

THE ALMOST ISOMORPHISM RELATION FOR SIMPLE REGULAR RINGS

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Abstract

A longstanding open problem in the theory of von Neumann regular rings is the question of whether every directly finite simple regular ring must be unit-regular. Recent work on this problem has been done by P. Menal, K.C. O'Meara, and the authors. To clarify some aspects of these new developments, we introduce and study the notion of almost isomorphism between finitely generated projective modules over a simple regular ring.

0. Introduction.

In the few past years, there have been some advances in the understanding of directly finite simple regular rings. In 1988, Menal and the second author [GM, Theorem 5.2] showed that if R is a directly finite regular algebra over an *uncountable* field, and if R contains no uncountable direct sums of nonzero right ideals, then it is unit-regular. As a consequence of this, any stably finite simple regular algebra R over an uncountable field is unit-regular [GM, Corollary 5.4]. More recently, O'Meara proved that a directly finite simple regular ring satisfying weak comparability is unit-regular [O, Theorem 1]. An alternative proof of O'Meara's Theorem was developed by the second author in privately circulated notes [G5]. We take the opportunity to present this proof here.

Our standard reference for the theory of regular rings is [G1], and for the theory of partially ordered abelian groups is [G4]. The reader can refer to these books for any undefined terms.

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Let R be an associative ring with 1. Denote by \mathcal{P} (resp. \mathcal{P}_0) the class of finitely generated projective right R -modules (resp. the class of nonzero finitely generated projective right R -modules). If R is a regular ring then $L(R_R)$ will denote the lattice of principal right ideals of R . For $A, B \in \mathcal{P}$, we will write $A \lesssim B$ (resp. $A < B$) if A is isomorphic to a submodule (resp. proper submodule) of B . For a positive integer k and module A , we let kA denote the direct sum of k copies of A .

A ring R is said to be *directly finite* if $xy = 1$ implies $yx = 1$, for $x, y \in R$. We say that R is *stably finite* if $M_n(R)$ is directly finite for all $n \geq 1$. It is not known whether directly finite regular rings are stably finite [G1, Open Problem 1]. The question is open even in the case of simple regular rings.

A ring R is said to be *unit-regular* if for any $x \in R$ there exists a unit $u \in R$ such that $x = xux$. Every unit-regular ring is stably finite, but there exist stably finite regular rings which are not unit-regular [G1, Proposition 5.2 and Example 5.10]. However, there are some interesting classes of regular rings for which it is known that direct finiteness implies unit-regularity. For example, this holds for regular rings satisfying general comparability [G1, Theorem 8.12], for right \aleph_0 -continuous regular rings [G2, Theorem 1.4], and for \aleph_0 -complete regular rings [Bu, Corollary 1.6]. An outstanding question in the theory is whether a directly finite simple regular ring is unit-regular [G1, Open Problem 3].

We say that a class of modules \mathcal{C} satisfies the *cancellation property* (with respect to the isomorphism relation) if $A \oplus C \cong B \oplus C$ implies $A \cong B$ for $A, B, C \in \mathcal{C}$. A regular ring R is unit-regular if and only if \mathcal{P} satisfies the cancellation property, see [G1, Theorem 4.5]. The main result of Section 1 states that a directly finite simple regular ring is unit-regular if and only if \mathcal{P} satisfies the cancellation property *with respect to the almost isomorphism relation* (defined in Section 1).

We say that R is *strictly unperforated* whenever $nA < nB$ implies $A < B$ for $A, B \in \mathcal{P}$ and $n \geq 1$. R is *unperforated* if $nA \lesssim nB$ implies $A \lesssim B$ for $A, B \in \mathcal{P}$ and $n \geq 1$.

Assume that R is a directly finite simple regular ring. It is an open question whether R is (strictly) unperforated. Strictly unperforated directly finite simple regular rings have a number of interesting properties. In particular they are unit-regular and, in the non-artinian case, they are close to being rings of matrices of any size (see Section 2). Some technical results needed to obtain the latter statement are included in an Appendix.

Let R be a stably finite simple regular ring. Then $(K_0(R), [R])$ is a partially ordered abelian group with order-unit, see [G1, Proposition

15.3]. Let $\Phi : K_0(R) \rightarrow \text{Aff}(S(K_0(R), [R]))$ be the natural map, see [G4, Chapter 7]. For any compact convex set S , denote the strict ordering on $\text{Aff}(S)$ by \ll , that is, $f \ll g$ if and only if $f(x) < g(x)$ for all $x \in S$. By [B3, Theorem 3.1.4] and [G4, Theorem 4.12], R is strictly unperforated if and only if $\Phi([A]) \ll \Phi([D])$ implies $A \prec D$ for $A, D \in \mathcal{P}$. We can consider the following weaker condition:

For any $D \in L(R_R)$, there exists $K \geq 1$ such that, for $A \in L(R_R)$, if $K\Phi([A]) \ll \Phi([D])$ then $A \prec D$.

We will see in Section 3 that R satisfies this condition if and only if R satisfies the following property:

The doubling condition (DD): For any $D \in L(R_R)$, there exists $K \geq 1$ such that, for $A \in L(R_R)$, if $A \leq D$ and $K\Phi([A]) \ll \Phi([D])$ then $2A \prec D$.

Similarly, the following two conditions are equivalent for a directly finite simple regular ring R :

Weak comparability (O'Meara): For any $D \in L(R_R)$ there exists $n \geq 1$ such that, for $A \in L(R_R)$, if $nA \lesssim R$ then $A \lesssim D$.

(dd): For any $D \in L(R_R)$ there exists $n \geq 1$ such that, for $A \in L(R_R)$, if $A \leq D$ and $nA \lesssim R$ then $2A \lesssim D$.

Observe, in particular, that the doubling condition implies weak comparability in any stably finite simple regular ring.

We close the paper by studying the effect of imposing comparability with respect to the (approximately) almost isomorphism relation on a stably finite simple regular ring.

1. Stable range of simple regular rings.

In this Section we study the stable range of simple regular rings, obtaining a restriction on the behaviour of the stable range on the family of finitely generated projective modules. It is easy to show by using our results that if there exists a simple regular ring of stable range 2, then there are corner rings of R with arbitrary finite stable range $n > 1$. So, the situation for simple regular rings differs very much from the situation for arbitrary regular rings, see [MM; GMM].

We will apply the results on stable range to give the new proof of O'Meara's Theorem [O, Theorem 1].

Recall that a ring R satisfies the *n-stable range condition* (for a given positive integer n) if whenever $a_1, \dots, a_{n+1} \in R$ with $a_1R + \dots + a_{n+1}R = R$, there exist elements $b_1, \dots, b_n \in R$ such that

$$(a_1 + a_{n-1}b_1)R + \dots + (a_n + a_{n+1}b_n)R = R.$$

If n is the least positive integer such that R satisfies the n -stable range condition, then R is said to have *stable range* n , and we write $sr(R) = n$. It is well-known that a regular ring has stable range one if and only if it is unit-regular [G1, Proposition 4.12]. The reader is referred to [V] for the basic properties of the stable range and to [W;MM;M] for the connections between cancellation properties of modules and the stable range of their endomorphism rings.

