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# $C_0$-FREDHOLM OPERATORS. IV

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*Abstract*

The purpose of this paper is to develop, in the context of operators of class $C_0$, a theory of Fredholm complexes analogous to that in [6], including an index stability result under perturbations. As a by-product, a simple proof of the additivity of the index for $C_0$-Fredholm operators will be given.

## 1. Introduction

In order to simplify both notation and statements we begin by reformulating certain facts in the theory of operators of class $C_0$ in terms of Hilbert modules over the algebra $H^\infty$ of bounded analytic functions in the unit disc. Let $K$ be a complex Hilbert space, and denote by $L(K)$ the algebra of bounded linear operators on $K$. If $T \in L(K)$ is an operator of class $C_0$ then we can turn $K$ into an $H^\infty$-module by setting

$$uk = u(T)k,\ u \in H^\infty,\ k \in K. \tag{1.1}$$

This module has the following properties:

(i) $\|uk\| \leq \|u\|_\infty \|k\|$, $u \in H^\infty$, $k \in K$ (in the terminology of [3], $K$ is a contractive module);

(ii) for each $k \in K$ the map $u \to uk$ is continuous if $H^\infty$ is given its weak* topology and $K$ is given its weak topology;

(iii) $K$ has nontrivial annihilator in $H^\infty$, i.e., $\{u \in H^\infty : uk = 0$ for all $k \in K\}$ is a nonzero ideal in $H^\infty$.

Conversely, if $K$ is a Hilbert $H^\infty$-module satisfying (i), (ii) and (iii), then (1.1) holds for some operator $T$ of class $C_0$. Therefore, a Hilbert $H^\infty$-module satisfying (i), (ii) and (iii) will be called a $C_0$-*module*. If $K$ is a $C_0$-module and $K' \subset K$ is a closed subspace such that $uk \in K'$ for all $u \in H^\infty$ and $k \in K'$, then $K'$ is called a *submodule* of $K$. Given a submodule $K' \subset K$ one can form the *quotient module* $K/K'$. With the quotient norm this is yet another $C_0$-module which can be identified as a Hilbert space with $K \ominus K'$ (the orthogonal complement of $K'$ in $K$). If $K_1$ and $K_2$ are $C_0$-modules then we denote by $\mathrm{Hom}(K_1, K_2)$ the Banach space of continuous $H^\infty$-module homomorphisms from $K_1$ to $K_2$. (These homomorphisms correspond, of course, with the intertwinings between the associated operators of class $C_0$.) We write $\mathrm{End}(K)$ for $\mathrm{Hom}(K,K)$. The modules $K_1$ and $K_2$ are *quasisimilar* if there exists a quasiaffinity in $\mathrm{Hom}(K_1,K_2)$; we recall that a *quasiaffinity* is an operator which is one-to-one and has dense range. It follows from well-known facts (cf. [1]) that quasisimilarity is indeed an equivalence relation for $C_0$-modules. We write $K_1 \sim K_2$ if $K_1$ and $K_2$ are quasisimilar.

Let $K$ be a $C_0$-module, and let $\langle\cdot,\cdot\rangle$ denote the scalar product in $K$. We define the *adjoint $C_0$-module* $K^*$ as follows. As a Hilbert space, $K^* = K$, and multiplication of $k \in K^*$ by $u \in H^\infty$, denoted $u\#k$, is given by

$$\langle u\#k, h\rangle = \langle k, u^\sim h\rangle,\ h \in K, \tag{1.2}$$

where $u^\sim(\lambda) = \overline{u(\overline{\lambda})}$. If $K$ determined via (1.1) by an operator $T$ of class $C_0$, then $K^*$ is likewise determined by the operator $T^*$. Clearly, if $\varphi \in \mathrm{Hom}(K_1,K_2)$, then the Hilbert space adjoint $\varphi^*$ belongs to $\mathrm{Hom}(K_2^*,K_1^*)$.

For every $C_0$-module $K$ we denote by $\mathrm{Lat}\ K$ the lattice of all submodules of $K$. Given a homomorphism $\varphi \in \mathrm{Hom}(K_1,K_2)$, there is an induced map $\varphi_+ : \mathrm{Lat}\ K_1 \to \mathrm{Lat}\ K_2$ given by $\varphi_+(M) = (\varphi M)^-$, $M \in \mathrm{Lat}\ K_1$. We say that $\varphi$ is a *lattice-isomorphism* if $\varphi_+$ is one-to-one and onto. Fix now a homomorphism $\varphi \in \mathrm{Hom}(K_1,K_2)$. Then $\ker\varphi$ is a submodule of $K_1$ and $(\mathrm{ran}\ \varphi)^-$ is a submodule of $K_2$. One can write now

$$\varphi = j\bar{\varphi}p, \tag{1.3}$$

where $j : (\mathrm{ran}\varphi)^- \to K_2$ denotes inclusion, $p : K_1 \to K_1/\ker\varphi$ denotes the canonical projection, and $\bar{\varphi}(k+\ker\varphi) = \varphi k$, $k \in K_1$. Clearly $\bar{\varphi}$ is a quasiaffinity. We say that $\varphi$ has *full range* if $\bar{\varphi}$ is a lattice isomorphism. We record for further use the following result from [1] (cf. Lemma 1.20 in Chapter 7).

