ if $T \subseteq |\mathcal{C}|$ is closed under quotients, extensions, coproducts and subobjects, we have that $(T, f(T))$ is a hereditary torsion theory.

2. Local smallness and local projectiveness of systems of generators

Local smallness and local projectiveness are introduced in this section, and we see that they become natural concepts in a Grothendieck category. $\mathcal{A}$ is always a category with arbitrary coproducts.

(2.1) Definition. Let $\mathcal{B} = \{A_i / i \in \mathbb{I}\}$ be a family of objects in $\mathcal{A}$. We shall say that $\mathcal{B}$ is a *locally projective* family in $\mathcal{A}$, if for each $f : A_i \to M'$ with $A_i \in \mathcal{B}$ and for every epimorphism $\varphi : M \to M'$ in $\mathcal{A}$ there exists an epimorphism $\xi : \coprod_{j \in \mathbb{I}_i^f} A_j \to A_i$, where A_j is also in $\mathcal{B}$ (henceforth a $\mathcal{B}$-covering $\xi = (\xi_j) : \coprod_{j \in \mathbb{I}_i^f} A_j \to A_i$), such that each $f \circ \xi_j$ factors through M, i.e. for each $j \in \mathbb{I}_i^f$ there is a $f_j : A_j \to M$ such that $\varphi \circ f_j = f \circ \xi_j$.

(2.2) Lemma. *If $\mathcal{A}$ is an abelian category with a system of generators $\mathcal{G}$, then $\mathcal{G}$ is a locally projective family.*

Proof: Since $\mathcal{A}$ has universal epimorphisms [3], if $\varphi : A \to A'$ is an epimorphism and $f \in \mathrm{Hom}_{\mathcal{A}}(G, A')$ we have a pullback diagram

$$\begin{array}{ccc} B & \xrightarrow[\varphi']{} & G \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\varphi]{} & A' \end{array}$$

with $\varphi' : B \to G$ epimorphism. We can cover B by $(\xi_j) : \coprod_j G_j \to B$ and $f' \circ \xi_j$ are the required factorizations. ∎

(2.3) Remark. An object A of $\mathcal{A}$ is said to be *small* if for any coproduct $\coprod_{\lambda\in\Lambda} M_\lambda$ and for any $f \in \mathrm{Hom}_{\mathcal{A}}(A, \coprod_{\lambda\in\Lambda} M_\lambda)$ there is a finite subset F of Λ such that f factors through $\coprod_{\lambda\in F} M_\lambda$ [3].

This is a strong and very restrictive concept. In fact, if $\mathcal{A}$ is a Grothendieck category with a small projective generator U, $\mathcal{A}$ is equivalent to R-mod where $R = \mathrm{Hom}_{\mathcal{A}}(U, U)$ [3].

Let us now set a weaker notion.

(2.4) Definition. We say that a family of objects $\mathcal{B} = \{A_i / i \in \mathbb{I}\}$ of $\mathcal{A}$ is *locally small* if for any $i \in \mathbb{I}$ and for every $f \in \mathrm{Hom}_{\mathcal{A}}(A_i, \coprod_{\lambda\in\Lambda} M_\lambda)$ there exists a $\mathcal{B}$-covering $(\xi_k) : \coprod_{k\in\mathbb{J}_i^f} A_k \to A_i$ such that each $f \circ \xi_k : A_k \to \coprod_{\lambda\in\Lambda} M_\lambda$ factors through a finite subcoproduct, i.e. for each $k \in \mathbb{J}_i^f$ there is a finite subset F_k of Λ and a $f_k \in \mathrm{Hom}_{\mathcal{A}}(A_k, \coprod_{\lambda\in F_k} M_\lambda)$ such that $j_{F_k} \circ f_k = f \circ \xi_k$, where $j_{F_k} : \coprod_{\lambda\in F_k} M_\lambda \to \coprod_{\lambda\in F_k} M_\lambda$ is the canonical inclusion.

The following example will be used in the main result of this section.