Lemma 1.1. *Let R be a non-artinian simple regular ring. Then for each $P \in \mathcal{P}_0$ and for all $k \geq 1$, there exists $Q \in \mathcal{P}_0$ such that $kQ \lesssim P$.*

Proof: Clearly we can assume that $P = eR$ for some nonzero idempotent $e \in R$. Since R is not artinian, $eR = e_1R \oplus e_2R$ for some nonzero idempotents e_1, e_2 . Since R is simple, $e_1R \lesssim n(e_2R)$ for some n and so $e_1R = A_1 \oplus \cdots \oplus A_n$ with $A_i \lesssim e_2R$ by [G1, Corollary 2.9]. Then $eR = A_1 \oplus (A_2 \oplus \cdots \oplus A_n \oplus e_2R)$ and clearly $2A_1 \lesssim eR$. Now, the result follows by induction. ■

The following result is patterned after an argument of Rieffel [R].

Theorem 1.2. *Let R be a simple regular ring such that $sr(eRe) \leq k$ for some $k \geq 1$ and all idempotents $e \in R$. Then R is unit-regular.*

Proof: If R is artinian, the result is well-known. So we can restrict ourselves to the non-artinian case. Assume that $P_1 \oplus nR \cong P_2 \oplus nR$ for some finitely generated projective modules P_1 and P_2 . If $P_1 = P_2 = 0$, we are done, so we can assume that $P_1 \neq 0$. By Lemma 1.1, $P_1 \cong kQ \oplus U$ for some $Q \in \mathcal{P}_0$, and clearly we can assume that $Q \cong eR$ for some idempotent $e \in R$. Now, we have

$$kQ \oplus U \oplus nR \cong P_2 \oplus nR.$$

Since R is simple we have $R \oplus V \cong sQ$ for some $s \geq 1$ and since $kQ \oplus U \oplus nR \oplus nV \cong P_2 \oplus nR \oplus nV$ we have $kQ \oplus U \oplus nsQ \cong P_2 \oplus nsQ$. By [W, Theorem 1.2], $kQ \oplus U \cong P_2$ and so $P_1 \cong P_2$. By [G1, Theorem 4.5], it follows that R is unit-regular. ■

We need the essentially-known fact that the finiteness of the stable range is Morita-invariant. We include a proof of this result, which is a straightforward application of the techniques in [W].

Lemma 1.3. *Let P and Q be finitely generated projective modules over a ring R . Assume that there exists $k \geq 1$ such that $Q \oplus U \cong kP$ and there exists $i \geq 1$ such that $P \oplus T \cong iQ$, that is, P and Q generate*

the same categories of modules. Then $sr(\text{End}_R(P)) < \infty$ if and only if $sr(\text{End}_R(Q)) < \infty$.

Proof: We will follow the proof in [W, Theorem 1.11]. Assume that $sr(\text{End}_R(Q)) = \tau < \infty$. Set $m = (r + i - 1)k$. Let

$$N = P_1 \oplus K = P'_1 \oplus \cdots \oplus P'_m \oplus L.$$

where $P_1 \cong P'_j \cong P$ for all j . Adding T to this relation we obtain

$$M := N \oplus T = Q_1 \oplus \cdots \oplus Q_i \oplus K = Q'_1 \oplus \cdots \oplus Q'_{i+r-1} \oplus V \oplus L$$

where $Q_p \cong Q'_q \cong Q$ for all p, q . Applying [W, Theorem 1.11], we get a submodule B of M such that

$$M = B \oplus K = B \oplus C \oplus L$$

where $C \subseteq Q'_1 \oplus \cdots \oplus Q'_{i+r-1} \oplus V$. Now we have

$$N = (B \cap N) \oplus K = [(B \oplus C) \cap N] \oplus L$$

and also $(B \oplus C) \cap N = (B \cap N) \oplus [(B \oplus C) \cap K]$. Let π be the projection of N onto $P'_1 \oplus \cdots \oplus P'_m$ along L . Then

$$[(B \oplus C) \cap K] \oplus L = \pi((B \oplus C) \cap K) \oplus L.$$

It follows that $N = (B \cap N) \oplus \pi((B \oplus C) \cap K) \oplus L$. By [W, Theorem 1.6], $sr(\text{End}_R(P)) \leq m < \infty$. ■

Remark 1.4. If $sr(\text{End}_R(Q)) = \tau$ then by the above proof we obtain the following bound: $sr(\text{End}_R(P)) \leq (r + i - 1)k$. In particular, if e is an idempotent of R and $R \lesssim n(eR)$ then $sr(R) \leq sr(eRe) + n - 1$. This is exactly the same bound obtained by Blackadar for C^* -algebras, see [B1, Lemma A6], [B2, p.33].

Our following result is an immediate consequence of Theorem 1.2 and Lemma 1.3.

Theorem 1.5. *Let R be a simple regular ring. Then one of the following possibilities occurs:*

- (1) R is unit-regular.
- (2) $sr(\text{End}_R(P)) = \infty$ for every $P \in \mathcal{P}_0$.
- (3) $sr(\text{End}_R(P))$ is finite for every $P \in \mathcal{P}$ and the set $\{sr(\text{End}_R(P)) \mid P \in \mathcal{P}\}$ is not bounded.

Remark 1.6. Let R be any simple ring which is not stably finite. By simplicity, we then have $2nR \lesssim nR$ for some $n \geq 1$. By [W, Theorem 1.2], $sr(\text{End}_R(nR)) = \infty$. By Lemma 1.3, this implies that $sr(\text{End}_R(P)) = \infty$ for every nonzero finitely generated projective R -module.

Theorem 1.7. *Let R be a simple regular ring. Assume that whenever $B, C_1, C_2 \in L(R_R)$ with $R \oplus B \lesssim R \oplus C_i$ for $i = 1, 2$, then $B \lesssim C_1 \oplus C_2$. Then R is unit-regular.*

Proof: Since it is easily seen that the hypothesis is inherited by the corner rings of R , it suffices by Theorem 1.2 to show that $sr(R) \leq 2$. Let $a, b, c \in R$ such that $aR + bR + cR = R$. There is an idempotent $e \in R$ such that $cR = eR \oplus [cR \cap (aR + bR)]$. Note that $e = ct$ for some $t \in R$, and $R = (aR + bR) \oplus eR$.

Now $R_R = C_1 \oplus D_1 = C_2 \oplus D_2$ where $C_1 = r.\text{ann}_R(a)$ and $C_2 = r.\text{ann}_R(b)$. We observe that left multiplication by a induces an isomorphism of D_1 onto aR , and similarly $D_2 \cong bR$. By using this we see that $R \oplus eR \lesssim R \oplus C_i$ for $i = 1, 2$. By our hypothesis, we deduce that $eR \lesssim C_1 \oplus C_2$, so $eR = E_1 \oplus E_2$ with each $E_i \lesssim C_i$.

Define $x \in R$ such that $xD_1 = 0$ and $xR = xC_1 = E_1$; note that $x = ex$. Since $aC_1 = 0$ and $aD_1 = aR$, we get $(a+x)R = aR + E_1$. Likewise, there exists $y \in eR$ such that $(b+y)R = bR + E_2$. Then we have $(a+ctx)R + (b+cty)R = (a+x)R + (b+y)R = aR + bR + E_1 + E_2 = aR + bR + eR = R$. This shows that $sr(R) \leq 2$. ■

We now introduce a key concept for this paper, namely the almost isomorphism relation.

Definitions. Let R be a regular ring and let $A, B \in \mathcal{P}$.

We say that A is *almost subisomorphic* to B , written $A \lesssim_a B$, if for all nonzero $C \in L(R_R)$ we have $A \lesssim B \oplus C$. We say that A is *almost isomorphic* to B , written $A \cong_a B$, if $A \lesssim_a B$ and $B \lesssim_a A$.