**1.4. Lemma.** *If $\varphi \in \mathrm{Hom}(K_1,K_2)$ then $\varphi_+$ is one-to-one if and only if $(\varphi^*)_+$ is onto. Thus $\varphi$ has full range if and only if*

(i) *$\varphi_+$ is onto* $\mathrm{Lat}\ ((\mathrm{ran}\ \varphi)^-)$; *and*
(ii) *$(\varphi^*)_+$ is onto* $\mathrm{Lat}\ ((\mathrm{ran}\ \varphi^*)^-)$.

The second part of the lemma is not stated in [1], but the reader should have no difficulty deducing it from the first part. Let us note that $\varphi$ has full range if it has closed range. Indeed, if $\varphi$ has closed range then the homomorphism $\bar{\varphi}$ in (1.3) is in fact invertible.

Next we introduce a notion that corresponds with property $(P)$ for operators of class $C_0$ (cf. Chapter 7 of [1]). A $C_0$-module $K$ is said to be *finite* if it is not quasisimilar to any of its proper submodules. An equivalent characterization is that for every $\varphi \in \mathrm{End}(K)$ we have $\ker\varphi = \{0\}$ if and only if $(\mathrm{ran}\ \varphi)^- = K$. We collect for further reference some basic facts about finite modules (cf. [1]).

**1.5. Proposition.**

(i) *The property of finiteness is preserved by quasisimilarity.*

(ii) *Let $K$ be a $C_0$-module and $K'$ a submodule. Then $K$ is finite if and only if both $K'$ and $K/K'$ are finite.*

(iii) *A $C_0$-module $K$ is finite if and only if $K^*$ is finite.*

(iv) *If $\varphi \in \operatorname{Hom}(K_1, K_2)$ and at least one of the modules $K_1$ and $K_2$ is finite then $\varphi$ has full range.*

A basic fact in the sequel is the following result which compensates for the fact that homomorphisms with full range do not usually have closed range. (see Proposition 6.9 and Corollary 6.10 in Chapter 7 of [**1**]).

**1.6. Proposition.** *Let $K$, $K'$ and $K''$ be $C_0$-modules, and $\alpha \in \operatorname{Hom}(K', K)$, $\beta \in \operatorname{Hom}(K'', K)$. Assume that $(\operatorname{ran}\beta)^-$ is finite and $\operatorname{ran}\alpha \subset (\operatorname{ran}\beta)^-$. Then*

(i) $(\alpha^{-1}(\operatorname{ran}\beta))^- = K'$;

(ii) $(\operatorname{ran}\alpha \cap \operatorname{ran}\beta)^- \supset \operatorname{ran}\alpha$; *and*

(iii) *if $K'$ is cyclic then $\alpha^{-1}(\operatorname{ran}\beta)$ contains a cyclic vector of $K'$.*

We recall that $K$ has a *cyclic vector* $k$ if $K = (H^\infty k)^-$. In general $K$ has *finite cyclic multiplicity* if there exist vectors $k_1, k_2, \ldots, k_n \in K$ such that $K = (H^\infty k_1 + H^\infty k_2 + \cdots + H^\infty k_n)^-$. It is known that modules with finite multiplicity, in particular cyclic modules, are finite.

Next we introduce an equivalence relation on the class of finite $C_0$-modules. Two modules $K_1$ and $K_2$ are *equivalent* if there exists a finite module $K$, and $\varphi \in \operatorname{End}(K)$ such that $K_1 \sim \ker\varphi$ and $K_2 \sim \operatorname{coker}\varphi = K/(\operatorname{ran}\varphi)^-$. It is shown in [1] that this is indeed an equivalence relation (the proof of transitivity was first done in [4]). We will write $[K]$ for the equivalence class of the module $K$, and we will write $[K] = \infty$ if $K$ is not a finite module. The operation

$$[K_1] + [K_2] = [K_1 \oplus K_2]$$

turns the set of equivalence classes into a commutative semigroup with unit (the zero module). We record for further use some results proved in [1].

**1.7. Lemma.**

(i) *If $K_1$ and $K_2$ are quasisimilar then $[K_1] = [K_2]$.*

(ii) *If $K'$ is a submodule of $K$ then $[K] = [K'] + [K/K']$.*

We finally define the notion of semi-Fredholm homomorphisms - these are precisely the $C_0$-semi-Fredholm operators defined in Chapter 7 of [1].

Let $K_1$ and $K_2$ be two $C_0$-modules, and $\varphi \in \operatorname{Hom}(K_1, K_2)$. Then $\varphi$ is said to be *semi-Fredholm* if

(i) $\varphi$ has full range; and

(ii) either $\ker\varphi$ or $\operatorname{coker}\varphi$ is finite.