(2.5) Example. If $\sigma : R\text{-mod} \to R\text{-mod}$ is an idempotent kernel functor, we denote by (R, σ)-mod the full subcategory of the σ-closed R-modules, i.e. the R-modules that are both σ-torsionfree and σ-injective. (R, σ)-mod is a Grothendieck category with generator $Q_\sigma(R) = \varinjlim_{I\in\mathcal{L}_\sigma} \mathrm{Hom}_R(I, R/\sigma(R))$, where $\mathcal{L}_\sigma$ denotes the Gabriel filter of left ideals I of R, such that $\sigma(R/I) = R/I$. In general $Q_\sigma(R)$ is not a small nor projective object in (R, σ)-mod, but it yields a locally small system of generators, if we take enough copies of $Q_\sigma(R)$. Explicitely, if $\{M_\lambda / \lambda \in \Lambda\}$ is a family of objects in (R, σ)-mod we denote by $\coprod_{\lambda\in\Lambda} M_\lambda$

his coproduct in (R,σ)-mod, i.e. $\coprod_{\lambda\in\Lambda} M_\lambda = Q_\sigma(\bigoplus_{\lambda\in\Lambda} M_\lambda)$ and, since the class of σ-torsionfree is closed under products and subobjects we have that the localization morphism $j_\Lambda : \bigoplus_{\lambda\in\Lambda} M_\lambda \to \coprod_{\lambda\in\Lambda} M_\lambda$ is injective. Let $f \in \mathrm{Hom}_R(Q_\sigma(R), \coprod_{\lambda\in\Lambda} M_\lambda)$ and $y = f(1)$. Since $\mathrm{coker}(j_\Lambda)$ is a σ-torsion module, there is an $I^f \in \mathcal{L}_\sigma$ such that for every $a \in I^f$ the morphism $f \circ \bar{a} : Q_\sigma(R) \to \coprod_{\lambda\in\Lambda} M_\lambda$ factors through $\bigoplus_{\lambda\in\Lambda} M_\lambda$, where $\bar{a} : Q_\sigma(R) \to Q_\sigma(R)$ is the right-multiplication by a. Thus we have, for each $a \in I^f$ a R-linear map $f_a : Q_\sigma(R) \to \bigoplus_{\lambda\in\Lambda} M_\lambda$ such that the diagram

$$\begin{tikzcd} Q_\sigma(R) \arrow[r, "\bar{a}"] \arrow[d, "f_a"'] & Q_\sigma(R) \arrow[d, "f"] \\ \bigoplus_{\lambda\in\Lambda} M_\lambda \arrow[r, "j_\Lambda"'] & \coprod_{\lambda\in\Lambda} M_\lambda \end{tikzcd}$$

commutes. Now, $f_a(1) \in \bigoplus_{i=1}^{n_a} M_{\lambda_i}$ and so we have $f_a(Q_\sigma(R)) \subseteq \bigoplus_{i=1}^{n_a} M_{\lambda_i} = \coprod_{i=1}^{n_a} M_{\lambda_i}$ since $\bigoplus_{i=1}^{n_a} M_{\lambda_i}$ is also σ-closed. The morphism $\xi = (\bar{a})_{a\in I_f} : \coprod_{I^f} Q_\sigma(R) \to Q_\sigma(R)$ is an epimorphism in (R,σ)-mod, i.e. $\mathrm{coker}(\xi)$ is a σ-torsion R-module, because $I^f \in \mathcal{L}_\sigma$. This proves that a family of enough copies of $Q_\sigma(R)$ is a locally small system of generators of (R,σ)-mod.

The former example provides a very surprising consequence via the Gabriel-Popescu theorem [4].

(2.6) Theorem. *Every Grothendieck category $\mathcal{C}$ with a generator U has a locally small system of generators.*

Proof: By the Gabriel-Popescu theorem, $\mathcal{C}$ is equivalent, by means of the functor $\mathrm{Hom}_{\mathcal{C}}(U,-) : \mathcal{C} \to R$-mod, to the category (R,σ)-mod, where $R = \mathrm{Hom}_{\mathcal{C}}(U,U)$ and σ is an idempotent kernel functor for which $R = Q_\sigma(R)$. U corresponding to $R = \mathrm{Hom}_{\mathcal{C}}(U,U)$ in this equivalence, and R being a σ-closed module, the result follows from (2.5) because the notion of local smallness is preserved by equivalences. It then suffices to take enough copies of U to obtain the required locally small system of generators. ∎

(2.7) Corollary. *If $\mathcal{C}$ is a Grothendieck category with a system of generators $\mathcal{G} = \{G_i / i \in \mathbb{I}\}$ then $\mathcal{G}$ is a locally small family.*