We say that A is *approximately almost subisomorphic* to B , written $A \lesssim_{aa} B$, if for all nonzero $C \in L(R_R)$ there exists $n \geq 1$ such that $nA \lesssim n(B \oplus C)$. We say that A is *approximately almost isomorphic* to B , written $A \cong_{aa} B$, if $A \lesssim_{aa} B$ and $B \lesssim_{aa} A$.

The above notions are specially useful when R is a simple regular ring which is not artinian. Since artinian simple regular rings are trivial for our theory, we will frequently assume that our simple regular rings are not artinian.

Lemma 1.8. *Let R be a non-artinian simple regular ring and let*

$A, B \in \mathcal{P}$. Then:

- (a) $A \lesssim_a B$ (resp. $A \lesssim_{aa} B$) if and only if $A \lesssim_a B \oplus C$ (resp. $A \lesssim_{aa} B \oplus C$) for all nonzero $C \in L(R_R)$.
 (b) The relation \lesssim_a (resp. \lesssim_{aa}) is transitive.

Proof: (a) Assume that $A \lesssim_a B \oplus C$ for all nonzero $C \in L(R_R)$. Fix a nonzero $D \in L(R_R)$. Then since R is not artinian $D = D_1 \oplus D_2$ for nonzero $D_1, D_2 \in L(R_R)$. So $A \lesssim (B \oplus D_1) \oplus D_2 = B \oplus D$. Thus $A \lesssim_a B$. The other implication is trivial. The proof for the relation \lesssim_{aa} is analogous. (b) is proved in a similar way. ■

Theorem 1.9. *Let R be a directly finite simple regular ring. Then the following conditions are equivalent:*

- (a) For all $A, B, C \in \mathcal{P}$, $A \oplus B \lesssim_a A \oplus C$ implies $B \lesssim_a C$.
 (b) For all $B, C \in L(R_R)$, $R \oplus B \lesssim R \oplus C$ implies $B \lesssim_a C$.
 (c) R is unit-regular.

Proof: (a) \Rightarrow (b): This is clear.

(b) \Rightarrow (c): Apply Theorem 1.7.

(c) \Rightarrow (a): This is immediate from the cancellation property of unit-regular rings. ■

Lemma 1.10. *Let A, B, C be finitely generated projective right modules over a regular ring, such that $A \oplus C \cong B \oplus C$. Let $n \in \mathbb{N}$. Then there exist decompositions $A = A' \oplus A''$ and $B = B' \oplus B''$ and $C = C' \oplus C''$ such that $A' \cong B'$ and $A'' \oplus C'' \cong B'' \oplus C''$, and also $n(A'' \oplus B'') \lesssim C'$.*

Proof: By [G3, Lemma 2.2], there are decompositions $A = A_{11} \oplus A_{12}$ and $B = B_{11} \oplus B_{12}$ and $C = C_{11} \oplus C_{12}$ such that $A_{11} \cong B_{11}$ and $A_{12} \oplus C_{12} \cong B_{12} \oplus C_{12}$, while also $A_{12} \cong C_{11}$. Applying this lemma repeatedly, we obtain decompositions $A_{i-1,2} = A_{i1} \oplus A_{i2}$ and $B_{i-1,2} = B_{i1} \oplus B_{i2}$ and $C_{i-1,2} = C_{i1} \oplus C_{i2}$, for $i = 2, 3, \dots$, such that $A_{i1} \cong B_{i1}$ and $A_{i2} \oplus C_{i2} \cong B_{i2} \oplus C_{i2}$, while also $A_{i2} \cong C_{i1}$.

Now set $A_1 = A_{11} \oplus A_{21} \oplus \dots \oplus A_{n1}$ and $A_2 = A_{n2}$, and define B_1, B_2, C_1, C_2 similarly. Then $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ and $C = C_1 \oplus C_2$, with $A_1 \cong B_1$ and $A_2 \oplus C_2 \cong B_2 \oplus C_2$. Since $A_2 = A_{n2} \leq A_{n-1,2} \leq \dots \leq A_{12}$, we also have $nA_2 \lesssim A_{12} \oplus A_{22} \oplus \dots \oplus A_{n2} \cong C_{11} \oplus C_{21} \oplus \dots \oplus C_{n1} = C_1$.

Finally, we apply the above procedure to the isomorphism $B_2 \oplus C_2 \cong A_2 \oplus C_2$. We obtain decompositions $B_2 = B_3 \oplus B_4$ and $A_2 = A_3 \oplus A_4$ and $C_2 = C_3 \oplus C_4$ such that $B_3 \cong A_3$ and $B_4 \oplus C_4 \cong A_4 \oplus C_4$, while $nB_4 \lesssim C_3$. Set $A' = A_1 \oplus A_3$ and $A'' = A_4$, and define B', B'', C', C''

similarly. Then $A = A' \oplus A''$ and $B = B' \oplus B''$ and $C = C' \oplus C''$, with $A' \cong B'$ and $A'' \oplus C'' \cong B'' \oplus C''$, while also $n(A'' \oplus B'') \lesssim nA_2 \oplus nB_4 \lesssim C_1 \oplus C_3 = C'$. ■

Now we can give a different proof of [O, Theorem 1].

Theorem 1.11. (O'Meara) *Let R be a directly finite simple regular ring satisfying weak comparability. Then R is unit-regular.*

Proof: We show that R satisfies condition (b) in Theorem 1.9. Let $B, C \in L(R_R)$ with $R \oplus B \lesssim R \oplus C$. Given $0 \neq D \in L(R_R)$ there exists, by weak comparability, a positive integer n such that $nT \lesssim R$ implies $T \lesssim D$ for any $T \in L(R_R)$.

By Lemma 1.10 there exists a decomposition $B = B' \oplus B''$ such that $B' \lesssim C$ and $nB'' \lesssim R$. Consequently $B'' \lesssim D$ and $B \lesssim C \oplus D$. It follows that $B \lesssim_a C$ and thus R is unit-regular by Theorem 1.9. ■

2. The almost isomorphism relation.

Let R be a non-artinian stably finite simple regular ring. By [B3, Theorem 3.1.4], the relation \lesssim_{aa} is cancellative, i.e. $A \oplus B \lesssim_{aa} A \oplus C$ implies $B \lesssim_{aa} C$. So the approximately almost isomorphism classes of finitely generated projective modules form a cancellative abelian semigroup S (since it is easy to show that direct sum gives a well-defined operation). Denote by $[A]_a$ the class of A in S . We define a partial order on S by $[A]_a \leq [B]_a$ if and only if $A \lesssim_{aa} B$. It is easy to show that this relation is well-defined and translation-invariant, so that S becomes a partially ordered abelian semigroup. Also, the relation \leq is cancellative, i.e. $x + y \leq z + y$ implies $x \leq z$ for $x, y, z \in S$, again by [B3, Theorem 3.1.4]. Let $K_0^a(R)$ be the abelian group obtained by adjoining inverses formally to S . Because of the cancellation property of \leq , the relation $x - y \leq z - t$ iff $x + t \leq z + y$ for $x, y, z, t \in S$ becomes a partial order in $K_0^a(R)$. It is easy to show that this partial order is translation-invariant and so $K_0^a(R)$ becomes a partially ordered abelian group, in which we fix the order-unit $[R]_a$.

