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NOTES ON A CLASS OF SIMPLE C^*-ALGEBRAS WITH REAL RANK ZERO

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Dedicat a la memòria d'en Pere Menal,
un bon amic i un bon matemàtic

Abstract

A construction method is presented for a class of simple C^*-algebras whose basic properties -including their real ranks- can be computed relatively easily, using linear algebra. A numerical invariant attached to the construction determines whether a given algebra has real rank 0 or 1. Moreover, these algebras all have stable rank 1, and each nonzero hereditary sub-C^*-algebra contains a nonzero projection, yet there are examples in which the linear span of the projections is not dense. (This phenomenon was first exhibited by Blackadar and Kumjian.) The construction also produces easy examples of simple C^*-algebras with real rank 0 and stable rank 1 for which K_0 fails to be unperforated.

Introduction

The concept of real rank for complex C^*-algebras was introduced by Brown and Pedersen [7] to provide an algebraic invariant that agrees with the topological dimension of the spectrum in the commutative case – unlike the stable rank, which in the commutative case equals one plus the integer part of half the dimension of the spectrum. In general, C^*-algebras with real rank zero (that is, with the property that the set of invertible self-adjoint elements in the algebra is dense in the set of all self-adjoint elements) have interesting parallels with zero-dimensional topological spaces. Somewhat suprisingly, many C^*-algebras constructed from higher-dimensional building blocks turn out to have real rank zero.

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In particular, simple C^*-inductive limits of homogeneous C^*-algebras of arbitrarily large dimension "often" have real rank zero (and stable rank one), as proved in work of Blackadar, Bratteli, Elliott, Kumjian [**4**], Blackadar, Dădărlat, Rørdam [**5**], and Dădărlat, Nagy, Némethi, Pasnicu [**8**]. For example, the main results of [**5**] show that if a simple unital C^*-algebra A can be written as the C^*-inductive limit of a sequence of homogeneous C^*-algebras $M_{n_k}(C(X_k))$ with "slow dimension growth" (meaning that the X_k are connected and $\lim_{k\to\infty} n_k^{-1} \dim X_k = 0$), then A has stable rank one (that is, the set of invertible elements is dense in the algebra), and A has real rank zero if and only if there are enough projections in A to separate the tracial states. These results are based on substantial topological arguments.

Our aim here is to make available a class of examples of C^*-algebras with low real rank, whose structure is more transparent than that of the examples mentioned above, and thus more open to further investigation. Namely, we construct a class of simple unital C^*-algebras whose basic properties – including their real ranks – can be computed relatively easily, using linear algebra rather than topological methods. A numerical invariant attached to the construction determines whether a given algebra has real rank 0 or 1. Moreover, these algebras all have stable rank 1, and each nonzero hereditary sub-C^*-algebra contains a nonzero projection. In some of these algebras the linear span of the projections is not dense, providing new instances of a phenomenon first exhibited by Blackadar and Kumjian in [**6**]. For all the algebras A constructed, the partially ordered abelian group $K_0(A)$ is weakly unperforated, yet there are examples (with real rank 0) for which it fails to be unperforated.

The basic C^*-algebra background that we shall need can be found in many introductory books, such as [**11**]. For some basic K-theoretic concepts, we refer the reader to [**3**], and for the basic theory of partially ordered abelian groups we refer to [**12**]. Further, we adopt some of the notation from [**3**]; in particular, given a C^*-algebra A, we write $M_\infty(A)$ for the non-unital algebraic direct limit of the matrix algebras $M_n(A)$ with connecting homomorphisms $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, and we view $M_\infty(A)$ as a directed union of the algebras $M_n(A)$. Finally, we denote the orthogonal sum of m copies of a projection $p \in M_\infty(A)$ by $m.p$. In case $p \in M_n(A)$, we view $m.p$ as a block diagonal $mn \times mn$ matrix with m blocks equal to p.

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1. Ingredients

We fix the following notation and assumptions on which our class of examples is based. Let X be a nonempty separable compact Hausdorff space (not necessarily connected), and choose elements $x_1, x_2, \dots \in X$ such that $\{x_n, x_{n+1}, \dots\}$ is dense in X for each n. For all positive integers n and k, let

$$\delta_n : M_k(C(X)) \longrightarrow M_k(\mathbf{C}) \subsetneq M_k(C(X))$$

be the C^*-homomorphism given by evaluation at x_n. Moreover, let $\nu(1), \nu(2), \dots$ be positive integers such that $\nu(n) \mid \nu(n+1)$ for all n, and set $A_n = M_{\nu(n)}(C(X))$ for each n. Next choose unital block diagonal homomorphisms $\phi_n : A_n \to A_{n+1}$ of the form

$$\text{diag}(\text{identity}, \dots, \text{identity}, \delta_n, \dots, \delta_n),$$

that is, $\phi_n(a) = \text{diag}(a, \dots, a, \delta_n(a), \dots, \delta_n(a))$ for $a \in A_n$. Let α_n denote the number of identity maps used in the definition of ϕ_n, and set $\phi_{s,n} = \phi_{s-1}\phi_{s-2}\cdots\phi_n : A_n \to A_s$ for $s > n$. Finally, let A be the C^*-inductive limit of the sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots,$$

and for each n let $\eta_n : A_n \to A$ be the natural map from A_n to the inductive limit.

Key Assumption. In each of the maps ϕ_n, at least one identity map and at least one δ_n occurs. In other words, $0 < \alpha_n < \nu(n+1)/\nu(n)$.

The multiplicity pattern of the homomorphisms in these examples (if we just count totals of identity maps plus evaluation maps) is that of a UHF algebra in whose Bratteli diagram all the multiplicities are at least 2. Similar examples can be constructed based on any simple infinite-dimensional AF C^*-algebra, as indicated in Section 7, Remark 3.

Because of our key assumption, given an element $a \in A_n$, both a and $\delta_n(a)$ appear as blocks in the block diagonal form of $\phi_n(a)$. Both $\phi_n(a)$ and $\delta_{n+1}\phi_n(a)$ appear in turn as blocks in $\phi_{n+1}\phi_n(a)$; hence, $\phi_{n+1}\phi_n(a)$ is a block diagonal sum of blocks of the forms a, $\delta_n(a)$, $\delta_{n+1}(a)$, and so on. Thus for $s > n$, the matrix $\phi_{s,n}(a)$ is a block diagonal sum of blocks of the forms a, $\delta_n(a)$, $\dots$, $\delta_{s-1}(a)$, and each of these blocks appears at least once.