Proof: Let $U = \coprod_{i \in \mathbb{I}} G_i$ be the *big* generator of $\mathcal{C}$. We denote by $p_i : U = G_i \times (\coprod_{j \neq i} G_j) \to G_i$ the canonical projection. Let $f : G_i \to \coprod_{\lambda \in \Lambda} M_\lambda$ be an arbitrary morphism and $f' = f \circ p_i$. Then by (2.6) we have a covering $\varphi : \coprod_k U_k \to U$ induced by $\varphi_k : U_k = U \to U$ such that for every $k \in \mathbb{K}$ there exists a $f_k : U_k \to \coprod_{\lambda \in F_k} M_\lambda$ (F_k is a finite subset of Λ) such that $j_{F_k} \circ f_k = f' \circ \varphi_k$. For every $k \in \mathbb{K}$ let $G_j^k = G_j$ and let $h_j^k : G_j^k \to U_k$ be the canonical monomorphism. If we put $f_j^k = f_k \circ h_j^k$, and $\xi_j^k = p_i \circ \varphi_k \circ h_j^k : G_j^k \to G_i$ we obtain a commutative diagram:

$$\begin{tikzcd}
G_j = G_j^k \arrow[r, "h_j^k"] \arrow[dr, "f_j^k"'] & U_k \arrow[r, "\varphi_k"] \arrow[d, "f_k"] & U \arrow[r, "p_i"] \arrow[d, "f'"] & G_i \arrow[dl, "f"] \\
& \coprod_{\lambda \in F_k} M_\lambda \arrow[r, "j_{F_k}"'] & \coprod_{\lambda \in \Lambda} M_\lambda &
\end{tikzcd}$$

Then $f \circ \xi_j^k$ factors through the finite coproduct $\coprod_{\lambda \in F_k} M_\lambda$, and furthermore, the morphism $\xi : \coprod_{k \in \mathbb{K}} \coprod_{j \in \mathbb{J}} G_j^k = \coprod_k U_k \to G_i$ induced by $(\xi_j^k)_{j,k} : G_j^k \to G_i$ is the epimorphism $p_i \circ \varphi$.

3. Gabriel filters and torsion theories

This part is devoted to establish the bijection between idempotent kernel functors (i.e. hereditary torsion theories) and some families of filters of subobjects of the generators in a Grothendieck category that we will denote by $\mathcal{C}$.

(3.1) Let $\mathcal{C}$ be a category and $\sigma : \mathcal{C} \to \mathcal{C}$ an idempotent kernel functor. We define, for each $M \in |\mathcal{C}|$, the Gabriel filter of subobjects of M, $\mathcal{L}_M^\sigma$, relative to σ, by $N \in \mathcal{L}_M^\sigma \Leftrightarrow M/N \in T_\sigma$.

(3.2) Proposition. *If $\mathcal{C}$ is a Grothendieck category with a system of generators $\mathcal{G} = \{G_i / i \in \mathbb{I}\}$ and if $\sigma : \mathcal{C} \to \mathcal{C}$ is an idempotent kernel functor then*

1) *$\mathcal{L}_G^\sigma \neq \phi$ for every $G \in \mathcal{G}$.*
2) *$I \subseteq J \subseteq G$, $I \in \mathcal{L}_G^\sigma \Rightarrow J \in \mathcal{L}_G^\sigma$.*
3) *If $G, G' \in \mathcal{G}$, $f \in \mathrm{Hom}_{\mathcal{C}}(G', G)$ and $I \in \mathcal{L}_G^\sigma$ then $f^{-1}(I) \in \mathcal{L}_{G'}^\sigma$,*

4) *Let $G \in \mathcal{G}$ and $I \subseteq J \in \mathcal{L}_G^\sigma$. If there is a $\mathcal{G}$-covering $(\xi_j) : \coprod_{j\in\mathbb{J}} G_j \to J$ such that $\xi_j^{-1}(I) \in \mathcal{L}_{G_j}^\sigma$ for each $j \in \mathbb{J}$, then $I \in \mathcal{L}_G^\sigma$.*

Proof: Let (T_σ, F_σ) be the hereditary torsion theory corresponding to σ. Then 1) is clear since $0 \in T_\sigma$. For 2) we consider the canonical epimorphism $G/I \to G/J$ and, since $G/I \in T_\sigma$ we have $G/J \in T_\sigma$ because T_σ is closed under quotients. 3) is a consequence of the facts that T_σ is closed under taking subobjects and that there is a monomorphism $G'/f^{-1}(I) \to G/I$.