A semi-Fredholm homomorphism $\varphi$ is *Fredholm* if

(iii) both $\ker\varphi$ and coker $\varphi$ are finite.

If $\varphi$ is a semi-Fredholm homomorphism, the *index* of $\varphi$ is defined as

$$\operatorname{ind}\varphi = [\ker\varphi] - [\operatorname{coker}\varphi].$$

It is important to note that the semigroup of equivalence classes of finite modules does not have the cancellation property, and so it cannot be embedded in a group. Therefore differences in that semigroup must be treated formally; thus,

$$[K_1] - [K_2] = [K_3] - [K_4]$$

simply means $[K_1]+[K_4] = [K_2]+[K_3]$. See Chapter 7 of [1] for an identification of this semigroup as the class of generalized inner functions.

We conclude this section with a userful elementary result about homomorphisms with full range.

**1.8. Lemma.**

(i) *Let $\varphi_1$ and $\varphi_2$ be $C_0$-module homomorphisms. Then $\varphi_1 \oplus \varphi_2$ has full range if and only if both $\varphi_1$ and $\varphi_2$ have full range.*

(ii) *Let $\varphi, \psi \in \operatorname{Hom}(K_1, K_2)$ be such that $(\operatorname{ran}\psi)^-$ is finite. If $\varphi$ has full range then $\varphi + \psi$ has full range.*

*Proof:* (i) Set $\varphi = \varphi_1 \oplus \varphi_2$ and note that $\bar{\varphi} = \bar{\varphi}_1 \oplus \bar{\varphi}_2$. Therefore it suffices to consider the case in which $\varphi_1$ and $\varphi_2$ are quasiaffinites. That $\varphi_1$ and $\varphi_2$ are lattice-isomorphisms if $\varphi$ is a lattice-isomorphism is easy to see, and left as an exercise for the reader. Assume that $\varphi_1$ and $\varphi_2$ are lattice-isomorphisms, say $\varphi_1 : K_1 \to K_1', \varphi_2 : K_2 \to K_2'$. To show that $\varphi_+$ is onto it suffices to show that its range contains every cyclic module $M' \subset K_1' \oplus K_2'$. But if $M'$ is such a module, there are cyclic modules $M_1' \subset K_1'$ and $M_2' \subset K_2'$ such that $M' \subset M_1' \oplus M_2'$. Choose submodules $M_1 \subset K_1$, $M_2 \subset K_2$ such that $\varphi_{1+}(M_1) = M_1', \varphi_{2+}(M_2) = M_2'$ and note that $\varphi_+(M_1 \oplus M_2) = M_1' \oplus M_2'$. Thus $\varphi : M_1 \oplus M_2 \to M_1' \oplus M_2'$ has dense range and, since $M_1' \oplus M_2'$ is finite, it must have full range by Proposition 1.5(iv). Thus there exists $M \subset M_1 \oplus M_2$ such that $\varphi_+(M) = M'$. It remains to be shown that $\varphi_+$ is one-to-one, but this follows at once from the first part of the argument applied to $\varphi^*$, and from Lemma 1.4.

(ii) We can assume without loss of generality that $(\operatorname{ran}\varphi + \operatorname{ran}\psi)^- = K_2$. Under this assumption, the homomorphism $p : (\operatorname{ran}\psi)^- \to \operatorname{coker}\varphi$, obtained by restricting the canonical projection to $(\operatorname{ran}\psi)^-$, has dense range, whence we deduce that coker $\varphi$ is finite. It follows that $\varphi$ is semi-Fredholm and hence $\varphi+\psi$ is semi-Fredholm by Theorem 7.1 in Chapter 7 of [1]. In particular, $\varphi+\psi$ has full range. ■

## 2. Complexes

We define a *complex* to be a homomorphism $\delta \in \operatorname{End}(K)$, where $K$ is some $C_0$-module, and $\delta^2 = 0$. Most of the complexes we will consider will be $\mathbb{Z}_2$-*graded*. This means that $K$ can be written as a direct sum $K = K_0 \oplus K_1$ such that $\delta K_0 \subset K_1$ and $\delta K_1 \subset K_0$. In this case it is convenient to denote $\delta_0 \in \operatorname{Hom}(K_0, K_1)$ and $\delta_1 \in \operatorname{Hom}(K_1, K_0)$ the restrictions of $\delta$ to the two summands. The *homology module* $H(\delta)$ of a complex $\delta$ is the $C_0$-module $\ker \delta/(\operatorname{ran} \delta)^-$. If $\delta$ is $\mathbb{Z}_2$-graded we have $H(\delta) = H_0(\delta) \oplus H_1(\delta)$, where $H_0(\delta) = \ker \delta_0/(\operatorname{ran} \delta_1)^-$ and $H_1(\delta) = \ker \delta_1/(\operatorname{ran} \delta_0)^-$.