Proposition 2.1. *Let R be a non-artinian stably finite simple regular ring. Let $\Phi : K_0(R) \rightarrow \text{Aff}(S(K_0(R), [R]))$ be the natural map. Assume that $\Phi(K_0(R))$ is endowed with the partial order $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in S(K_0(R), [R])$. Then $\Phi(K_0(R)) \cong K_0^a(R)$ as partially ordered abelian groups with order-unit.*

Proof: We have a surjective positive homomorphism $\alpha : K_0(R) \rightarrow K_0^a(R)$ given by $\alpha([A] - [B]) = [A]_a - [B]_a$. We will show that $\text{Ker}(\alpha) =$

$\text{Ker}(\Phi)$. It is clear that $\Phi([A] - [B]) = 0$ whenever $A \cong_{aa} B$, but this happens exactly when $[A]_a = [B]_a$. Conversely assume that $\Phi([A] - [B]) = 0$. Let $0 \neq C \in L(R_R)$. Then $\Phi([A \oplus C] - [B]) \gg 0$ and so, by [G4, Theorem 4.12] there exists $m \geq 1$ such that $m([A \oplus C] - [B]) > 0$. By using [B3, Theorem 3.1.4] we have that there exists $n \geq 1$ such that $nmB \lesssim nm(A \oplus C)$. Consequently $B \lesssim_{aa} A$. Analogously $A \lesssim_{aa} B$ and so $[A]_a - [B]_a = 0$.

It follows that we have a group isomorphism $\gamma : K_0^a(R) \longrightarrow \Phi(K_0(R))$ given by $\gamma([A]_a - [B]_a) = \Phi([A] - [B])$. Since R is not artinian γ is positive. Conversely, if $\Phi([A] - [B]) \geq 0$ then by the same argument as before we obtain that $B \lesssim_{aa} A$ and consequently $[A]_a - [B]_a \geq 0$. ■

Henceforth, we will identify $K_0^a(R)$ with $\Phi(K_0(R))$. The proof of the following lemma is straightforward.

Lemma 2.2. *Let R be a stably finite simple regular ring.*

- (a) *Let $S = \varinjlim M_n(R)$. Then $K_0(S) = K_0(R) \otimes \mathbf{Q}$.*
- (b) *Let $S_m = \varinjlim M_{m^k}(R)$, for a fixed $m \geq 1$. Then $K_0(S_m) = K_0(R) \otimes \mathbf{D}_m$ where $\mathbf{D}_m = \{a/m^k \mid a \in \mathbf{Z}, k \geq 1\}$.*

By [B3, Theorem 3.1.4], the rings $S = \varinjlim M_n(R)$ and all $S_m = \varinjlim M_{m^k}(R)$ are unperforated unit-regular simple rings provided R is a stably finite simple regular ring. It follows that $K_0(S)$ and all $K_0(S_m)$ are simple dimension groups. Therefore we can use [G4, Theorem 14.14] and Lemma 2.2 to study the question of when $\Phi(K_0(R)^+)$ is dense in $\text{Aff}(S(K_0(R), [R]))^+$, leading to the following lemma.

Recall that for any partially ordered abelian group G and any subgroup H of \mathbf{Q} , the tensor product $G \otimes H$ is a partially ordered abelian group with positive cone $(G \otimes H)^+ = \{x \otimes y \mid x \in G^+, y \in H^+\}$.

Lemma 2.3. *Let $H = \mathbf{Q}$ (resp. $H = \mathbf{D}_m$), endowed with the usual order. Let (G, u) be a partially ordered simple abelian group with order-unit, and assume that $G \otimes H$ is a simple dimension group. Let $\Phi : G \longrightarrow \text{Aff}(S(G, u))$ be the natural map. Then the following properties are equivalent:*

- (a) $\Phi(G^+)$ is dense in $\text{Aff}(S(G, u))^+$.
- (b) For each $0 \neq x \in G^+$, for each $n \geq 1$ (resp. for each $n = m^k$, with $k \geq 0$), and for each $\epsilon > 0$ there exist $y_1, y_2 \in G^+$ such that

$$\begin{aligned} n\Phi(y_2) &\ll \Phi(x) \ll n\Phi(y_1) \\ n\Phi(y_1) - \Phi(x) &\ll \epsilon \\ \Phi(x) - n\Phi(y_2) &\ll \epsilon \end{aligned}$$

- (c) For each $0 \neq x \in G^+$, for each $n \geq 1$ (resp. for each $n = m^k$, with $k \geq 0$), and for each $\epsilon > 0$ there exists $y_1 \in G^+$ such that $\Phi(x) \ll n\Phi(y_1)$ and $n\Phi(y_1) - \Phi(x) \ll \epsilon$.

Proof:

(a) \Rightarrow (b):

Assume that $\Phi(G^+)$ is dense in $\text{Aff}(S(G, u))^+$. Now fix $0 \neq x \in G^+$, $n \geq 1$ and $\epsilon > 0$. Since G is simple, $\Phi(x) \gg 0$.

Hence, after possibly replacing ϵ by a smaller positive real number, we may assume that $\Phi(x) \gg \frac{\epsilon}{2}$. Thus the functions $f_1 = \frac{1}{n}\Phi(x) + \frac{\epsilon}{2n}$ and $f_2 = \frac{1}{n}\Phi(x) - \frac{\epsilon}{2n}$ are positive. Since $\Phi(G^+)$ is dense in $\text{Aff}(S(G, u))^+$, there exist $y_1, y_2 \in G^+$ such that $\|\Phi(y_i) - f_i\| < \frac{\epsilon}{2n}$ for $i = 1, 2$. Then

$$\begin{aligned}\Phi(x) &= nf_1 - \frac{\epsilon}{2} \ll n\Phi(y_1) \ll nf_1 + \frac{\epsilon}{2} = \Phi(x) + \epsilon \\ \Phi(x) - \epsilon &= nf_2 - \frac{\epsilon}{2} \ll n\Phi(y_2) \ll nf_2 + \frac{\epsilon}{2} = \Phi(x).\end{aligned}$$

So, $y_1, y_2 \in G^+$ satisfy the required conditions.

Obviously, (b) \Rightarrow (c).

(c) \Rightarrow (a):

Let $\tau : G \rightarrow G \otimes H$ be the map given by $\tau(x) = x \otimes 1$. τ induces $\tau_* : S(G \otimes H, u \otimes 1) \rightarrow S(G, u)$. If $s = \tau_*(s')$ with $s' \in S(G \otimes H, u \otimes 1)$ then $s'(x \otimes 1/n) = \frac{1}{n}s(x)$ for $\frac{1}{n} \in H$. This implies that τ_* is injective. On the other hand, if $s \in S(G, u)$ then the expression $s'(x \otimes \frac{1}{n}) = \frac{1}{n}s(x)$ defines $s' \in S(G \otimes H, u \otimes 1)$ such that $\tau_*(s') = s$. It follows that τ_* is an affine homeomorphism. Now we have the following commutative diagram:

$$\begin{array}{ccc}\text{Aff}(S(G, u)) & \xrightarrow{\text{Aff}(\tau_*)} & \text{Aff}(S(G \otimes H, u \otimes 1)) \\ \uparrow \Phi & & \uparrow \Phi' \\ G & \xrightarrow{\tau} & G \otimes H\end{array}$$

and $\text{Aff}(\tau_*)$ is an isomorphism of partially ordered abelian groups with order-unit. Since $G \otimes H$ is a simple dimension group, $\Phi'((G \otimes H)^+)$ is dense in $\text{Aff}(S(G \otimes H, u \otimes 1))^+$ by [G4, Theorem 14.14]. So $\Phi(G^+)$ is dense in $\text{Aff}(S(G, u))^+$ if and only if $\text{Aff}(\tau_*)(\Phi(G^+))$ is dense in $\Phi'((G \otimes H)^+)$.