Lemma 1. *A is a simple unital C^*-algebra.*

Proof: Obviously A is unital. To prove simplicity, it suffices to show that the algebraic direct limit of this system is simple (see e.g. [**1**, Lemma 4.5]). Thus given any nonzero element $a \in A_n$, we need to show that $A_s\phi_{s,n}(a)A_s = A_s$ for some $s > n$. Since $\{x_n, x_{n+1}, \dots\}$ is dense in X, there is some $s > n$ for which $\delta_{s-1}(a) \neq 0$. This nonzero constant matrix appears as a diagonal block in $\phi_{s,n}(a)$, from which we conclude that $A_s\phi_{s,n}(a)A_s = A_s$ as desired. ∎

2. Stable rank one

We show that A has stable rank 1, using a method of Rørdam [**18**], which is based on the observation that all nilpotent elements in a unital C^*-algebra B lie in the closure of $GL(B)$ (cf. [**18**, (4.1)]).

Lemma 2. *Let $a \in A_n$ and $\epsilon > 0$. If a is not invertible, there exist $a' \in A_n$ and unitaries $v, w \in M_{\nu(s)}(\mathbf{C})$ for some $s > n$ such that $\|a' - a\| < \epsilon$ and $v\phi_{s,n}(a')w$ is nilpotent.*

Proof: Since a is not invertible, $a(x)$ is a singular matrix for some $x \in X$. Moreover, since $\{x_n, x_{n+1}, \dots\}$ is dense in X, there exists $j \geq n$ such that $\|a(x_j) - a(x)\| < \epsilon$. Set $a' = a + a(x) - a(x_j)$; then $\|a' - a\| < \epsilon$ and $\delta_j(a') = a(x)$ is singular.

Let $\nu = \nu(n)$. Since $\delta_j(a')$ is singular, there exist unitary matrices $u_1, u_2 \in M_\nu(\mathbf{C})$ such that $u_1\delta_j(a')u_2$ is a block diagonal matrix with a 1×1 zero block and a second $(\nu - 1) \times (\nu - 1)$ block.

Let $b = \phi_{j+1,n}(a')$ and observe that b is block diagonal with blocks $\nu \times \nu$ or smaller. Moreover, b has at least one block equal to $\delta_j(a')$. Thus there exist unitaries $u_3, u_4 \in M_{\nu(j+1)}(\mathbf{C})$ such that u_3bu_4 is block diagonal with at least one 1×1 zero block and all blocks at most $\nu \times \nu$.

Next set $s = j+1+\nu$ and $c = \phi_{s,j+1}(u_3bu_4)$, and observe that c is block diagonal with at least 2^ν blocks being 1×1 zero blocks and all blocks being at most $\nu \times \nu$. Conjugation of c by an appropriate permutation matrix can move ν of the zero blocks into adjacent positions on the diagonal. Thus there is a unitary $u_5 \in M_{\nu(s)}(\mathbf{C})$ such that $u_5cu_5^*$ is block diagonal with a $\nu \times \nu$ zero block in the upper left corner and all blocks at most $\nu \times \nu$.

Now $u_5cu_5^*$ can be multiplied on the left by a permutation matrix that moves the first ν rows to the bottom and shifts the remaining rows up, resulting in a strictly upper triangular matrix. Thus there exists a unitary $u_6 \in M_{\nu(s)}(\mathbf{C})$ such that $u_6 u_5 c u_5^*$ is strictly upper triangular, hence nilpotent.

We complete the proof by setting $v = u_6 u_5 \phi_{s,j+1}(u_3)$ and $w = \phi_{s,j+1}(u_4) u_5^*$. ■

Theorem 3. *The stable rank of A is 1.*

Proof: We show that $GL(A)$ is dense in A. For this, it suffices to show that $\eta_n(A_n)$ is contained in $GL(A)^-$ for each n.

Consider $a \in A_n$. If a is invertible, then $\eta_n(a) \in GL(A)$. If not, let $\epsilon > 0$ and use Lemma 2 to find $a_\epsilon \in A_n$ and unitaries $v, w \in M_{\nu(s)}(\mathbf{C})$ for some $s > n$ such that $\|a_\epsilon - a\| < \epsilon$ and $v\phi_{s,n}(a_\epsilon)w$ is nilpotent. Then $\eta_s(v)\eta_n(a_\epsilon)\eta_s(w)$ is a nilpotent element of A, whence $\eta_s(v)\eta_n(a_\epsilon)\eta_s(w) \in GL(A)^-$, and consequently $\eta_n(a_\epsilon) \in GL(A)^-$. Since $\|\eta_n(a_\epsilon) - \eta_n(a)\| < \epsilon$, we conclude that $\eta_n(a) \in GL(A)^-$. ■

In case $\dim X < \infty$, Theorem 3 follows immediately from [**8**, Theorem 3.6]. (Cf. also [**5**, Theorem 1].) Even if $\dim X = \infty$, it is still possible to apply [**5**, Theorem 1] in the following manner, as pointed out by Blackadar. First use [**16**, Theorem VII.3] to write X as an inverse limit of finite CW-complexes X_n. By repeating some X_n's if necessary, we may assume that $\dim X_n \le \nu(n)/n$ for all n. Now A is isomorphic to the C^*-inductive limit of a sequence of the form

$$M_{\nu(1)}(C(X_1)) \longrightarrow M_{\nu(2)}(C(X_2)) \longrightarrow M_{\nu(3)}(C(X_3)) \longrightarrow \cdots,$$

and this sequence satisfies the slow dimension growth hypothesis of [**5**, Theorem 1].

3. Projections in hereditary subalgebras

We next show that A has a large supply of projections – in particular, every nonzero hereditary sub-C^*-algebra of A contains a nonzero projection. This property was labelled (SP) in [**2**] and was investigated in [**6**] and [**17**]. Blackadar [private communication] has observed that (SP) holds in any C^*-inductive limit satisfying the slow dimension growth condition.

Lemma 4. *If $b \in A_n$ is self-adjoint with $\|b\| > 1$, there exists a positive element $c \in A_s$ for some $s > n$ such that $\|c\| < 1$ and $\phi_{s,n}(b)c\phi_{s,n}(b)$ is a nonzero projection.*

Proof: Since $\|b\| > 1$, there exists $j \ge n$ such that $\|\delta_j(b)\| > 1$. Since $\delta_j(b)$ is a self-adjoint matrix, $\delta_j(b) = \lambda_1 p_1 + \cdots + \lambda_m p_m$ for some nonzero orthogonal projections $p_i \in M_{\nu(n)}(\mathbf{C})$ and some $\lambda_i \in \mathbf{R}$ with $|\lambda_1| > 1$.