We can prove 4) by considering the commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[dd] & \\
0 \arrow[dd] & & 0 \arrow[dd] \\
 & \xi_j^{-1}(I) \arrow[dl, dotted] \arrow[dr, dashed] \arrow[dd] & \\
\coprod_j \xi_j^{-1}(I) \arrow[dd] \arrow[rr] & & I \arrow[dd] \\
 & G_j \arrow[dl] \arrow[dr, dashed, "\xi_j"] \arrow[dd] & \\
\coprod_{j\in\mathbb{J}} G_j \arrow[dd] \arrow[rr, "\xi"] & & J \arrow[dd] \\
 & G_j/\xi_j^{-1}(I) \arrow[dl, dashed] \arrow[dr, dashed] \arrow[d] & \\
\coprod_{j\in\mathbb{J}} G_j/\xi_j^{-1}(I) \arrow[d] \arrow[rr, "\xi'"] & 0 & J/I \arrow[d] \\
0 & & 0
\end{tikzcd}
$$

with exact columns because the coproducts are exact in $\mathcal{C}$. $\xi_j' : G_j/\xi_j^{-1}(I) \to J/I$ is defined by ξ_j, and ξ' is an epimorphism by commutativity. Then, as T_σ is closed under coproducts and quotients, $J/I \in T_\sigma$ since $G_j/\xi_j^{-1}(I) \in T_\sigma$ by hypothesis. As well, the canonical exact sequence $0 \to J/I \to G/I \to G/J \to 0$ and the fact that T_σ is closed under extensions, shows that $G/I \in T_\sigma$ if G/J is so.

Note that if $I, J \in \mathcal{L}_G^\sigma$ then $I \cap J \in \mathcal{L}_G^\sigma$. In fact, if $(\xi_i) : \coprod_{i\in\mathbb{I}} G_i \to I$ is a $\mathcal{G}$-covering of I then $\xi_j^{-1}(I \cap J) = \psi_j^{-1}(J)$, being $\psi_j = h \circ \xi_j$ with h :

$I \to G$ the inclusion. Hence, condition 4) works because $\psi_j^{-1}(J) \in \mathcal{L}^{\sigma}_{G_j}$ for every $j \in \mathbb{I}$, by condition 3).

Conversely:

(3.3) Proposition. *If $\mathcal{C}$ is a Grothendieck category with a system of generators $\mathcal{G}$, and if $\{\mathcal{L}_G / G \in \mathcal{G}\}$ is a family of filters of subobjects of the objects G such that verifies jointly the properties 1) to 4) of (3.2) then there exists an idempotent kernel functor $\sigma : \mathcal{C} \to \mathcal{C}$ such that $\mathcal{L}^{\sigma}_G = \mathcal{L}_G$ for every $G \in \mathcal{G}$.*

Proof: We define a class T of objects in $\mathcal{C}$ saying that A is a member of T iff $\ker(f) \in \mathcal{L}_G$ for every $f \in \mathrm{Hom}_{\mathcal{C}}(G, A)$ and $G \in \mathcal{G}$.

With this definition T is a hereditary torsion class.

In fact, T is closed by subobjects because if $A' \subseteq A \in T$ and if $f' \in \mathrm{Hom}_{\mathcal{C}}(G, A')$, $\ker(f) = \ker(f')$ being $f = i \circ f'$, $i : A' \subseteq A$.

To show that T is closed under epimorphic images we shall use that $\mathcal{G}$ is a locally projective system of generators in $\mathcal{C}$ (2.2). If $\varphi : A \to A'$ is an epimorphism with $A \in T$ and if $f \in \mathrm{Hom}_{\mathcal{C}}(G, A')$ with $G \in \mathcal{G}$ let $(\xi_g) : \coprod_{g \in \mathbf{G}} G_g \to G$ be a $\mathcal{G}$-covering such that $f \circ \xi_g$ factors through $f_g : G_g \to A$. Let be $I = \ker(f)$ and put $I_g = \xi_g^{-1}(I)$. So $I_g \in \mathcal{L}_{G_g}$ since $\ker(f_g) \subseteq I_g$, and the condition 2) works. Also, by condition 4) applied to $I \subseteq G \in \mathcal{L}_G$ we obtain $I \in \mathcal{L}_G$. So $A' \in T$ and T is closed by quotients.