Let $0 \neq \Phi'(x \otimes 1/n)$ and $\epsilon > 0$, with $x \in G^+$ and $1/n \in H$. Then there exists $y_1 \in G^+$ with $\Phi(x) \ll n\Phi(y_1)$ and $n\Phi(y_1) - \Phi(x) \ll \epsilon$. It follows that

$$\|\text{Aff}(\tau_*)(\Phi(y_1)) - \Phi'(x \otimes \frac{1}{n})\| < \frac{\epsilon}{n} \leq \epsilon$$

and consequently $\Phi(G^+)$ is dense in $\text{Aff}(S(G, u))^+$. ■

For a stably finite simple regular ring R , the condition $\Phi([A]) \ll \Phi([B])$ is equivalent to $tA \prec tB$ for some $t \geq 1$, by [G4, Theorem 4.12] and [B3, Theorem 3.1.4]. By using this and Lemma 2.3, we obtain the following Proposition.

Proposition 2.4. *Let R be a stably finite simple regular ring. Then the following conditions are equivalent:*

- (1) $\Phi(K_0(R)^+)$ is dense in $\text{Aff}(S(K_0(R), [R]))^+$.
- (2) For any $A \in \mathcal{P}_0$, $n \geq 1$ and $\epsilon > 0$ there exist $B, C \in \mathcal{P}$ such that $tA \prec ntB$ for some $t \geq 1$ while $n[B]_a - [A]_a \ll \epsilon$; and $nrC \prec rA$ for some $r \geq 1$ while $[A]_a - n[C]_a \ll \epsilon$.
- (3) There exists $m \geq 2$ such that for all $A \in \mathcal{P}$, $k \geq 1$ and $\epsilon > 0$, there exists $B \in \mathcal{P}$ such that $tA \prec m^k tB$ for some $t \geq 1$ while $m^k [B]_a - [A]_a \ll \epsilon$.

We will call condition (1) in the above Proposition condition (D).

Corollary 2.5. *If R is a stably finite simple regular ring and there exists $m \geq 2$ such that eRe is an $m \times m$ matrix ring for each idempotent $e \in R$, then R satisfies condition (D).*

Corollary 2.6. *Let R be a stably finite simple regular ring satisfying condition (D). Then $(K_0^a(R), \ll)$ is a simple dimension group.*

Proof: We observe that $\Phi(K_0(R)^+) \subseteq K_0^a(R)^+$. Since R satisfies (D), $K_0^a(R)^+$ is dense in $\text{Aff}(S(K_0(R), [R]))^+$. Consequently $K_0^a(R)$ is dense in $\text{Aff}(S(K_0(R), [R]))$ and thus by [G4, Prop. 14.15] $(K_0^a(R), \ll)$ is a simple dimension group. ■

The following corollary is a consequence of the fact that $(K_0^a(R), \ll)$ is an interpolation group whenever R satisfies condition (D).

Corollary 2.7. *Let R be a stably finite simple regular ring satisfying condition (D). If $nA \prec n(B \oplus C)$ for $A, B, C \in \mathcal{P}_0$, then for each $\epsilon > 0$ there exists $A_1, A_2, A'_1, A'_2 \in \mathcal{P}$ and $m \geq 1$ such that $m(A'_1 \oplus A'_2) \prec mA \prec m(A_1 \oplus A_2)$, and $mA'_1 \prec mA_1 \prec mB$ and $mA'_2 \prec mA_2 \prec mC$, while also $[A_1]_a + [A_2]_a - [A]_a \ll \epsilon$ and $[A]_a - [A'_1]_a - [A'_2]_a \ll \epsilon$.*

Proof: If $nA \prec n(B \oplus C)$, then we have $[A]_a \ll [B]_a + [C]_a$. Since $(K_0^a(R), \ll)$ is an interpolation group by Corollary 2.6, there exist $X'_1, X'_2 \in K_0^a(R)^{++}$ such that $[A]_a = X'_1 + X'_2$ while $X'_1 \ll [B]_a$ and $X'_2 \ll [C]_a$. Clearly, one of the inequalities must be strict, so assume that $X'_1 \ll [B]_a$.

If also $X'_2 \ll [C]_a$, put $X_1 = X'_1$, $X_2 = X'_2$. If $X'_2 = [C]_a$ then write $X'_1 = [Z_1]_a - [Z_2]_a$ for some $Z_1, Z_2 \in \mathcal{P}$, and observe that there exists $k \geq 1$ such that $kZ_2 \prec kZ_1$ and $kZ_1 \prec k(B \oplus Z_2)$. Choose $Z \in \mathcal{P}_0$ such that $Z \prec C$ and $k(Z_1 \oplus Z) \prec k(B \oplus Z_2)$. Set $X_1 = X'_1 + [Z]_a$ and $X_2 = X'_2 - [Z]_a$ and observe that $X_1, X_2 \in K_0^g(R)^{++}$ satisfy $[A]_a = X_1 + X_2$, and $X_1 \ll [B]_a$ and $X_2 \ll [C]_a$.

Now write $X_1 = [T_1]_a - [T_2]_a$ for some $T_1, T_2 \in \mathcal{P}$. Then we have $kT_2 \prec kT_1$ for some $k \geq 1$ and so we can write $kT_1 \cong kT_2 \oplus S$ for some $S \in \mathcal{P}_0$. Observe that $X_1 = \frac{1}{k}[S]_a$. Choose $\epsilon_1 > 0$ such that $\epsilon_1 \ll [B]_a - X_1$ and $\epsilon_1 \leq \epsilon/2$. By Proposition 2.4 there exist $A'_1, A_1 \in \mathcal{P}$ such that $ktA'_1 \prec tS \prec ktA_1$ for some $t \geq 1$ while also $k[A_1]_a - [S]_a \ll \epsilon_1$ and $[S]_a - k[A'_1]_a \ll \epsilon_1$.

We have $[A'_1]_a \ll X_1 \ll [A_1]_a$ and also $[A_1]_a - X_1 \ll \epsilon_1/k \leq \epsilon_1 \leq \epsilon/2$ and $X_1 - [A'_1]_a \ll \epsilon_1/k \leq \epsilon_1 \leq \epsilon/2$. It follows that $[B]_a \gg X_1 + \epsilon_1 \gg [A_1]_a$. Similarly there exist $A_2, A'_2 \in \mathcal{P}$ such that $[A'_2]_a \ll X_2 \ll [A_2]_a$ with $[A_2]_a - X_2 \ll \epsilon/2$ and $X_2 - [A'_2]_a \ll \epsilon/2$, while also $[C]_a \gg [A_2]_a$. Consequently $[A'_1 \oplus A'_2]_a \ll X_1 + X_2 = [A]_a \ll [A_1 \oplus A_2]_a$ and also $[A]_a - [A'_1]_a - [A'_2]_a \ll \epsilon$ and $[A_1]_a + [A_2]_a - [A]_a \ll \epsilon$, as required. ■

We say that R is *strictly m -unperforated* if $mA \prec mB$ implies $A \prec B$ for all $A, B \in \mathcal{P}$. So, R is strictly unperforated if it is strictly m -unperforated for all $m \geq 1$.