Set $b' = \phi_{j+1,n}(b)$, and note that b' is block diagonal with at least one block equal to $\delta_j(b)$. Thus there is a nonzero projection $q \in A_{j+1}$ such that $qb' = b'q = \lambda_1 q$. Then $c = \lambda_1^{-2} q$ is a positive element of A_{j+1} such that $\|c\| = |\lambda_1|^{-2} < 1$ and $b'cb' = \lambda_1^{-2} b'qb' = q$. ∎

Theorem 5. *Each nonzero hereditary sub-C^*-algebra of A contains a nonzero projection.*

Proof: It suffices to show that for any positive element $a \in A$ with $\|a\| = 1$, there is a nonzero projection in $(aAa)^-$. Choose a positive element $b_0 \in A_n$ for some n such that $\|\eta_n(b_0) - a\| < \frac{1}{128}$. Then $\frac{127}{128} < \|b_0\| < \frac{129}{128}$, and since $b_0 \geq 0$ we see that the element $b = b_0 + \frac{1}{128}$ satisfies $1 < \|b\| < \frac{65}{64}$. Hence, b is a positive element of A_n such that $\|\eta_n(b) - a\| < \frac{1}{64}$.

By Lemma 4, there exists a positive element $c \in A_s$ for some $s > n$ such that $\|c\| < 1$ and $\phi_{s,n}(b)c\phi_{s,n}(b)$ is a nonzero projection. Thus $p = \eta_n(b)\eta_s(c)\eta_n(b)$ is a nonzero projection in A, and $d = a\eta_s(c)a$ is a positive element of aAa such that

$$\|d-p\| \leq \|a\|\cdot\|c\|\cdot\|a-\eta_n(b)\|+\|a-\eta_n(b)\|\cdot\|c\|\cdot\|b\| < \frac{1}{64}+\frac{1}{64}\cdot\frac{65}{64} < \frac{1}{16}.$$

Consequently, $\|d\| > 1/2$ and

$$\|d^2-d\| \leq \|d\|\cdot\|d-p\|+\|d-p\|\cdot\|p\|+\|p-d\| < \frac{17}{16}\cdot\frac{1}{16}+\frac{1}{16}+\frac{1}{16} < \frac{1}{4}.$$

Therefore there exists a nonzero projection in the C^*-algebra $(aAa)^-$ (see e.g. [**11**, Lemma 19.8]). ∎

4. The linear span of the projections

In this section, we show that the linear span of the projections in A need not be dense in A, that is, A need not satisfy (LP), in the terminology of [**2**], [**17**]. It follows that A can have real rank 1.

For $s > n$, note that $\phi_{s,n}$ is a block diagonal map consisting of $\alpha_n\alpha_{n+1}\cdots\alpha_{s-1}$ identity maps together with maps from the list $\delta_n, \ldots, \delta_{s-1}$. Set

$$\omega_{s,n} = \alpha_n\alpha_{n+1}\cdots\alpha_{s-1}\frac{\nu(n)}{\nu(s)};$$

we might call this number the *weighted identity ratio* for $\phi_{s,n}$. Observe that

$$\omega_{s+1,n} = \omega_{s,n}\alpha_s\frac{\nu(s)}{\nu(s+1)}$$

and that $0 < \alpha_s \nu(s)/\nu(s+1) < 1$, whence $0 < \omega_{s+1,n} < \omega_{s,n}$. Thus there exists a limit for the sequence $\{\omega_{s,n}\}$ as $s \to \infty$.

Use $\operatorname{tr} a$ to denote the trace of a $\nu \times \nu$ matrix a, and recall that $|\operatorname{tr} a| \le \nu\|a\|$.

Theorem 6. *Assume that* $\lim_{t\to\infty} \omega_{t,1} = \epsilon > 0$, *and that* X *is not totally disconnected. Then the linear span of the projections in* A *is not dense, and* A *has real rank 1.*

Proof: Since A has stable rank 1, its real rank is at most 1 [**7**, Proposition 1.2]. In a C^*-algebra of real rank 0, the self-adjoint elements can be approximated by real linear combinations of orthogonal projections [**7**, Theorem 2.6], and so the complex linear span of the projections is dense in the algebra. Hence, the second conclusion of the theorem will follow from the first.

Since X is not totally disconnected, there exist distinct $y, z \in X$ which cannot be separated by clopen sets. By Urysohn's Lemma, there exists $f \in C(X)$ such that $f(y) = 0$ and $f(z) = 1$. Let $a = \operatorname{diag}(f, f, \ldots, f) \in A_1$.

Suppose there exists $b \in A$ such that b is a linear combination of projections and $\|b - \eta_1(a)\| < \epsilon/4$. Then there exists $c \in A_s$ for some $s > 1$ such that c is a linear combination of projections and $\|\eta_s(c) - b\| < \epsilon/4$. Thus $\|c - \phi_{s,1}(a)\| < \epsilon/2$.

Now $\omega_{s,1} > \lim_{t\to\infty} \omega_{t,1} = \epsilon$, and so $\phi_{s,1}(a)$ is a diagonal matrix with more than $\epsilon\nu(s)$ diagonal entries equal to f, while the remaining diagonal entries are constant. Evaluating at y and z and subtracting, we find that $\phi_{s,1}(a)(z) - \phi_{s,1}(a)(y)$ is a diagonal matrix with more than $\epsilon\nu(s)$ diagonal entries equal to 1, while the remaining entries are zero. Thus

$$\operatorname{tr}\big(\phi_{s,1}(a)(z) - \phi_{s,1}(a)(y)\big) > \epsilon\nu(s).$$

For any projection $p \in M_{\nu(s)}(C(X))$, the function $x \mapsto \operatorname{tr} p(x)$ is a continuous map from X to $\mathbf{Z}$, and so $\operatorname{tr} p(y) = \operatorname{tr} p(z)$. This equality must hold for linear combinations of projections as well, and hence $\operatorname{tr} c(y) = \operatorname{tr} c(z)$.