A $\mathbb{Z}_2$-graded complex $\delta$ will be called a *semi-Fredholm complex* if

(i) $\delta$ has full range; and

(ii) either $H_0(\delta)$ or $H_1(\delta)$ is finite

A semi-Fredhollm complex $\delta$ is *Fredholm* if

(iii) both $H_0(\delta)$ and $H_1(\delta)$ are finite.

The *index* of the semi-Fredholm complex $\delta$ is defined as

$$\operatorname{ind}(\delta) = [H_0(\delta)] - [H_1(\delta)].$$

To see the relationship between semi-Fredholm homomorphisms and semi-Fredholm complexes, we can associate with every homomorphism $\varphi \in \operatorname{Hom}(K_0, K_1)$ a complex $\delta \in \operatorname{End}(K_0 \oplus K_1)$ by setting $\delta_0 = \varphi$ and $\delta_1 = 0$. Then $\varphi$ is semi-Fredholm if and only if $\delta$ is semi-Fredholm, and $\operatorname{ind}(\delta) = \operatorname{ind}(\varphi)$.

**2.1. Proposition.** *Let $\delta \in \operatorname{End}(K_0 \oplus K_1)$ be a $\mathbb{Z}_2$-graded complex. If at least one of the modules $K_0$ and $K_1$ is finite then $\delta$ is semi-Fredholm and* $\operatorname{ind}(\delta) = [K_0] - [K_1]$. *If both $K_0$ and $K_1$ are finite then $\delta$ is Fredholm.*

*Proof:* If either $K_1$ or $K_2$ is finite then we know that $\delta_0$ and $\delta_1$ must have full range. Thus $\delta$ has full range by Lemma 1.8. If $K_0$ is finite then $\ker \delta_0 \subset K_0$ is also finite, and hence $H_0(\delta) = \ker \delta_0/(\operatorname{ran} \delta_1)^-$ is finite. Analogously we conclude that $\delta$ must be semi-Fredholm if $K_1$ is finite, and Fredholm if both $K_0$ and $K_1$ are finite. To calculate the index we note that $K_0/\ker \delta_0 \sim (\operatorname{ran} \delta_0)^-$ so that

$$[K_0] = [\ker \delta_0] + [(\operatorname{ran} \delta_0)^-] = [H_0(\delta)] + [(\operatorname{ran} \delta_1)^-] + [(\operatorname{ran} \delta_0)^-]$$

Analogously,

$$[K_1] = [H_1(\delta)] + [(\operatorname{ran} \delta_1)^-] + [(\operatorname{ran} \delta_0)^-],$$

whence

$$[K_0] + [H_1(\delta)] = [K_1] + [H_0(\delta)],$$

and this immediately gives the index of $\delta$. ∎

The preceding proposition has some immediate consequences pertaining to exact sequences. A sequence

$$K_0 \xrightarrow{\varphi_0} K_1 \xrightarrow{\varphi_1} K_2 \tag{2.2}$$

of homomorphisms will be said to be $C_0$-*exact* if

(i) $\varphi_0$ has full range; and
(ii) $\ker \varphi_1 = (\operatorname{ran} \varphi_0)^-$.

Recall that (2.2) is exact if $\ker \varphi_1 = \operatorname{ran} \varphi_0$, so that exactness implies $C_0$-exactness but not conversely. Analogously, a complex $\delta$ is $C_0$-exact if it has full range and $H(\delta) = \{0\}$.

**2.3. Corollary.** *Let $\delta \in \operatorname{End}(K_0 \oplus K_1)$ be a $\mathbb{Z}_2$-graded complex. Suppose that $\delta$ is $C_0$-exact and at least one of the modules $K_0$ and $K_1$ is finite. Then both $K_0$ and $K_1$ are finite, and $[K_0] = [K_1]$.*

*Proof:* We have $\operatorname{ind}(\delta) = 0$, and hence $[K_0] = [K_1]$ by Proposition 2.1. ■

**2.4. Corollary.** *Let*

$$0 \longrightarrow K_0 \xrightarrow{\varphi_0} K_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{n-1}} K_n \longrightarrow 0$$

*be a $C_0$-exact sequence of homomorphisms. Then $[K_0]-[K_1]+\cdots+(-1)^n[K_n] = 0$.*

*Proof:* Define $M_0 = K_0 \oplus K_2 \oplus \dots$ and $M_1 = K_1 \oplus K_3 \oplus \dots$, and define a complex $\delta \in \operatorname{End}(M_0 \oplus M_1)$ by

$$\delta_0(k_0 \oplus k_2 \oplus \dots) = \varphi_0 k_0 \oplus \varphi_2 k_2 \oplus \dots,$$
$$\delta_1(k_1 \oplus k_3 \oplus \dots) = \varphi_1 k_1 \oplus \varphi_3 k_3 \oplus \dots.$$