Corollary 2.8. *Let R be a non-artinian directly finite simple regular ring which is strictly m -unperforated for some $m \geq 2$. Then R is unit-regular, strictly unperforated, satisfies property (D) and, moreover, for any $A \in \mathcal{P}_0$, $n \geq 1$ and $\epsilon > 0$, the following conditions hold:*

- (a) *If $B \in \mathcal{P}$ and $B \prec A$, then there exists $T \in \mathcal{P}$ such that $B \prec nT \prec A$ and $[A]_a - n[T]_a \ll \epsilon$.*
- (b) *If $D \in \mathcal{P}$ and $A \prec D$, then there exists $V \in \mathcal{P}$ such that $A \prec nV \prec D$ and $n[V]_a - [A]_a \ll \epsilon$.*

Proof: R is unit-regular by [O, Theorem 1].

R is strictly unperforated by [B3, Theorem 2.1.11]. By Proposition A1, R satisfies property (D).

Since R is strictly unperforated, (a) and (b) follows from Proposition 2.4 by using the same arguments as in Corollary 2.7. ■

The following observation is a consequence of Corollary 2.8.

Remark 2.9. Let R be a directly finite simple regular ring which is strictly m -unperforated for some $m \geq 2$. Then $A \lesssim_{aa} B$ if and only if $A \lesssim_a B$, for $A, B \in \mathcal{P}$.

3. The doubling condition.

We will say that a stably finite regular ring R satisfies the *doubling condition* (DD) if for every nonzero $D \in L(R_R)$ there exists $K \geq 1$ such that, for $A \in L(R_R)$, if $A \leq D$ and $K[A]_a \ll [D]_a$ then $2A \lesssim D$.

Lemma 3.1. *Let R be a stably finite simple regular ring. Then R satisfies (DD) if and only if for every $D \in \mathcal{P}$ there exists $K \geq 1$ such that, for $A \in \mathcal{P}$, if $A \leq D$ and $K[A]_a \ll [D]_a$ then $2A \lesssim D$.*

Proof: Assume that R satisfies (DD). Write $D = D_1 \oplus \cdots \oplus D_r$, where $0 \neq D_i \lesssim R$. For $i = 1, \dots, r$, there exist $K_i \geq 1$ such that, for $A_i \in L(R_R)$, if $A_i \leq D_i$ and $K_i[A_i]_a \ll [D_i]_a$ then $2A_i \lesssim D_i$. Take

$$K \geq \max\{K_i[D]_a/[D_i]_a \mid i = 1, \dots, r\}. \quad \blacksquare$$

Lemma 3.2. *Let R be a directly finite simple regular ring such that $n(xR) \prec n(yR)$ implies $xR \prec yR$ for all $x, y \in R$. Then R satisfies the doubling condition with $K = 2$.*

Proof: It follows from [O, Theorem 1] that R is unit-regular. Put $D = yR$ and $A = xR$, and assume that $A \leq D$. Write $D = A \oplus T$. Assume that $2[A]_a \ll [D]_a$. Then $2nA \prec nD \cong nA \oplus nT$ for some $n \geq 1$. Since R is unit-regular $nA \prec nT$. By hypothesis $A \prec T$. So we obtain property (DD) with $K = 2$. \blacksquare

Lemma 3.3. *Let R be a regular ring and $B, D \in \mathcal{P}$ with $B \lesssim rD$ for some $r \geq 1$. Then there exists a decomposition $B = B_1 \oplus \cdots \oplus B_r$ such that $B_1 \lesssim B_2 \lesssim \cdots \lesssim B_r \lesssim D$.*

Proof: In case $r = 1$, there is nothing to prove. Now assume that $r > 1$, and choose a decomposition $B = B'_1 \oplus \cdots \oplus B'_r$ such that $B'_i \lesssim D$ for all i . Choose $D'_i \leq D$ such that $B'_i \cong D'_i$, and set $D' = D'_1 + \cdots + D'_r \leq D$. Then there exists an epimorphism $\phi : B \rightarrow D'$ induced by the isomorphisms from B'_i onto D'_i . Since $B'_r \cap \text{Ker}(\phi) = 0$, we must have $B = B'_r \oplus \text{Ker}(\phi) \oplus T$ for some T . Note that $B_r := B'_r \oplus T$ is isomorphic to D' . Also, $T' := \text{Ker}(\phi) \oplus T$ satisfies $T' \oplus B'_r = B = (B'_1 \oplus \cdots \oplus B'_{r-1}) \oplus B'_r$ and so $T' \cong B'_1 \oplus \cdots \oplus B'_{r-1}$. Now $\text{Ker}(\phi) \leq T' \lesssim (r-1)D' \cong (r-1)B_r$. By induction, there exists a decomposition $\text{Ker}(\phi) = B_1 \oplus \cdots \oplus B_{r-1}$ such that $B_1 \lesssim B_2 \lesssim \cdots \lesssim B_{r-1} \lesssim B_r$. Thus $B = B_1 \oplus \cdots \oplus B_{r-1} \oplus B_r$ and the result follows. \blacksquare

We thank E. Pardo for a simplification of the original version of the following proposition.

Proposition 3.4. *Let R be a stably finite simple regular ring. Then the following conditions are equivalent:*

- (a) R satisfies (DD).
- (b) For any $D \in L(R_R)$ there exists $K \geq 1$ such that, for $A \in L(R_R)$, if $K[A]_a \ll [D]_a$ then $A \prec D$.
- (c) For any $D \in \mathcal{P}$ there exists $K \geq 1$ such that, for $A \in \mathcal{P}$, if $K[A]_a \ll [D]_a$ then $A \prec D$.

Proof: Obviously (c) \Rightarrow (b).

(b) \Rightarrow (a): Clearly, we can suppose that R is not artinian. Let $D \in L(R_R)$ and write $D = E_1 \oplus E_2 \oplus E_3$, where $E_1 \neq 0$ and $E_2 \cong E_1$. There exists $K' \geq 1$ such that, for $B \in L(R_R)$, if $K'[B]_a \ll [E_1]_a$ then $B \prec E_1$. Choose a constant $K \geq K'[D]_a/[E_1]_a$. Then it is clear that, for $A \in L(R_R)$, if $K[A]_a \ll [D]_a$ then $2A \prec E_1 \oplus E_2 \leq D$.

(a) \Rightarrow (c): We may suppose that $D \neq 0$. Choose a nonzero $E \in L(R_R)$ such that $E \lesssim D$. By simplicity, $D \lesssim mE$ for some $m \geq 1$. By (DD), there exists $K' \geq 1$ such that for $A \in L(R_R)$, if $A \leq E$ and $K'[A]_a \ll [E]_a$ then $2A \lesssim E$.

Set $K = K'm$, and consider $A \in \mathcal{P}$ such that $K[A]_a \ll [D]_a$; then $K'[A]_a \ll [E]_a$. By simplicity, $A \lesssim rE$ for some $r \geq 1$. Hence, by Lemma 3.3 there exists a decomposition $A = A_1 \oplus \cdots \oplus A_r$ such that $A_1 \lesssim A_2 \lesssim \cdots \lesssim A_r \lesssim E$. Since $A_r \lesssim E$ and $K'[A_r]_a \leq K'[A]_a \ll [E]_a$, we obtain $2A_r \lesssim E$. Now $A_{r-1} \oplus A_r \lesssim 2A_r \lesssim E$ and $K'[A_{r-1} \oplus A_r]_a \leq K'[A]_a \ll [E]_a$, whence $2(A_{r-1} \oplus A_r) \lesssim E$. Continuing by induction, we find that $2A \lesssim E$, and therefore $A \prec D$. ■

Let R be a directly finite simple regular ring. We say that R satisfies (dd) if for any $D \in L(R_R)$ there exists $n \geq 1$ such that, for $A \in L(R_R)$, if $A \leq D$ and $nA \lesssim R$ then $2A \lesssim R$. The proof of the following proposition is analogous to that of Proposition 3.4, so that we will omit it.