Finally, we have

$$|\operatorname{tr} c(x) - \operatorname{tr} \phi_{s,1}(a)(x)| \le \nu(s)\|c(x) - \phi_{s,1}(a)(x)\| < \nu(s)\epsilon/2$$

for all $x \in X$. However, since $\operatorname{tr} c(y) = \operatorname{tr} c(z)$, this implies that

$$|\operatorname{tr} \phi_{s,1}(a)(z) - \operatorname{tr} \phi_{s,1}(a)(y)| < \nu(s)\epsilon,$$

contradicting our previous estimate.

Therefore $\eta_1(a)$ cannot be approximated to within $\epsilon/4$ by a linear combination of projections. ■

In case X is connected, Theorem 6 can be obtained from [**4**, Theorem 1.3]. (Cf. also [**5**, Theorem 2]. If $\dim X = \infty$, the argument at the end of Section 2 is needed.) However, to use [**4**, Theorem 1.3] requires verifying that the projections in A do not separate the tracial states, which is about as labor-intensive as proving Theorem 6.

When the hypotheses of Theorem 6 are satisfied, A provides an example of a simple unital C^*-algebra satisfying (SP) but not (LP). The first examples of this phenomenon were constructed by Blackadar and Kumjian [**6**, Example 1.6, Corollary 1.10]. An explicit example in our format may be constructed as follows.

Example 7. *A simple unital C^*-algebra A with stable rank 1 and real rank 1 such that each nonzero hereditary sub-C^*-algebra of A contains a nonzero projection, but such that the linear span of the projections is not dense in A.*

Proof: Choose integers $m_1, m_2, \cdots \geq 2$ such that

$$\prod_{k=1}^{\infty} \frac{m_k - 1}{m_k} > 0.$$

Construct A as above, where X is not totally disconnected, $\nu(1) = 1$ and $\nu(n+1) = m_1 m_2 \cdots m_n$, and $\alpha_n = m_n - 1$. Then

$$\omega_{t,1} = \alpha_1 \alpha_2 \cdots \alpha_{t-1} \frac{\nu(1)}{\nu(t)} = \prod_{k=1}^{t-1} \frac{m_k - 1}{m_k}$$

for $t > 1$, whence $\lim_{t\to\infty} \omega_{t,1} > 0$. The desired properties of A follow from Theorems 3, 5, 6. ■

5. Real rank zero

We now determine exactly when A has real rank 0. First we record an easy observation that appears, for instance, in the proof of [**8**, Lemma 3.3].

Lemma 8. *Let q be a projection in a unital C^*-algebra B such that $q \precsim 1-q$, and let b be a self-adjoint element of qBq. Then b is a limit of invertible self-adjoint elements from B.*

Proof: Let $\epsilon > 0$. Write $1-q$ as an orthogonal sum of projections q', q'' with $q' \sim q$. If x is an invertible self-adjoint element of $(q+q')B(q+q')$ such that $\|x-b\| < \epsilon$, then $x+\epsilon q''$ is an invertible self-adjoint element of B such that $\|x+\epsilon q''-b\| < \epsilon$. Thus without loss of generality, $q \sim 1-q$. Hence, we may identify B with a matrix algebra $M_2(C)$, where C is a unital C^*-algebra (isomorphic to qBq), and $b = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ for some self-adjoint element $c \in C$. Then $a = \begin{pmatrix} c & \epsilon \\ \epsilon & 0 \end{pmatrix}$ is an invertible self-adjoint element of B such that $\|a-b\| \le \epsilon$. ■

Theorem 9. *A has real rank* 0 *if and only if either* $\lim_{t\to\infty} \omega_{t,1} = 0$ *or X is totally disconnected.*

Proof: If the given limit is positive and X is not totally disconnected, then A has real rank 1 by Theorem 6. If X is totally disconnected, then $\dim X = 0$ and so A has real rank 0 by [**7**, Propositions 1.1, 3.1, Theorem 2.10]. Now assume that the given limit is zero. Since $\omega_{t,1} = \omega_{t,n}\alpha_1\alpha_2\cdots\alpha_{n-1}\nu(1)/\nu(n)$ for $t > n$, we also have

$$\lim_{t\to\infty} \omega_{t,n} = 0$$

for all n.

To show that A has real rank 0, it suffices to show that for any self-adjoint element $a \in A_n$ and any $\epsilon > 0$, there is an invertible self-adjoint element $z \in A$ such that $\|z - \eta_n(a)\| < \epsilon$.

We are done if a is invertible, so assume not. Then $\det a$ vanishes somewhere on X, and so there exists $s > n$ such that $|\det \delta_s(a)| < (\epsilon/2)^{\nu(n)}$. Now $\delta_s(a)$ is a $\nu(n) \times \nu(n)$ matrix, and the product of its eigenvalues (with multiplicities) equals $\det \delta_s(a)$, so $\delta_s(a)$ must have an eigenvalue λ with $|\lambda| < \epsilon/2$.

Consider the element $b = \phi_{s+1,n}(a) \in A_{s+1}$. Then b is block diagonal with blocks of the form a or $\delta_j(a)$, and at least one block equal to $\delta_s(a)$. Set $\alpha = \alpha_n\alpha_{n+1}\cdots\alpha_s$; then b has α blocks equal to a.

Since $\lim_{t\to\infty} \omega_{t,s+1} = 0$, there exists $t > s+1$ such that $\omega_{t,s+1} < 1/(\alpha\nu(n)+1)$. Set $c = \phi_{t,n}(a) \in A_t$, and note that c is block diagonal with "big blocks" of the form b or $\delta_j(b)$. Each of these big blocks is itself block diagonal with "small blocks" of the form a or $\delta_k(a)$.

Now $\alpha_{s+1}\alpha_{s+2}\cdots\alpha_{t-1}\nu(s+1)/\nu(t) = \omega_{t,s+1} < 1/(\alpha\nu(n)+1)$, and hence

$$\alpha\nu(n)\alpha_{s+1}\alpha_{s+2}\cdots\alpha_{t-1} < \frac{\nu(t)}{\nu(s+1)} - \alpha_{s+1}\alpha_{s+2}\cdots\alpha_{t-1}.$$

The right hand side of this inequality is the number of big blocks of the form $\delta_j(b)$ in c. Each b-block contains only α small blocks equal to a, while each $\delta_j(b)$-block contains at least one small block equal to $\delta_s(a)$. Therefore in the small block decomposition of c, we see that $\nu(n)$ times the number of a-blocks is less than the number of $\delta_s(a)$-blocks.

We next break up each of the small blocks in c into three pieces as follows. Diagonalize $\delta_s(a)$, and write $\delta_s(a) = 0 + \lambda q_a + a'$ for some rank 1 projection q_a and some a' in $(1-q_a)M_{\nu(n)}(\mathbf{C})(1-q_a)$. Write each of the other $\delta_j(a)$-blocks as $0+0+\delta_j(a)$. Finally, write each a-block as $a+0+0$.