Then $\delta$ is $C_0$-exact so that $[M_0] = [M_1]$ by Corollary 2.4. ■

This last corollary allows one to give an easy proof, in the spirit of [7], of the fact that $\operatorname{ind}(\psi\varphi) = \operatorname{ind}(\psi) + \operatorname{ind}(\varphi)$ if $\varphi$ and $\psi$ are Fredholm homomorphisms, say $\varphi \in \operatorname{Hom}(K_0, K_1)$ and $\psi \in \operatorname{Hom}(K_1, K_2)$. Indeed, one can form the sequence

$$\begin{aligned} 0 \longrightarrow \ker\varphi \xrightarrow{\varphi_0} \ker(\psi\varphi) \xrightarrow{\varphi_1} \ker\psi \xrightarrow{\varphi_2} \operatorname{coker}\varphi& \\ \xrightarrow{\varphi_3} \operatorname{coker}(\psi\varphi) \xrightarrow{\varphi_4} \operatorname{coker}\psi \longrightarrow 0&, \end{aligned} \tag{2.5}$$

where $\varphi_0$ is inclusion, $\varphi_1 k = \varphi k$ if $k \in \ker(\psi\varphi)$, $\varphi_2$ is the canonical projection onto $\operatorname{coker}\varphi = K_1/(\operatorname{ran}\varphi)^-$, $\varphi_3(k+(\operatorname{ran}\varphi)^-) = \psi k+(\operatorname{ran}(\psi\varphi))^-$, and $\varphi_4(k+(\operatorname{ran}(\psi\varphi))^-) = k+(\operatorname{ran}\varphi)^-$. The index formula follows at once from Corollary 2.4 and the following result.

**2.6. Lemma.** *The sequence (2.5) is $C_0$-exact.*

*Proof:* Since all modules in (2.5) are finite, all homomorphisms $\varphi_j$ have full range. Clearly $\varphi_0$ is one-to-one and ran $\varphi_0 = \ker\varphi_1$. That $\varphi_2\varphi_1 = 0$ is immediate. Now, clearly $\ker\varphi_2 = \ker\psi \cap (\text{ran } \varphi)^-$, and since $\varphi$ is Fredholm, there exists a submodule $M \subset K_0$ such that $(\varphi M)^- = \ker\varphi_2$. Since $(\varphi M)^- \subset \ker\psi$, we have $M \subset \ker(\psi\varphi)$ and hence

$$(\text{ran } \varphi_1)^- \supset (\varphi_1 M)^- = (\varphi M)^- = \ker\varphi_2.$$

Next note that

$$\ker\varphi_3 = \{k + (\text{ran } \varphi)^- : \psi k \in (\text{ran } (\psi\varphi))^-\}, \text{ and}$$
$$\text{ran } \varphi_2 = \{k + (\text{ran } \varphi)^- : \psi k = 0\},$$

so that clearly $\varphi_3\varphi_2 = 0$. Suppose that $k$ is such that $\psi k \in (\text{ran}(\psi\varphi))^-$, and denote by $M_1 \subset K_1$ and $M_2 \subset K_2$ the cyclic modules generated by $k$ and $\psi k$, respectively. Since $\psi\varphi$ has full range, there exists $M_0 \subset K_0$ such that $(\psi\varphi M_0)^- = M_2$. By Proposition 1.6 we have that $M_1 \cap \psi^{-1}(\psi\varphi M_0)$ is dense in $M_1$. Thus, given $\varepsilon > 0$, there exists $k' \in M_1$ such that $\|k - k'\| < \varepsilon$ and $\psi k' = \psi\varphi h$ for some $h \in K_0$. Then we can write

$$k' + (\text{ran } \varphi)^- = k' - \varphi h + (\text{ran } \varphi)^- \in \text{ran } \varphi_2$$

because $\psi(k' - \varphi h) = 0$. Since $\varepsilon > 0$ is arbitrary, it follows that ran $\varphi_2$ is dense in $\ker\varphi_3$. We have

$$\text{ran } \varphi_3 = \{\psi k + (\text{ran } (\psi\varphi))^- : k \in K_1\}, \text{ and}$$
$$\ker\varphi_4 = \{k' + (\text{ran } (\psi\varphi))^- : k' \in (\text{ran } \varphi)^-\},$$

and it is immediate that ran $\varphi_3$ is dense in $\ker\varphi_4$. Finally, $\varphi_4$ is onto. ■

## 3. Homomorphisms between complexes

Let $\delta' \in \text{End}(K')$ and $\delta \in \text{End}(K)$ be two complexes. A *homomorphism* $\varphi : \delta' \to \delta$ is simply an element $\varphi \in \text{Hom}(K', K)$ such that $\varphi\delta' = \delta\varphi$. If $\delta$ and $\delta'$ are $\mathbb{Z}_2$-graded with decompositions $K' = K_0' \oplus K_1'$ and $K = K_0' \oplus K_1'$, we will also require that $\varphi K_j' \subset K_j$, $j = 0, 1$. If $\varphi : \delta' \to \delta$ is a homomorphism, there is an induced homomorphism $\varphi_* \in \text{Hom}(H(\delta'), H(\delta))$ defined by $\varphi_*(k' + (\text{ran } \delta')^-) = \varphi k' + (\text{ran } \delta)^-$, $k' \in \ker\delta'$. This homomorphism is well-defined since $\varphi \ker\delta' \subset \ker\delta$ and $\varphi(\text{ran } \delta')^- \subset (\text{ran } \delta)^-$.

