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ITERATED SERIES AND THE HELLINGER-TOEPLITZ THEOREM

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Abstract

We show that an iterated double series condition due to Antosik implies the uniform convergence of the double series. An application of Antosik's condition is given to the derivation of a vector form of the Hellinger-Toeplitz Theorem.

A problem which is frequently encountered in analysis is the interchanging of the summations in an iterated double series. That is, if $a_{ij} \in \mathbb{R}$, $i, j \in \mathbb{N}$ is a double sequence, when does the equality $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ hold? For example, if $a_{ij} \geq 0$ for all i, j, then this condition holds (where the sums may be infinite), and, in general, if $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| < \infty$, then the equality holds ([4]). Another condition which guarantees the equality is the existence of the double limit, $\lim_{m,n} \sum_{i=1}^{m}\sum_{i=1}^{n} a_{ij}$ along with the convergence of the series $\sum_{j=1}^{\infty} a_{ij}$, $\sum_{i=1}^{\infty} a_{ij}$ ([4]). The existence of this double limit is often difficlut to verify; one possible way to guarantee the existence of the double limit is to show that one of the iterated series is uniformly convergent, but this is also often difficult to verify. In this note we would like to point out the existence of a condition due to P. Antosik which involves only the iterated series and which guarantees the existence of the double limit and, hence, the equality of the two iterated series ([1]). Antosik's condition works equally well for vector-valued series so we present this version. To illustrate the utility of Antosik's condition, we establish a Hellinger-Toeplitz type theorem

concerning the continuity of matrix transformations between sequence spaces.

Throughout this note G will denote an Abelian topological group. Let $x_{ij} \in G$ for $i, j \in \mathbb{N}$. We assume that the series $\sum_{j=1}^{\infty} x_{ij} (\sum_{i=1}^{\infty} x_{ij})$ converges for each $i \in \mathbb{N}$ ($j \in \mathbb{N}$) and seek conditions which guarantee the equality (and existence) of the two iterated series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij}$, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij}$. One such condition is the existence of the double limit $\lim_{m,n} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}$ (denoted by $\sum_{i,j} x_{ij}$ and called the double series generated by x_{ij}) ([4]). We give a condition which involves only iterated series and which guarantees the existence of the double series. Since this condition involves only the iterated series, it may sometimes be easier to check than the existence of the double limit. We give an example of such a situation in proving the Hellinger-Toeplitz result given in Theorem 3.

Recall that a series $\sum_{i=1}^{\infty} x_i$ in G is subseries convergent if the series $\sum_{i=1}^{\infty} x_{n_i}$ converges in G for every subsequence $\{n_i\}$. The principal tool used in the proof of our main result on double series is the vector-valued generalization for subseries convergent of the classical Schur Lemma on weakly convergent series in l^1 ([**2**, 8.1]).

Theorem 1. *Suppose* $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{im_j}$ *converges for each increasing sequence of positive integers* $\{m_j\}$. *Then the double series* $\sum_{i,j} x_{ij}$ *converges and* $\sum_{i,j} x_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij}$.

Proof: Note that the series $\sum_{i=1}^{\infty} x_{ij}$ converges for each j (consider the difference between the two series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{in_j}$ and $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{im_j}$, where $n_k = k$ for each k and $\{m_k\}$ is the subsequence $\{1, \ldots, j-1, j+1, \ldots\}$).

Thus, if $\sigma \subset \mathbb{N}$ is finite, the meaning of $\sum_{i=1}^{\infty}\sum_{j\in\sigma}^{\infty} x_{ij}$ is clear; if $\sigma \subset \mathbb{N}$ is infinite, arrange the elements of σ into a subsequence $\{n_1, n_2, \dots\}$ and set $\sum_{i=1}^{\infty}\sum_{j\in\sigma}^{\infty} x_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{in_j}$. Set $z_{mj} = \sum_{i=1}^{m} x_{ij}$. Then for $\sigma \subset \mathbb{N}$, $\sum_{j\in\sigma} z_{mj} = \sum_{i=1}^{m}\sum_{j\in\sigma} x_{ij}$ converges to $\sum_{i=1}^{\infty}\sum_{j\in\sigma}^{\infty} x_{ij}$ as $m \to \infty$. By Schur's Lemma ([2, 8.1]), the series $\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} x_{ij})$ is subseries convergent and $\lim_m \sum_{i=1}^{\infty}\sum_{j\in\sigma} x_{ij} = \sum_{j\in\sigma}\sum_{i=1}^{\infty} x_{ij}$ uniformly for $\sigma \subset \mathbb{N}$. Hence, the double limit $\lim_{m,n} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}$ exists and equals $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij}$. ■

Example 2. The condition in Theorem 1 is sufficient for the existence of the double series (and the equality of the 2 iterated series), but it is not necessary. In the scalar case the hypothesis of Theorem 1 implies that the rows of the matrix $[x_{ij}]$ are absolutely convergent, so the matrix $x_{ij} = (-1)^{j+1}/i^2 j$ fails to satisfy the hypothesis of Theorem 1, but the double series $\sum_{i,j} x_{ij}$ converges.