Collecting and summing the projections corresponding to these decompositions, we obtain orthogonal projections q_1, q_2, q_3 in $M_{\nu(t)}(\mathbf{C})$ such that

$q_1 + q_2 + q_3 = 1$;

q_1, q_2, q_3 all commute with c;

$q_1 c$ is a block diagonal sum of a-blocks;

$q_2 c = \lambda q_2$;

$q_3 c$ is a constant matrix.

The matrix size of q_2 equals the number of $\delta_s(a)$-blocks in c, which we have arranged to be greater than $\nu(n)$ times the number of a-blocks in c, hence greater than the matrix size of q_1. Thus $q_1 \lesssim q_2$.

By Lemma 8, there is an invertible self-adjoint element x in $(q_1+q_2)A_t(q_1+q_2)$ such that $\|x - q_1 c\| < \epsilon/2$. Since $|\lambda| < \epsilon/2$, we get $\|x - (q_1+q_2)c\| < \epsilon$. Finally, $q_3 c$ lies in a finite-dimensional sub-C^*-algebra of $q_3 A_t q_3$, and so there is an invertible self-adjoint element $y \in q_3 A_t q_3$ such that $\|y - q_3 c\| < \epsilon$. Hence, $x+y$ is an invertible self-adjoint element of A_t such that $\|x+y-c\| < \epsilon$.

Therefore $\eta_t(x+y)$ is an invertible self-adjoint element of A that lies within ϵ of $\eta_n(a)$. ∎

In case X is connected and $\dim X < \infty$, Theorem 9 can be obtained from [**5**, Theorem 2]; if $\dim X \leq 2$, it can also be obtained from [**4**, Theorem 1.3]. (Cf. the comments following Theorem 6.) It is also possible to apply [**5**, Theorem 2] in the general case by writing A as a C^*-inductive limit of a different sequence of homogeneous C^*-algebras, as discussed at the end of Section 2.

It is easy to choose the parameters $\nu(n)$ and α_n such that the algebra A has real rank 0. For example, if $\nu(n) = 2^n$ and $\alpha_n = 1$ for all n, then $\omega_{t,1} = 1/2^{t-1}$ for all t, and thus A has real rank 0 by Theorem 9.

6. Perforation in K_0

In this section, we show that perforation occurs in $K_0(A)$ only if this group has torsion. We also provide an explicit calculation of $K_0(A)$ in the case that X is connected. Note that because A is simple, every nonzero projection in $M_\infty(A)$ is full. Consequently, every nonzero element of $K_0(A)^+$ is an order-unit, i.e., $K_0(A)$ is a simple ordered group.

Recall that a simple partially ordered abelian group G is *weakly unperforated* provided that whenever $m \in \mathbf{N}$ and $x \in G$ satisfy $mx > 0$, it follows that $x > 0$. Thus G is unperforated, that is, $mx \geq 0$ implies $x \geq 0$, precisely if G is both weakly unperforated and torsionfree. See [**20**, Section 8] for a discussion of weak unperforation in non-simple ordered groups.)

We write $p \prec q$ for projections $p, q \in M_\infty(A)$ to mean that $p \precsim q$ while $p \not\sim q$.

Theorem 10. *$K_0(A)$ is weakly unperforated.*

Proof: Since A has stable rank 1, it has cancellation of projections, and so we just need to show that if p, q are projections in $M_\infty(A)$ with $m.p \prec m.q$ for some integer $m \geq 2$, then $p \prec q$. There is a nonzero projection $r \in M_\infty(A)$ such that $m.p \oplus r \sim m.q$.

After replacing p, q, r by equivalent projections, we may assume that there are projections e, f, g in $M_k(A_n)$ for some k, n such that $\eta_n(e) = p$, $\eta_n(f) = q$, $\eta_n(g) = r$. After increasing n if necessary, we may also assume that $m.e \oplus g \sim m.f$. Note that $g \neq 0$, and that $m.\delta_j(e) \oplus \delta_j(g) \sim m.\delta_j(f)$ for all j. Thus $\delta_j(e) \precsim \delta_j(f)$, and $\delta_j(e) \prec \delta_j(f)$ if $\delta_j(g) \neq 0$.

Choose $s > n$ such that $\delta_{s-1}(g) \neq 0$. Set $e' = \phi_{s,n}(e)$ and $f' = \phi_{s,n}(f)$ in $M_k(A_s)$. Then e' is block diagonal with blocks of the form e or $\delta_j(e)$. Hence, we can write $e' = e_1 \oplus e_2$ where e_1 is block diagonal with all blocks of the form e while e_2 is block diagonal with all blocks of the form $\delta_j(e)$. Write $f' = f_1 \oplus f_2$ in the same manner. Then $m.e_1 \prec m.f_1$ and $e_2 \prec f_2$. More precisely, e_2 and f_2 are constant projections, and $\operatorname{rank} f_2 > \operatorname{rank} e_2$.

Choose $t > s$ such that $2^{t-s} \geq mk\nu(n)$, and set $e'' = \phi_{t,n}(e)$ and $f'' = \phi_{t,n}(f)$ in $M_k(A_t)$. Write $e'' = e_3 \oplus e_4$ where e_3 is block diagonal with all blocks of the form e while e_4 is block diagonal with all blocks of the form $\delta_j(e)$. Write $f'' = f_3 \oplus f_4$ in the same manner. Then $m.e_3 \prec m.f_3$, while e_4 and f_4 are constant projections with $\operatorname{rank} f_4 \geq 2^{t-s} + \operatorname{rank} e_4$.