Consider now an exact (not just $C_0$-exact!) sequence

$$0 \longrightarrow \delta' \xrightarrow{\varphi} \delta \xrightarrow{\psi} \delta'' \longrightarrow 0 \tag{3.1}$$

of homomorphisms between complexes. By analogy with ususal homological algebra [2] we will define a *connecting homomorphism* $\partial : H(\delta'') \to H(\delta')$ as follows. Consider an element $k'' \in \ker\delta''$. Since $\psi$ is onto, we have $k'' = \psi k$ for some $k$, and $\psi\delta k = \delta''\psi k = \delta'' k'' = 0$. Therefore $\delta k = \varphi k'$ for some $k'$, and $\varphi\delta' k' = \delta\varphi k' = \delta\delta k = 0$ so that $\delta' k' = 0$ because $\varphi$ is one-to-one. We set $\partial(k'' + (\text{ran } \delta'')^-) = k' + (\text{ran } \delta')^-$.

**3.2. Lemma.** *The map $\partial$ is a well defined homomorphism in* $\mathrm{Hom}(H(\delta''), H(\delta'))$.

*Proof:* Let $k', k$ and $k''$ be such that $\delta'' k'' = 0$, $\psi k = k'$, and $\varphi k' = \delta k$. It suffices to prove that there exists a constant $C > 0$ such that $\operatorname{dist}(k'', (\operatorname{ran} \delta'')^-) < 1$ implies $\operatorname{dist}(k', (\operatorname{ran} \delta')^-) < C$. To do this observe that since $\psi$ has closed range, there exists a constant $A > 0$ such that $\operatorname{dist}(k, \ker \psi) \leq A\|\psi k\|$. Analogously, since $\varphi$ has closed range, $B\|\varphi k\| \geq \|k'\|$ for all $k'$. Assume that $\operatorname{dist}(k'', (\operatorname{ran} \delta'')^-) < 1$, and choose $k_1''$ such that $\|k'' - \delta'' k_1''\| < 1$. Choose next $k_1$ such that $\psi k_1 = k_1''$ and note that we must have

$$\operatorname{dist}(k - \delta k_1, \ker \psi) < A.$$

By exactness, we must be able to find $k_1'$ such that $\|k - \delta k_1 - \varphi k_1'\| < A$. We have then

$$\|\delta k - \delta\varphi k_1'\| = \|\delta(k - \delta k_1 - \varphi k_1')\| \leq A\|\delta\|$$

so that

$$\|k' - \delta' k_1'\| \leq B\|\varphi(k' - \delta' k_1')\| = B\|\delta k - \delta\varphi k_1'\| \leq BA\|\delta\|.$$

Thus $\operatorname{dist}(k', (\operatorname{ran} \delta')^-) \leq BA\|\delta\|$. We conclude that $\partial$ is well-defined and $\|\partial\| \leq BA\|\delta\|$. ■

If $\delta \in \mathrm{End}(K)$ is a complex then $\delta^* \in \mathrm{End}(K^*)$ is also a complex. Moreover, since $\ker \delta^* = (\operatorname{ran} \delta)^\perp$ and $(\operatorname{ran} \delta^*)^- = (\ker \delta)^\perp$ we see upon identifying $H(\delta)$ with $\ker \delta \ominus (\operatorname{ran} \delta)^-$ that $H(\delta^*) = H(\delta)^*$. Now, if $\varphi : \delta' \to \delta$ is a homomorphism, then $\varphi^* : \delta^* \to \delta'^*$ is another homomorphism and hence there is an induced $(\varphi^*)_* \in \mathrm{Hom}(H(\delta^*), H(\delta'^*))$. If we identify $H(\delta^*) = H(\delta)^*$ as above, it is immediate that $(\varphi^*)_* = (\varphi_*)^*$. The following result is of a similar nature, but somewhat more difficult to verify.

**3.3. Lemma.** *Let $\partial$ be the connecting homomorphism of the exact sequence (3.1). Then the connecting homomorphism of the exact sequence*

$$0 \longrightarrow \delta''^* \xrightarrow{\psi^*} \delta^* \xrightarrow{\varphi^*} \delta'^* \longrightarrow 0$$