Proposition 3.5. *Let R be a directly finite simple regular ring. Then R satisfies weak comparability if and only if R satisfies (dd).*

Corollary 3.6. *Let R be a stably finite simple regular ring satisfying (DD). Then $M_n(R)$ satisfies weak comparability for all $n \geq 1$. In particular, R is unit-regular.*

Proof: By Lemma 3.1, $M_n(R)$ satisfy (DD) for all $n \geq 1$. Since property (DD) obviously implies property (dd), the result follows from Proposition 3.5 and Theorem 1.11. ■

4. Simple regular rings with a unique rank function.

Let R be a regular ring and let s, t be positive integers. Consider the following comparability conditions:

aa-comparability: For $x, y \in R$, either $xR \lesssim_{aa} yR$ or $yR \lesssim_{aa} xR$.

a-comparability: For $x, y \in R$, either $xR \lesssim_a yR$ or $yR \lesssim_a xR$.

(s:t)-comparability ([G1, p. 275]): For any $x, y \in R$, either $t(xR) \lesssim s(yR)$ or $t(yR) \lesssim s(xR)$.

approximate (s:t)-comparability ([G1, p. 275]): For any $x, y \in R$, there exists a positive integer n such that either $nt(xR) \lesssim ns(yR)$ or $nt(yR) \lesssim ns(xR)$.

In case $t = 1$ we abbreviate the two latter terms to "s-comparability" and "approximate s-comparability" respectively.

Proposition 4.1. (cf. [G1, Theorem 18.17]) *Let R be a stably finite simple regular ring. Then the following conditions are equivalent:*

- (a) R has a unique rank function.
- (b) R satisfies the aa-comparability condition.
- (c) R satisfies approximate (s:t)-comparability for all integers $0 < t < s$.
- (d) R satisfies approximate (s:t)-comparability for some integers $s \geq t > 0$.

Proof: We can assume that R is not artinian.

By [G1, Theorem 18.3], R has a rank function N .

(a) \Rightarrow (b): Assume that N is the unique rank function on R . Then, given $x, y \in R$ either $[xR]_a - [yR]_a > 0$, $[yR]_a - [xR]_a > 0$ or $[xR]_a = [yR]_a$. This corresponds to $n(yR) < n(xR)$ some n , $m(xR) < m(yR)$ some m , or $xR \cong_{aa} yR$, respectively. Thus, in any case, either $xR \lesssim_{aa} yR$ or $yR \lesssim_{aa} xR$.

(b) \Rightarrow (c): Let s and t be integers such that $0 < t < s$ and let $0 \neq x, y \in R$. Assume that $xR \lesssim_{aa} yR$. Choose $0 \neq z \in R$ such that $t(zR) \lesssim yR$. There exists $n \geq 1$ such that $n(xR) \lesssim n(yR \oplus zR)$ and so

$$nt(xR) \lesssim nt(yR) \oplus nt(zR) \lesssim ns(yR).$$

Consequently, R satisfies approximate (s:t)-comparability.

(c) \Rightarrow (d): Obvious

(d) \Rightarrow (a): The same proof as in [G1, Theorem 18.17] applies. ■

Corollary 4.2. *Let R be a stably finite simple regular ring. If R satisfies the aa-comparability condition then so does $M_n(R)$ for all $n \geq$*

1. Moreover, given finitely generated projective modules A, B , either $nA < nB$ for some $n \geq 1$, $mB < mA$, for some $m \geq 1$, or $A \cong_{aa} B$

Proof: Since $M_n(R)$ has a unique rank function the above proof applies to finitely generated projective modules. ■

Proposition 4.3. *Let R be a directly finite simple regular ring satisfying a -comparability. Then R is unit-regular.*

Proof: It is clear that R satisfies 2-comparability. By [O, Corollary 2], R is unit-regular. ■

By a slight modification of the proof in [G1, Proposition 8.2], we obtain the following result.

Proposition 4.4. *Let R be a directly finite simple regular ring. If R satisfies the a -comparability condition, then so does $M_n(R)$ for all $n \geq 1$.*

Proof: We can assume that R is not artinian.

We will prove that for finitely generated projective modules A, B , either $A \lesssim_a B$ or $B \lesssim_a A$. By induction, assume the result is true for proper submodules of $(n-1)R$, and let A, B with $A, B < nR$.

Write $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ with $A_1, A_2, B_1, B_2 < (n-1)R$. Now either $A_1 \lesssim_a B_1$ or $B_1 \lesssim_a A_1$, and either $A_2 \lesssim_a B_2$ or $B_2 \lesssim_a A_2$. We need only consider the case where $A_1 \lesssim_a B_1$ and $B_2 \lesssim_a A_2$. Let $0 \neq C \in L(R_R)$ be such that $B_1 \oplus C < (n-1)R$ and $A_2 \oplus C < (n-1)R$. Then $A_1 \lesssim B_1 \oplus C$ and $B_2 \lesssim A_2 \oplus C$, so $B_1 \oplus C = B'_1 \oplus B''_1$, $A_2 \oplus C = A'_2 \oplus A''_2$ with $B'_1 \cong A_1$ and $A'_2 \cong B_2$. So either $B''_1 \lesssim_a A'_2$ or $A''_2 \lesssim_a B''_1$. Assume that $B''_1 \lesssim_a A'_2$. Then $B_1 \oplus B_2 \oplus C \cong B'_1 \oplus B''_1 \oplus A'_2 \lesssim_a B'_1 \oplus A''_2 \oplus A'_2 \cong A_1 \oplus A_2 \oplus C$, and so $B_1 \oplus B_2 \lesssim_a A_1 \oplus A_2$, since R is unit-regular by Proposition 4.3. ■

Proposition 4.5. *Let R be a simple regular ring with a unique rank function N . Then the following are equivalent:*

- (a) R satisfies the a -comparability axiom.
- (b) R is strictly unperforated.
- (c) For $x, y \in R$, $n(xR) < n(yR)$ implies $xR < yR$

Proof: We can assume that R is not artinian.

(a) \Rightarrow (b): Assume that $nA < nB$. If $B \lesssim_a A$ then write $nB = T_1 \oplus T_2$ with $T_1 \cong nA$ and $T_2 \neq 0$. Choose $0 \neq T$ with $nT < T_2$. Then $B \lesssim A \oplus T$ so $nB \lesssim nA \oplus nT < T_1 \oplus T_2 = nB$, contradiction. So $A \lesssim_a B$. Applying

the same argument to $A \oplus T$, we see that $A \oplus T \lesssim_a B$. Consequently, $A \oplus T \prec B \oplus T$ and so, $A \prec B$ because R is unit-regular.

(b) \Rightarrow (c): Obvious.

(c) \Rightarrow (a): Since R has a unique rank function, we see from Proposition 4.1 that either $xR \lesssim_{aa} yR$ or $yR \lesssim_{aa} xR$. If $xR \lesssim_{aa} yR$, then for $0 \neq C \in L(R_R)$ we have $n(xR) \prec n(yR \oplus C)$, and so $xR \prec yR \oplus C$ provided that $yR \oplus C \lesssim R$. We can always assume this except in the case where $yR = R$. But if $yR = R$ then $xR \lesssim_a R = yR$ obviously. Consequently, R satisfies a-comparability. ■

Appendix.