If $\alpha = \alpha_n \alpha_{n+1} \cdots \alpha_{t-1}$, then e_3 is block diagonal with exactly α blocks equal to e, and so $e_3 \sim \alpha.e$. Similarly, $f_3 \sim \alpha.f$. Write $\alpha = m\beta + \gamma$ for some nonnegative integers β, γ with $\gamma < m$. Since $m.e \prec m.f$, we get $m\beta.e \prec m\beta.f$. On the other hand, $\gamma k\nu(n) < mk\nu(n) \leq 2^{t-s}$, and so

$$\operatorname{rank} f_4 > \gamma k\nu(n) + \operatorname{rank} e_4.$$

Since e is equivalent to a projection in $M_{k\nu(n)}(C(X))$, it is subequivalent to a constant projection of rank $k\nu(n)$. Hence, $e_4 \oplus \gamma.e \prec f_4$, and thus

$$e_3 \oplus e_4 \sim m\beta.e \oplus \gamma.e \oplus e_4 \prec m\beta.f \oplus f_4 \prec \alpha.f \oplus f_4 \sim f_3 \oplus f_4.$$

Therefore $e'' \prec f''$, and consequently $p \prec q$, as desired. ∎

Theorem 10 can also be obtained from trace arguments, namely by proving that if p and q are any projections in $M_\infty(A)$ such that $\tau(p) < \tau(q)$ for all tracial states τ, then $p \lesssim q$. The latter result follows immediately from [**15**, Theorem 3.7] in case X is connected and $\dim X < \infty$; in general, it follows after modifying the sequence $A_1 \to A_2 \to \ldots$ in the manner described at the end of Section 2.

Given a partially ordered abelian group G with an order-unit u, we write $S(G,u)$ for the set of states on (G,u), that is, the positive real-valued group homomorphisms s on G satisfying $s(u) = 1$.

The following corollary can also be obtained from the argument at the end of Section 2 together with the arguments in [**9**, Lemma 2.1.10, Proposition 2.2.3, Corollary 2.2.4].

Corollary 11. *The partial ordering on $K_0(A)$ equals the strict ordering inherited from the states, that is,*

$$K_0(A)^+ = \{0\} \cup \{x \in K_0(A) \mid s(x) > 0 \text{ for all } s \in S(K_0(A),[1_A])\}.$$

Proof: If x is a nonzero element of $K_0(A)^+$, then $x = [p]$ for some nonzero projection $p \in M_\infty(A)$. Since A is simple, p is full, and so there exists a positive integer k such that $1_A \lesssim k.p$. Then $[1_A] \leq kx$ in $K_0(A)$, and hence $s(x) \geq 1/k$ for all $s \in S(K_0(A),[1_A])$.

On the other hand, if y is an element of $K_0(A)$ such that $s(y) > 0$ for all s in $S(K_0(A),[1_A])$, it follows from [**12**, Theorem 4.12] that there exists a positive integer m such that $my > 0$. Therefore $y > 0$ by Theorem 10. ∎

The proof of the following example can be streamlined in case the space X is connected, by appealing to Theorem 13. However, we prefer to exhibit the easy direct argument.

Example 12. *A simple unital C^*-algebra A, with stable rank 1 and real rank 0, such that $K_0(A)$ is weakly unperforated but not unperforated.*

Proof: Choose X so that $K_0(C(X))$ has 2-torsion; for example, $X = \mathbf{RP}_n$ for any $n \geq 2$ [**14**, IV.6.47]. Then there exist projections $p, q, r' \in M_\infty(C(X))$ such that $2.p \oplus r' \sim 2.q \oplus r'$ while $p \oplus r \not\sim q \oplus r$ for all projections $r \in M_\infty(C(X))$. After replacing p and q by $p \oplus r'$ and $q \oplus r'$, we may assume that $2.p \sim 2.q$.

Now construct A as above with $\nu(n) = 2^n$ and $\alpha_n = 1$ for all n. Then A has stable rank 1 by Theorem 3 and $K_0(A)$ is weakly unperforated by Theorem 10. Since $\omega_{t,1} = 2^{1-t}$ for all $t > 1$, we also have $\lim_{t\to\infty} \omega_{t,1} = 0$, and so A has real rank 0 by Theorem 9.

We may view p and q as projections in $M_{2^n}(C(X)) = A_n$ for some n. Then $\eta_n(p)$ and $\eta_n(q)$ are projections in A such that $2.\eta_n(p) \sim 2.\eta_n(q)$ and so $2[\eta_n(p)] = 2[\eta_n(q)]$ in $K_0(A)$.

Suppose that $[\eta_n(p)] = [\eta_n(q)]$. Then $\eta_n(p) \sim \eta_n(q)$ because A has stable rank 1. Hence, there is some $s > n$ such that $\phi_{s,n}(p) \sim \phi_{s,n}(q)$. Also, $2.\phi_{s,n}(p) \sim 2.\phi_{s,n}(q)$. Observe that $\phi_{s,n}(p) = p \oplus p'$ and $\phi_{s,n}(q) = q \oplus q'$ where p' and q' are constant projections. Then

$$2.p \oplus 2.p' \sim 2.q \oplus 2.q',$$

whence $2.\delta_1(p) \oplus 2.p' \sim 2.\delta_1(q) \oplus 2.q'$. Since $2.p \sim 2.q$ implies that $2.\delta_1(p) \sim 2.\delta_1(q)$, we conclude that $p' \sim q'$. However, we also have

$$p \oplus p' = \phi_{s,n}(p) \sim \phi_{s,n}(q) = q \oplus q',$$

contradicting our assumption that $p \oplus r \not\sim q \oplus r$ for all projections $r \in M_\infty(C(X))$.

Thus $[\eta_n(p)] \neq [\eta_n(q)]$, and consequently $[\eta_n(p)] - [\eta_n(q)]$ is a nonzero torsion element in $K_0(A)$. Therefore $K_0(A)$ is not unperforated. ∎

Blackadar has pointed out another example with the properties of the one just constructed, namely the tensor product of the CAR algebra with the simple C^*-algebra B given in [**3**, Exercise 10.11.2].

We conclude this section by computing $K_0(A)$ in terms of $K_0(C(X))$ when X is connected. Recall that in this case there is a unique state on $(K_0(C(X)), [1_{C(X)}])$ (see [**3**, Exercise 6.10.3]). This state is obtained by taking ranks of projections, and so it is integer-valued. The kernel of this state is called the *reduced* K_0 group of $C(X)$; it is usually denoted $\widetilde{K}_0(C(X))$, and it is naturally isomorphic to the corresponding reduced K^0-group of the space X, denoted $\widetilde{K}^0(X)$.

Since the computation of $K_1(A)$ in terms of $K_1(C(X))$ is quite easy (whether or not X is connected), we include this as a companion to our computation of $K_0(A)$.

Theorem 13. *Let* $U = \bigcup_{n=1}^{\infty} \mathbf{Z}\nu(n)^{-1} \subseteq \mathbf{Q}$ *and* $V = \bigcup_{n=1}^{\infty} \mathbf{Z}(\alpha_1\alpha_2\cdots\alpha_n)^{-1} \subseteq \mathbf{Q}$.