*is precisely $\partial^*$.*

*Proof:* Assume that $\delta' \in \mathrm{End}(K')$, $\delta \in \mathrm{End}(K)$ and $\delta'' \in \mathrm{End}(K'')$. There is a unique linear map $\psi^- : K'' \to K \ominus \ker \psi$ such that $\psi\psi^- = I_{K''}$, and a unique map $\varphi^- : K \to K'$ such that $\varphi^-\varphi = I_{K'}$ and $\ker \varphi^- = K \ominus \operatorname{ran} \varphi$. In addition, one can verify easily that $(\varphi^*)^- = (\varphi^-)^*$ and $(\psi^*)^- = (\psi^-)^*$. Upon identifying $H(\delta')$ and $H(\delta'')$ as subspaces of $K'$ and $K''$, respectively, we claim that

$$\partial = P_{H(\delta')}\varphi^-\delta\psi^-|H(\delta''),$$

where $P_M$ denotes orthogonal projection onto $M$. Indeed, if $k'' \in \ker \delta''$ then $k = \psi^- k''$ satisfies $\psi k = k''$ and hence $k' = \varphi^- \delta k$ satisfies $\varphi k' = \delta k$. Now the lemma becomes obvious because the connecting homomorphism of the adjoint sequence is

$$\begin{aligned} P_{H(\delta''^*)}(\psi^*)^- \delta^* (\varphi^*)^- | H(\delta'^*) &= P_{H(\delta'')}(\psi^-)^* \delta^* (\varphi^-)^* | H(\delta') \\ &= \left(P_{H(\delta')} \varphi^- \delta \psi^- | H(\delta'')\right)^* = \partial^*. \ \blacksquare \end{aligned}$$

With these technical lemmas out of the way, we can prove the $C_0$-exactness of the long homology sequence.

**3.4. Theorem.** *Let $\partial$ be the connecting homomorphism of the exact sequence (3.1). If $\delta', \delta$ and $\delta''$ have full range then the triangle*

$$\begin{array}{ccc} H(\delta') & \xrightarrow{\varphi_*} & H(\delta) \\ \partial \nwarrow & & \nearrow \psi_* \\ & H(\delta'') & \end{array}$$

*is $C_0$-exact.*

*Proof:* The equalities $\partial \psi_* = 0$, $\psi_* \varphi_* = 0$, and $\varphi_* \partial = 0$ are immediate. We will prove that the lattice maps $(\varphi_*)_+$, $(\psi_*)_+$, and $\partial_+$ are onto $\mathrm{Lat}(\ker \psi_*)$, $\mathrm{Lat}(\ker \partial)$, and $\mathrm{Lat}(\ker \varphi_*)$ respectively, and the theorem will follow from Lemmas 1.4, 3.3, and the remarks preceding Lemma 3.3. Notice that it suffices to show that the range of $(\varphi_*)_+, \ldots$ contains every cyclic submodule of $\mathrm{Lat}(\ker \psi_*)$, ... .

Let $k + (\text{ran } \delta)^- \in \ker \psi_*$, i.e., $k \in \ker \delta$ and $\psi k \in (\text{ran } \delta'')^-$. Denote by $M \subset \ker \delta$ the cyclic submodule generated by $k$, and note that since $\psi M \subset (\text{ran } \delta'')^-$ and $\delta''$ has full range, we have by Proposition 1.6 that $M \cap \psi^{-1}(\text{ran } \delta'')$ contains a cyclic vector $k_1$ for $M$. Thus $\psi k_1 \in \text{ran } \delta''$, say $\psi k_1 = \delta'' k''$. Now $\psi$ is onto, so we have $\psi k_2 = k''$ for some $k_2$, whence $\psi(k_1 - \delta k_2) = 0$. Thus $k_1 - \delta k_2 = \varphi k'$ for some $k'$, and $\varphi(\delta' k') = \delta(k_1 - \delta k_2) = 0$. Therefore $k' \in \ker \delta'$ and

$$k_1 + (\text{ran } \delta)^- = \varphi_*(k' + (\text{ran } \delta')^-) \in \text{ran } \varphi_*.$$

We conclude that the cyclic module generated by $k_1 + (\text{ran } \delta)^-$ belongs to the range $(\varphi_*)_+$. Clearly though this cyclic module coincides with that generated by $k + (\text{ran } \delta)^-$, and this shows that $(\varphi_*)_+$ is onto $\mathrm{Lat}(\ker \psi_*)$.

Next consider an element $k'' + (\text{ran } \delta'')^- \in \ker \partial$. Thus $k'' \in \ker \delta''$ and if $k$, $k'$ are such that $\psi k = k''$ and $\varphi k' = \delta k$ then $k' \in (\text{ran } \delta')^-$. Since $\delta'$ has full range, there is a cyclic module $M'$ such that $k' \in (\delta' M')^-$. Now notice that

$$\delta k = \varphi k' \in (\varphi \delta' M')^- = (\delta \varphi M')^- = (\delta(\varphi M' + \ker \delta)^-)^-,$$

and since $\delta$ has full range we deduce that $k \in (\varphi M' + \ker \delta)^-$. Denote by $M$ the cyclic module generated by $K$, and by $p : (\varphi M' + \ker \delta)^- \to (\varphi M' +$

$\ker\delta)^-/\ker\delta$ the canonical projection. Then $(\varphi M' + \ker\delta)^-/\ker\delta$ is the closure of $\operatorname{ran}(p\varphi|M')$ and hence it is finite. By Proposition 1.6 there exists a cyclic vector $k_1$ for $M$ such that $pk_1 \in \operatorname{ran}(p\varphi|M')$ and this clearly implies that $k_1 \in \operatorname{ran}\varphi + \ker\delta$. If $k_1 = k_2 + k_3$ with $k_2 \in \operatorname{ran}\varphi$ and $k_3 \in \ker\delta$, then $k'' = \psi k$ and $\psi k_3 = \psi k_1$ generate the same cyclic space. Thus the cyclic space generated by $k'' + (\operatorname{ran}\delta'')^-$ is the range under $(\psi_*)_+$ of the cyclic space generated by $k_3 + (\operatorname{ran}\delta)^-$. Therefore $(\psi_*)_+$ is onto $\operatorname{Lat}(\ker\partial)$.