Let us now take $0 \to A' \xrightarrow{\psi} A \xrightarrow{\varphi} A'' \to 0$ an exact sequence with A', $A'' \in T$ and let $f \in \mathrm{Hom}_{\mathcal{C}}(G, A)$ for $G \in \mathcal{G}$. We must have $I = \ker(f) \in \mathcal{L}_G$. To show this we observe that, if $f'' = \varphi \circ f$, $\ker(f'')$ is the pullback of f and ψ

$$\begin{tikzcd}
& \xi_j^{-1}(I) \arrow[dl] \arrow[dr] & & \\
G_j \arrow[dr, "\xi_j"] \arrow[ddr, dotted, "f_j"'] & & I \arrow[dl] \arrow[d] & \\
& \ker(f'') \arrow[r] \arrow[d, "f'"] & G \arrow[d, "f"] \arrow[dr, "f''"] & \\
& A' \arrow[r, "\psi"'] & A \arrow[r, "\varphi"'] & A''
\end{tikzcd}$$

Let $(\xi_j) : \coprod_j G_j \to \ker(f'')$ be a $\mathcal{G}$-covering and put $f_j = f' \circ \xi_j$. Then

$\xi_j^{-1}(I) = \ker(f_j)$ and since $A' \in T$ we have $\xi_j^{-1}(I) \in \mathcal{L}_{G_j}$. Since $A'' \in T$ we obtain $\ker(f'') \in \mathcal{L}_G$ and so, the condition 4) applied to $I \subseteq \ker(f'')$ provides $I \in \mathcal{L}_G$. Hence $A \in T$ and so T is closed under extensions.

The local smallness property of $\mathcal{G}$ (2.7) is now applied in proving that T is closed under coproducts. Indeed, if $A_\lambda \in T$ for every $\lambda \in \Lambda$ and $f \in \mathrm{Hom}_{\mathcal{C}}(G, \coprod_\lambda A_\lambda)$ there exists a $\mathcal{G}$-covering $(\xi_j) : \coprod_{j \in \mathbb{J}_f} G_j \to G$ such that for every $j \in \mathbb{J}$ the morphism $\xi_j \circ f$ factors through a finite subcoproduct. So, for every $j \in \mathbb{J}$ there is a finite subset Λ_j of Λ and $f_j \in \mathrm{Hom}_{\mathcal{C}}(G_j, \coprod_{\lambda \in \Lambda_j} A_\lambda)$ such that $f \circ \xi_j = h_j \circ f_j$, $h_j : \coprod_{\lambda \in \Lambda_j} A_\lambda \to \coprod_{\lambda \in \Lambda} A_\lambda$ being the canonical monomorphism. As T is closed by extensions we have $\coprod_{\lambda \in \Lambda_j} A_\lambda \in T$ and so $\ker(f_j) = \xi_j^{-1}(\ker(f)) \in \mathcal{L}_{G_j}$. The condition 4) applied to $\ker(f) \subseteq G$ yields $\ker(f) \in \mathcal{L}_G$. Hence $\coprod_{j \in \Lambda} A_\lambda \in T$ and so T is closed under arbitrary coproducts.

Finally, let us see that for every $G \in \mathcal{G}$, if I is a subobject of G, $I \in \mathcal{L}_G$ iff $G/I \in T$. From the definition of T, we have $G/I \in T$ implies $I \in \mathcal{L}_G$. Conversely, suppose $I \in \mathcal{L}_G$ and let $f : G' \to G/I$ be an arbitrary morphism with $G' \in \mathcal{G}$ also. We must prove that $\ker(f) \in \mathcal{L}_{G'}$. Let $(\xi_j) : \coprod_{j \in \mathbb{J}^f} G_j \to G'$ be a $\mathcal{G}$-covering such that for each $j \in \mathbb{J}^f$ the morphism $f \circ \xi_j : G_j \to G/I$ factors through the canonical projection $\varphi : G \to G/I$. Let $f_j : G_j \to G$ be this factorization. Then we have $f_j^{-1}(I) = \ker(\varphi \circ f_j) = \ker(f \circ \xi_j) = \xi_j^{-1}(\ker(f))$, and so, as $I \in \mathcal{L}_G$, condition 3) yields $f_j^{-1}(I) \in \mathcal{L}_{G_j}$ for every $j \in \mathbb{J}^f$. As a consequence of condition 4) applied to $\ker(f) \subseteq G'$ we have $\ker(f) \in \mathcal{L}_{G'}$.