(a) $K_1(A) \cong V \bigotimes K_1(C(X))$.

(b) *Now assume that* X *is connected. Set* $(G,u) = (K_0(C(X)), [1_{C(X)}])$, *let* t *be the unique state on* (G,u), *and let* $W = U \bigoplus (V \bigotimes \ker t)$. *Set* $w = (1,0) \in W$, *and give* W *the strict ordering from the component* U, *that is, make* W *into a partially ordered abelian group with positive cone*

$$W^+ = \{(0,0)\} \cup \{(a,b) \in W \mid a > 0\}.$$

Then $(K_0(A),[1_A]) \cong (W,w)$ *as partially ordered abelian groups with order-unit.*

Proof: Set $\alpha_0 = 1$, and set $\beta_n = \nu(n+1)\nu(n)^{-1} - \alpha_n$ for $n \geq 1$.

(a) Let $G_1 = K_1(C(X))$, and identify $K_1(A_n)$ with G_1 for all n. Let us use additive notation for these abelian groups. For any unitary matrix $u \in U_{\nu(n)}(C(X))$, the induced map $K_1(\phi_n) : G_1 \to G_1$ sends the class $[u]$ to the class $\alpha_n[u] + \beta_n[\delta_n(u)]$. Since the unitary group of $M_{\nu(n+1)}(\mathbf{C})$ is connected, the class of $\delta_n(u)$ vanishes in $K_1(\mathbf{C})$, and hence it also vanishes in G_1. Thus $K_1(\phi_n)([u]) = \alpha_n[u]$. In other words, the homomorphism $K_1(\phi_n)$ is just multiplication by α_n. Since the functor K_1 preserves direct limits (e.g., [**3**, p. 68]), $K_1(A)$ is isomorphic to the direct limit, in the category of abelian groups, of the sequence

$$G_1 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_2} G_1 \xrightarrow{\alpha_3} \cdots .$$

Therefore we conclude that $K_1(A) \cong V \bigotimes G_1$.

(b) We may identify $(K_0(A_n),[1_{A_n}])$ with $(G,\nu(n)u)$ for all n. Set

$$f_n = K_0(\phi_n) : (G,\nu(n)u) \longrightarrow (G,\nu(n+1)u).$$

For any projection $p \in M_\infty(A_n)$, observe that $\phi_n(p)$ is a block diagonal matrix consisting of α_n blocks equal to p followed by β_n blocks equal to a constant projection of the same rank as p. Thus $\phi_n(p) \sim \alpha_n.p \oplus \beta_n(\operatorname{rank} p).1_{C(X)}$, and consequently $f_n([p]) = \alpha_n[p] + \beta_n t([p])u$. Therefore

$$f_n(x) = \alpha_n x + \beta_n t(x)u$$

for all $x \in G$.

Next let (H,v) be the direct limit, in the category of partially ordered abelian groups with order-unit, of the sequence

$$(G,\nu(1)u) \xrightarrow{f_1} (G,\nu(2)u) \xrightarrow{f_2} (G,\nu(3)u) \xrightarrow{f_3} \cdots ,$$

with natural maps $g_n : (G, \nu(n)u) \to (H, v)$. Since the functor K_0 preserves direct limits (e.g. [**11**, Theorem 19.9]), we have $(K_0(A), [1_A]) \cong (H, v)$.

Let $K = \ker t$. Since t is a group homomorphism from G to $\mathbf{Z}$ and $t(u) = 1$, there is a direct sum decomposition $G = \mathbf{Z}u \bigoplus K$ (as abelian groups, not necessarily as ordered groups). From the form of the maps f_n, we see that $f_n(\mathbf{Z}u) \subseteq \mathbf{Z}u$ and $f_n(K) \subseteq K$ for all n. Hence, as an abelian group H is the direct sum of the direct limits of the sequences

$$\mathbf{Z}u \xrightarrow{f_1} \mathbf{Z}u \xrightarrow{f_2} \cdots \quad \text{and} \quad K \xrightarrow{f_1} K \xrightarrow{f_2} \cdots .$$

The direct limit of the first sequence is just U, with natural maps $\mathbf{Z}u \to U$ sending u to $\nu(n)^{-1}$. The direct limit of the second sequence is $V \bigotimes K$, with natural maps $K \to V \bigotimes K$ sending x to $(\alpha_1\alpha_2 \cdots \alpha_{n-1})^{-1} \otimes x$. Consequently, there is an abelian group isomorphism $h : H \to W$ such that $hg_n(u) = (\nu(n)^{-1}, 0)$ and $hg_n(x) = (0, (\alpha_1\alpha_2 \cdots \alpha_{n-1})^{-1} \otimes x)$ for all n and all $x \in K$. Thus

$$hg_n(y) = (\nu(n)^{-1}t(y),\ (\alpha_1\alpha_2 \cdots \alpha_{n-1})^{-1} \otimes (y - t(y)u))$$

for all n and all $y \in G$.

Any nonzero element $x \in G^+$ satisfies $t(x) > 0$ by Corollary 11, whence $hg_n(x) > 0$ for each n. Thus $hg_n(G^+) \subseteq W^+$ for all n. Since $H^+ = \bigcup_{n=1}^{\infty} g_n(G^+)$, it follows that $h(H^+) \subseteq W^+$. Note also that $h(v) = hg_1(\nu(1)u) = w$.

The projection $t' : W \to U \subseteq \mathbf{R}$ is a state on (W, w), and since h is a positive homomorphism sending v to w, it follows that $t'h$ is a state on (H, v). There is just one state on $(G, \nu(n)u)$ for each n, and so $t'h$ must be the only state on (H, v). By Corollary 11,

$$H^+ = \{0\} \cup \{x \in H \mid t'h(x) > 0\},$$

and thus $h(H^+) = W^+$.

Therefore $h : (H, v) \to (W, w)$ is an isomorphism of partially ordered abelian groups with order-unit. ■

Observe that the partially ordered abelian group W given in Theorem 13 satisfies the Riesz interpolation property, and hence also the equivalent Riesz decomposition property [**12**, Proposition 2.1]. In fact, $K_0(A)$ satisfies Riesz decomposition whether or not X is connected, as follows from Theorem 10 and [**13**, Corollary 4.7].

7. Concluding remarks

1. Gong and Lin have proved that when A has real rank zero, its exponential rank is at most $1+\epsilon$ [**10**, Theorem 1.3], that is, the set of exponential unitaries in A is dense in the connected component of the identity of the unitary group of A.