Theorem 1 was proven for series in a space equipped with a sequential convergence structure satisfying certain convergence properties by Antosik in [1, 3.3]. An interesting aspect of the theorem, even for scalar-valued series, is that the condition in the hypothesis of the theorem only involves iterated series and, therefore, can sometimes be easily checked. We give an example of such a condition in the proof of the Hellinger-Toeplitz result below.

The classical Hellinger-Toeplitz Theorem asserts that any matrix which maps l^2 into l^2 is (norm) continuous ([5]); we seek conditions on sequence spaces which will guarantee that matrix transformations between the sequence spaces are continuous. Since Theorem 1 is valid for vector-valued series, we consider vector-valued sequence spaces. Let X, Y be Hausdorff topological vector spaces and let $L(X, Y)$ be the space of all continuous linear operators from X into Y. Let $E(F)$ be a vector space of X-valued (Y-valued) sequences; if $x \in E$, we denote the kth coordinate of x by x_k so $x = \{x_k\}$. The β-dual of E (with respect to Y), denoted by $E^{\beta Y}$, is the space of all sequences $\{T_k\} = T \subset L(X, Y)$ such

that the series $\sum_{k=1}^{\infty} T_k x_k$ converges for each $x \in E$ ([9], Maddox does not require that the operators are continuous); if Y is the scalar field, we write $E^{\beta Y} = E^{\beta}$. If $x \in E$ and $T \in E^{\beta Y}$, we write $T \cdot x = \sum_{k=1}^{\infty} T_k x_k$. This gives a map $x \to T \cdot x$ from E into Y, and we let $\sigma(E, E^{\beta Y})$ be the weakest topology on E such that all of theses maps for $T \in E^{\beta Y}$ are continuous; when X and Y are the scalar field, this is just the weak topology $\sigma(E, E^{\beta})$ from the duality between E and its β-dual, E^{β}.

Let $A_{ij} \in L(X, Y)$ for $i, j \in \mathbb{N}$ and let A be the operator-valued matrix $[A_{ij}]$. We say that A maps E into F or $A \in (E, F)$ if for each $x \in E$, $i \in \mathbb{N}$, the series $\sum_{j=1}^{\infty} A_{ij} x_j$ converges and the sequence $\{\sum_{j=1}^{\infty} A_{ij} x_j\} \in F$, i.e., if the formal matrix product $Ax = \{\sum_{j=1}^{\infty} A_{ij} x_j\}$ belongs to F for each $x \in E$. We are interested in conditions which guarantee that a matrix $A \in (E, F)$ is continuous with respect to appropiate topologies on E and F. For example, the classical Hellinger-Toeplitz Theorem asserts that any (scalar) matrix $A \in (l^2, l^2)$ is norm continuous ([5]). Toeplitz and Köthe generalized this result to other sequence spaces ([8], [7, 34.7(7)]). We now use Theorem 1 to give a further generalization of the Toeplitz-Köthe result to vector-valued sequence spaces.

The pair (X, Y) is said to have the *Banach-Steinhaus property* if whenever $T_k \in L(X, Y)$ converges pointwise, $\lim T_k x = Tx$, for $x \in X$, then the limit operator T is continuous. For example, if X is an F-space or if X is a barrelled locally convex space and Y is a locally convex space, (X, Y) has the Banach-Steinhaus property.

The sequence space E is said to be *monotone* if $m_0 E = E$, where m_0 is the scalar sequence space consisting of all sequences with finite range and $m_0 E$ is the coordinatewise product of sequences in m_0 and sequences in E ([3]). In particular, any normal (scalar) sequence space is monotone ([6, 30.1]).

Further, $c_{00}(X)$ denotes the space of X-valued sequences which are 0 eventually; if X is the scalar field we write $c_{00}(X) = c_{00}$.

We now establish our Hellinger-Toeplitz result, which asserts that a matrix transformation $A \in (E, F)$ is continuous with respect to the weak topologies $\sigma(E, E^{\beta Y})$, $\sigma(F, F^{\beta Y})$ of E and F under appropriate conditions on E and F.

Theorem 3. *Let E be monotone and contain $c_{00}(X)$ and let (X, Y)*

have the Banach-Steinhaus Property. If the matrix $A = [A_{ij}]$ maps E into F, then A is $\sigma(E, E^{\beta Y}) - \sigma(F, F^{\beta Y})$ continuous.