We prove the following result:

Proposition A1. *Let R be a strictly unperforated non-artinian simple unit-regular ring. Let $\Phi : K_0(R) \rightarrow \text{Aff}(S(K_0(R), [R]))$ be the natural map. Then $\Phi(K_0(R)^+)$ is dense in $\text{Aff}(S(K_0(R), [R]))^+$.*

To prove Proposition A1, it clearly suffices to prove a corresponding result for partially ordered abelian groups (Theorem A3).

Note that if G is a partially ordered abelian group then its torsion subgroup T is a convex subgroup, and so G/T is a partially ordered abelian group with respect to the induced ordering. A special case of a result of Elliott [E, Theorem 4.5] says that if G is a strictly unperforated interpolation group, then G/T is an unperforated interpolation group. Since the proof of this case is much easier than the proof of [E, Theorem 4.5], we give the details.

Proposition A2. (Elliott) *Let G be a directed strictly unperforated interpolation group, and let T be its torsion subgroup. Then G/T is a dimension group.*

Proof: It is clear that since G is directed, so is G/T .

First consider $x \in G$ and $n \in \mathbf{N}$ such that $n(x + T) \geq 0$. If $x \in T$, then $x + T = 0$, and so we may assume that $x \notin T$. Now $nx + T = y + T$ for some $y \in G^+$, and $y > 0$ because $x \notin T$. Then $k(nx - y) = 0$ for some $k \in \mathbf{N}$, whence $knx = ky > 0$. Since G is strictly unperforated, $x > 0$, and hence $x + T > 0$. Thus G/T is unperforated.

Now consider $x_1, x_2, y_1, y_2 \in G$ such that $x_i + T \leq y_j + T$ for all i, j . If $x_r + T = y_s + T$ for some r, s , then $x_i + T \leq x_r + T \leq y_j + T$ for all i, j . Hence, we may assume that $x_i + T < y_j + T$ for all i, j . Consequently, there are nonzero elements $w_{ij} \in G^+$ such that $x_i + w_{ij} + T = y_j + T$. There is some $k \in \mathbf{N}$ such that $k(x_i + w_{ij} - y_j) = 0$ for all i, j , whence

$kx_i < ky_j$ and so $x_i < y_j$ for all i, j , by strict unperforation. Thus there exists $z \in G$ such that $x_i \leq z \leq y_j$ for all i, j , and hence $x_i + T \leq z + T \leq y_j + T$ for all i, j . Therefore G/T is an interpolation group. ■

Theorem A3. *Let (G, u) be a strictly unperforated simple interpolation group with order-unit, such that G^+ contains no atoms. Let $\Phi : G \rightarrow \text{Aff}(S)$ be the natural map, where $S = S(G, u)$. Then $\Phi(G^+)$ is dense in $\text{Aff}(S)^+$.*

Proof: Let T be the torsion subgroup of G , and note that G/T is simple and that the element $u' := u + T$ is an order-unit in G/T . By Proposition A2, G/T is a dimension group.

Suppose that $(G/T)^+$ contains an atom, say $x+T$ where $x \in G^+$. Since x cannot be an atom in G^+ , there exists $y \in G$ such that $0 < y < x$. But then $0 + T < y + T < x + T$ (because $T \cap G^+ = \{0\}$), contradicting our assumption about $x + T$. Therefore $(G/T)^+$ contains no atoms.

Let $\pi : G \rightarrow G/T$ be the quotient map, and set $S' = S(G/T, u')$. The induced map $\pi^* : S' \rightarrow S$ is an affine homeomorphism, and hence the induced map $\pi^{**} : \text{Aff}(S) \rightarrow \text{Aff}(S')$ is an isomorphism of ordered Banach spaces. There is a commutative diagram as follows, where Φ' is the natural map.

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & \text{Aff}(S) \\ \pi \downarrow & & \downarrow \pi^{**} \\ G/T & \xrightarrow{\Phi'} & \text{Aff}(S') \end{array}$$

Since $(G/T)^+ = \pi(G^+)$, it suffices to prove that $\Phi'((G/T)^+)$ is dense in $\text{Aff}(S')^+$. Thus there is no loss of generality in assuming that G is a simple dimension group, with no atoms in G^+ .

Since G^+ has no atoms, G is not cyclic. Therefore, by [G4, Theorem 14.14], $\Phi(G^+)$ is dense in $\text{Aff}(S)^+$. ■

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Note added in proof. E. Pardo has proved that any non-artinian stably finite simple regular ring satisfies condition (D).

References

- [B1] B. BLACKADAR, A stable cancellation theorem for simple C^* -algebras, *Proc. London Math. Soc.* **47** (1983), 303–305.
- [B2] B. BLACKADAR, Comparison theory for simple C^* -algebras, in “Operator Algebras and Application”, D.E. Evans and M. Takesaki (eds.), LMS Lecture Notes Series **135**, Cambridge Univ. Press, 1988, pp. 21–54.
- [B3] B. BLACKADAR, Rational C^* -algebras and nonstable K -theory, *Rocky Mountain J. Math.* **20** (1990), 285–316.
- [Bu] C. BUSQUÉ, Directly finite aleph-nought-complete regular rings are unit-regular, in Proceedings of the First International Meeting on Ring Theory, Granada, Spain 1986, J.L. Bueso, P. Jara and B. Torrecillas (eds.), Lecture Notes in Math. **1328**, Springer-Verlag, Berlin-New York, 1988, pp. 38–49.
- [E] G. A. ELLIOTT, Dimension groups with torsion, *Internat. J. Math.* **1** (1990), 361–380.
- [G1] K. R. GOODEARL, “Von Neumann regular rings,” Pitman, London, 1979.
- [G2] K. R. GOODEARL, Directly finite aleph-nought-continuous regular rings, *Pacific J. Math.* **100** (1982), 105–122.
- [G3] K. R. GOODEARL, Metrically complete regular rings, *Trans. Amer. Math. Soc.* **272** (1982), 275–310.
- [G4] K. R. GOODEARL, “Partially ordered abelian groups with interpolation,” Math. Surveys and Monographs **20**, Amer. Math. Soc., Providence, 1986.
- [G5] K. R. GOODEARL, Unpublished notes on simple regular rings, 1990.
- [GM] K. R. GOODEARL AND P. MENAL, Stable range one for rings with many units, *J. Pure Applied Algebra* **54** (1988), 261–287.
- [GMM] K. R. GOODEARL, P. MENAL AND J. MONCASI, Free and residually artinian regular rings, *J. Algebra* (to appear).
- [M] J. MONCASI, Rang estable en anells regulars, Ph.D. Thesis, Universitat Autònoma de Barcelona, 1984.
- [MM] P. MENAL AND J. MONCASI, On regular rings with stable range 2, *J. Pure Applied Algebra* **24** (1982), 25–40.
- [O] K. C. O’MEARA, Simple regular rings satisfying weak comparability, *J. Algebra* **141** (1991), 162–186.

- [R] M. RIEFFEL, The cancellation Theorem for projective modules over irrational rotation C^* -algebras, *Proc. London Math. Soc.* **47** (1983), 285–302.
- [V] L. N. VASERSTEIN, Stable rank of rings and dimensionality of topological spaces, *Func. Anal. Applic.* **5** (1971), 102–110.
- [W] R. B. WARFIELD, JR., Cancellation of modules and groups and stable range of endomorphism rings, *Pacific J. Math.* **91** (1980), 457–485.

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