2. Blackadar has raised the interesting question of how extensive the class of C^*-algebras constructed here might be. More precisely, if B is a simple unital C^*-inductive limit of a sequence of C^*-algebras of the form $M_{n_k}(C(X_k))$ (with arbitrary unital connecting homomorphisms), is B isomorphic to one of the algebras A above? In particular, does this hold when B has real rank zero, or when B is obtained from a C^*-inductive limit with slow dimension growth in the sense of [**5**]? Of course if B is to be isomorphic to an algebra A constructed using a connected space X, then $K_0(B)$ and $K_1(B)$ must necessarily have the forms described in Theorem 13.

Thomsen has observed that for simple C^*-inductive limits with real rank 1, the question above has a negative answer, as follows. On the one hand, given any metrizable Choquet simplex S, there exists a simple C^*-algebra B, obtained as the C^*-inductive limit of a sequence of algebras of the form $M_{n_k}(C([0,1]))$, such that the tracial state space $T(B)$ is affinely homeomorphic to S [**19**, Theorem 3.9]. As long as S contains more than one point, it follows from [**19**, Theorem 1.4] and [**7**, Proposition 1.2] that B has real rank 1. On the other hand, when an algebra A as constructed above has real rank 1, it can be shown that $T(A)$ is affinely homeomorphic to the Bauer simplex $M_1^+(X)$ of all probability measures on X; in particular, the extreme boundary of $T(A)$ is compact. Thus if the simplex S is chosen with non-compact extreme boundary, the algebra B cannot be isomorphic to any of the algebras A above.

3. As mentioned earlier, we can also carry out our construction following the pattern of any simple infinite-dimensional AF C^*-algebra. (These are precisely the AF C^*-algebras that can be obtained from Bratteli diagrams in which all the multiplicities are at least 2.) The algebras A_n would then have the form

$$M_{\nu(n,1)}(C(X)) \times \cdots \times M_{\nu(n,r(n))}(C(X)),$$

and the homomorphisms ϕ_n would be tuples of block diagonal sums of homomorphisms

$$M_{\nu(n,i)}(C(X)) \longrightarrow M_\bullet(C(X))$$

of the form diag(identity, ..., identity, $\delta_n, \ldots, \delta_n$) with at least one identity map and at least one δ_n. The resulting C^*-inductive limits enjoy the same properties as the examples discussed above, provided the weighted identity ratios $\omega_{s,n}$ for $s > n$ are redefined as follows.

First, write $\phi_{s,n}$ as a $\tau(s)$-tuple of homomorphisms $A_n \to M_{\nu(s,j)}(C(X))$, and write the j^{th} component of $\phi_{s,n}$ as a block diagonal sum of homomorphisms

$$\phi_{s,n}^{j,i} : M_{\nu(n,i)}(C(X)) \longrightarrow M_\bullet(C(X)).$$

Let $\alpha_{s,n}^{j,i}$ be the number of identity maps used in $\phi_{s,n}^{j,i}$, and define

$$\omega_{s,n} = \max\Big\{ \sum_{i=1}^{\tau(n)} \alpha_{s,n}^{j,i} \frac{\nu(n,i)}{\nu(s,j)} \mid j = 1, \ldots, \tau(s) \Big\}.$$

Finally, the condition "$\lim_{t\to\infty} \omega_{t,1} = \epsilon > 0$" in Theorem 6 should be changed to the condition "$\lim_{t\to\infty} \omega_{t,n} = \epsilon > 0$ for some n", and the statement of Theorem 9 should be changed similarly.

References

1. B. BLACKADAR, Infinite tensor products of C^*-algebras, *Pacific J. Math.* **72** (1977), 313–334.
2. B. BLACKADAR, Notes on the structure of projections in simple C^*-algebras, Semesterbericht Funktionalanalysis, Wintersemester 1982/83, Universität Tübingen.
3. B. BLACKADAR, "*K-Theory for operator algebras*," M.S.R.I. Publ. **5**, Springer-Verlag, New York, 1986.
4. B. BLACKADAR, O. BRATTELI, G. A. ELLIOTT, AND A. KUMJIAN, Reduction of real rank in inductive limits of C^*-algebras, *Math. Annalen* **292** (1992), 111–126.
5. B. BLACKADAR, M. DĂDĂRLAT, AND M. RØRDAM, The real rank of inductive limit C^*-algebras, *Math. Scand.* **69** (1991), 211–216.
6. B. BLACKADAR AND A. KUMJIAN, Skew products of relations and the structure of simple C^*-algebras, *Math. Zeitschrift* **189** (1985), 55–63.
7. L. G. BROWN AND G. K. PEDERSEN, C^*-algebras of real rank zero, *J. Func. Anal.* **99** (1991), 131–149.
8. M. DĂDĂRLAT, G. NAGY, A. NÉMETHI, AND C. PASNICU, Reduction of topological stable rank in inductive limits of C^*-algebras, *Pacific J. Math.* **153** (1992), 267–276.
9. M. DĂDĂRLAT AND A. NÉMETHI, Shape theory and (connective) K-theory, *J. Operator Theory* **23** (1990), 207–291.

10. G. GONG AND H. LIN, The exponential rank of inductive limit C^*-algebras (to appear).
11. K. R. GOODEARL, "*Notes on Real and Complex C^*-Algebras,*" Shiva, Nantwich (Cheshire), 1982.
12. K. R. GOODEARL, "*Partially Ordered Abelian Groups with Interpolation,*" Surveys and Monographs **20**, Amer. Math. Soc., Providence, 1986.
13. K. R. GOODEARL, Riesz decomposition in inductive limit C^*-algebras (to appear).
14. M. KAROUBI, "*K Theory: An Introduction,*" Grundlehren der Math. Wissenschaften **226**, Springer-Verlag, Berlin, 1978.
15. M. MARTIN AND C. PASNICU, Some comparability results in inductive limit C^*-algebras (to appear).
16. J.-I. NAGATA, "*Modern General Topology,*" North-Holland, Amsterdam, 1968.
17. G. K. PEDERSEN, The linear span of projections in simple C^*-algebras, *J. Operator Theory* **4** (1980), 289–296.
18. M. RØRDAM, On the structure of simple C^*-algebras tensored with a UHF-algebra, *J. Func. Anal.* **100** (1991), 1–17.
19. K. THOMSEN, Inductive limits of interval algebras: the tracial state space (to appear).
20. K. R. GOODEARL, K_0 of multiplier algebras of C^*-algebras with real rank zero, to appear.

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