Proof: Let $B = \{B_i\} \in F^{\beta Y}$ and let A^i be the ith row of A so $B \cdot Ax = \sum_i B_i(A^i \cdot x) = \sum_i \sum_j B_i A_{ij} x_j$ for $x \in E$. Note for each j the series $\sum_i B_i A_{ij}$ converges in the strong operator topology of $L(X, Y)$ (Fix j and for $x \in X$ let $\bar{x}$ be the vector in E with x in the jth coordinate and 0 elsewhere. Then $\sum_k A_{ik}\bar{x}_k = A_{ij}x$ so $\{A_{ij}x\}_i \in F$ and since $B \in F^{\beta Y}$, $\sum_i B_i A_{ij} x$ converges, i.e., $\sum_i B_i A_{ij}$ converges in the strong operator topology to an element of $L(X, Y)$ since (X, Y) has the Banach-Steinhaus Property). Since E is monotone, the series $\sum_i \sum_j B_i A_{in_j} x_{n_j}$ converges for each $x \in E$ and subsequence $\{n_j\}$. By the Interchange Theorem 1, if we set $C_j = \sum_i B_i A_{ij}$ and $C = \{C_j\}$, then $C \in E^{\beta Y}$ and $B \cdot Ax = \sum_i \sum_j B_i A_{ij} x_j = \sum_j \sum_i B_i A_{ij} x_j = C \cdot x$ so if $\{x^\delta\}$ is a net in E which is $\sigma(E, E^{\beta Y})$ convergent to 0, then Ax^δ is $\sigma(F, F^{\beta Y})$ convergent to 0 and A is continuous with respect to these topologies.

The (scalar) space l^2 obviously satisfies the hypothesis of Theorem 3 and if $A \in (l^2, l^2)$, then A is weakly continuous and, hence, norm continuous; this is just the classical Hellinger-Toeplitz Theorem. More generally, if E and F are scalar sequences and E is monotone and contains c_{00}, then any matrix map $A : E \to F$ is continuous with respect to the weak topologies $\sigma(E, E^\beta)$ and $\sigma(F, F^\beta)$. In particular, if E is normal (i.e., if $x \in E$ and $|y_k| \leq |x_k|$ for all k, then $y \in E$), then E is obviously monotone so the result holds in this case; this is essentially the version of the Hellinger-Toeplitz Theorem for sequence spaces due to Köthe ([**7**, 34.7.(7)]). Köthe's result uses α-duals (the α-dual of a scalar sequence space E is the set of all sequences $\{y_j\}$ such that $\sum |x_j y_j| < \infty$ for all $x \in E$; for monotone spaces $E^\alpha = E^\beta$) so is somewhat weaker than Theorem 3 since $\sigma(F, F^\beta)$ is stronger than $\sigma(F, F^\alpha)$. In the scalar case the proof of Theorem 1 above uses only the scalar version of the Schur Lemma ([**2**, 8.2]) and the proof of Theorem 3 is then much more elementary than that given by Köthe which uses results on projective limit topologies for locally convex spaces. A scalar version of Theorem 3 has been established in [**11**].

If Y is the scalar field, the spaces E and $E^{\beta Y} = E^\beta$ are in duality with

respect to the bilinear pairing $x \cdot y$, where $x \in E$ and y is an X'-valued sequence belonging to E^β ([10]). If E is monotone and X is barrelled, the hypothesis of Theorem 3 are satisfied so any matrix map A from E into F is continuous with respect to the weak topologies $\sigma(E, E^\beta)$ and $\sigma(F, F^\beta)$. In this case, which includes the case when X is also the scalar field, the matrix map A from E into F is also continuous with respect to the Mackey (strong) topologies of E and F, respectively ([10]). Moreover, the computation in Theorem 3 shows that the transpose of the operator A, $A' : F^\beta \to E^\beta$, is given by the matrix $[A'_{ji}]$. In this case, the transpose map, A', is continuous with respect to the weak (Mackey, strong) topologies of F^β and E^β, respectively ([7, 32.2]).

There are abundant examples of vector sequence spaces satisfying the hypothesis of Theorem 3. For example, $c_{00}(X)$ or $c_0(X)$, the vector space of all X-valued sequences which converge to 0, or $m_0(X)$, the vector space of all X-valued sequences with finite range, or $l^\infty(X)$, the space of all X-valued bounded sequences, are all monotone sequence spaces containing $c_{00}(X)$. If X is a normed space and $1 \leq p < \infty$, the space $l^p(X)$ consisting of all X-valued sequences such that $\sum_{k=1}^{\infty} \|x_k\|^p < \infty$ is monotone and contains $c_{00}(X)$.

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