The assertions (1.5) and (1.4) can now be applied, and the proof is finished. ■

So we can state:

(3.4) Corollary. *If $\mathcal{C}$ is a Grothendieck category, there exists a bijection between hereditary torsion theories in $\mathcal{C}$ and families of Gabriel filters $\mathcal{L}_{G_i}$ of subobjects of the generators provided that they satisfy the conditions 1) to 4) of (3.2).*

Note that if the generators are projective and small, condition 4) of (3.2) can be reformulated.

(3.5) Proposition. *If $\mathcal{C}$ is a Grothendieck category with a system $\mathcal{G}$ of small and projective generators, then a family of filters $\{\mathcal{L}_G / G \in \mathcal{G}\}$ verifies the conditions 1) to 4) of (3.2) if, and only if, satisfies the conditions 1), 2), 3) and the following 4').*

4') Let $G \in \mathcal{G}$ and $I \subseteq J \in \mathcal{L}_G$ such that $f^{-1}(I) \in \mathcal{L}_{G'}$ for every $f \in \mathrm{Hom}_{\mathcal{C}}(G', J)$, with $G' \in \mathcal{G}$, then $I \in \mathcal{L}_G$.

Proof: It's obvious that 1), 2), 3), 4) imply 4'). Reciprocally, suppose that conditions 1), 2), 3), 4') hold, and let $I \subseteq J \in \mathcal{L}_G$. We must prove that if $(\xi_j) : \coprod_{j \in \mathbb{J}} G_j \to J$ is a $\mathcal{G}$-covering such that $\xi_j^{-1}(I) \in \mathcal{L}_{G_j}$ for each $j \in \mathbb{J}$, then $I \in \mathcal{L}_G$. For this, let $G' \in \mathcal{G}$ and $f \subset \mathrm{Hom}_{\mathcal{C}}(G', J)$. Since G' is projective, there exists an $f' \in \mathrm{Hom}_{\mathcal{C}}(G', \coprod_{j \in \mathbb{J}} G_j)$ such that $(\xi_j) \circ f' = f$. As G' is small, there is a finite subcoproduct $\coprod_{j \in F} G_j$ and a $g \in \mathrm{Hom}_{\mathcal{C}}(G', \coprod_{j \in F} G_j)$ such that $f' = h_F \circ g$, $h_F : \coprod_{j \in F} G_j \to \coprod_{j \in F} G_j$ being the canonical monomorphism. If $p_j : \coprod_{j \in F} G_j \to G_j$ is the canonical projection, let $g_j = p_j \circ g$. Now $g^{-1}(\coprod_{j \in F} \xi_j^{-1}(I)) = \cap_{j \in F} g_j^{-1}(\xi_j^{-1}(I))$ and so $g^{-1}(\coprod_{j \in F} \xi_j^{-1}(I)) \in \mathcal{L}_G$, by 2) because $g_j^{-1}(\xi_j^{-1}(I)) \in \mathcal{L}_G$, for every $j \in F$ by condition 3). Also, $g^{-1}(\coprod_{j \in F} \xi_j^{-1}(I)) \subseteq f^{-1}(I)$ and then $f^{-1}(I) \in \mathcal{L}_G$, by 2). Hence, condition 4') yields $I \in \mathcal{L}_G$.

(3.6) Remark. Note also that if the generators $\mathcal{G}$ are not small, one can prove that conditions 1), 2), 3) and 4') do not imply 4). In fact, there is a counterexample that we take from [**1**].

Counterexample. If $R = \prod_{n \in \mathbb{N}} K_n$ is a denumerable product of fields, then R is a self-injective and regular ring, and $R = Q_g(R)$ where g is the torsion theory defined by the Goldie topology in R. R is a projective generator in the Grothendieck category (R, g)-mod but it is not a small object because the Gabriel filter of g has not a basis of finitely generated ideals. We take now the class $\mathcal{L}$ of ideals of the form $\prod_{n \in C} K_n$ where $C \subseteq \mathbb{N}$ is a cofinite set. Then $\mathcal{L}$ verifies the properties 1) to 4') but not condition 4). In fact, if we take, in (R, g)-mod, the covering $(f_i) : \coprod_i R_i \to R$, defined by $f_i = h_i \circ p_i$, where $R_i = R$, $p_i : R \to K_i$ is the i^{th}-projection and $h_i : K_i \to R$ is the i^{th}-injection, for all i, we obtain $f_i^{-1}(0) = \prod_{n \in \mathbb{N} - \{i\}} K_n \in \mathcal{L}$. If condition 4) holds, this yields that $0 \in \mathcal{L}$, a contradiction.

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