Finally let $k' + (\operatorname{ran}\delta')$ belong to $\ker\varphi_*$, i.e., $\varphi k' \in (\operatorname{ran}\delta)^-$. Denote by $M'$ the cyclic module generated by $k'$. Since $\delta$ has full range, $M' \cap \varphi^{-1}(\operatorname{ran}\delta)$ contains a cyclic vector $k_1'$ for $M'$. Thus $\varphi k_1' \in \operatorname{ran}\delta$, say $k_1' = \delta k$. We see then that $\delta''\psi k = \psi\delta k = \psi\varphi k_1' = 0$ so that in fact $k_1' + (\operatorname{ran}\delta')^- = \partial(\psi k + (\operatorname{ran}\delta'')^-)$. This implies immediately that $\partial_+$ is onto $\operatorname{Lat}(\ker\varphi_*)$. ■

**3.5. Corollary.** *Assume that the complexes $\delta'$, $\delta$, and $\delta''$ in the exact sequence (3.1) are semi-Fredholm. Then* $\operatorname{ind}(\delta) = \operatorname{ind}(\delta') + \operatorname{ind}(\delta'')$.

*Proof:* Since the complexes in question are $\mathbb{Z}_2$-graded, the triangle in Theorem 3.4 becomes a hexagon

$$\begin{array}{ccccc}
H_0(\delta') & \xrightarrow{\varphi_*} & H_0(\delta) & \xrightarrow{\psi_*} & H_0(\delta'') \\
\partial\big\uparrow & & & & \big\downarrow\partial \\
H_1(\delta'') & \xleftarrow{\psi_*} & H_1(\delta) & \xleftarrow{\varphi_*} & H_1(\delta')
\end{array}$$

and one can deduce as in Corollary 2.4 that

$$[H_0(\delta')] - [H_0(\delta)] + [H_0(\delta'')] - [H_1(\delta')] + [H_1(\delta)] - [H_1(\delta'')] = 0.$$

This implies immediately the index formula. ■

## 4. Stability of the index

As we mentioned above, the semigroup of all classes of finite $C_0$-modules does not have the cancellation property. One can nevertheless cancel under certain circumstances. If $K_1$ and $K_2$ are finite modules we will write $[K_1] \le [K_2]$ if $[K_2] = [K_1] + [K_3]$ for some finite module $K_3$. The following result is proved in [1] (see Lemma 6.2 in Chapter 7).

**4.1. Lemma.** *Let $K_1$, $K_2$ and $K_3$ be finite modules. If $[K_1] + [K_3] = [K_2] + [K_3]$, $[K_3] \le [K_1]$, and $[K_3] \le [K_2]$, then $[K_1] = [K_2]$.*

We need an additional lemma in order to prove the main result in this section.

**4.2. Lemma.** *If $K_1 \xrightarrow{\varphi} K_2 \xrightarrow{\psi} K_3$ is a $C_0$-exact sequence of $C_0$-modules then $[K_2] \leq [K_1] + [K_3]$.*

*Proof:* Using Lemma 1.7 we have

$$\begin{aligned}[K_2] = [\ker \psi] + [K_2/\ker \psi] &= [(\text{ran } \varphi)^-] + [(\text{ran } \psi)^-] \\ &= [K_1/\ker \varphi] + [(\text{ran } \psi)^-] \leq [K_1] + [K_2]. \ \blacksquare\end{aligned}$$

**4.3. Theorem.** *Let $\delta$, $\delta_1 \in \operatorname{End}(K)$ be two $\mathbb{Z}_2$-graded complexes. Assume that $\delta$ is semi-Fredholm and $(\operatorname{ran}(\delta_1 - \delta))^-$ is finite. Then $\delta_1$ is also semi-Fredholm and*

$$\operatorname{ind}(\delta_1) + [(\operatorname{ran}(\delta_1 - \delta))^-] = \operatorname{ind}(\delta) + [(\operatorname{ran}(\delta_1 - \delta))^-].$$

*Proof:* Let us set $\varepsilon = \delta_1 - \delta$, and denote by $K'$ the submodule of $K$ generated by ran $\varepsilon$ and ran $(\delta\varepsilon)$. Note that if $K = K_0 \oplus K_1$ is the gradation of $K$, then $K' = K'_0 \oplus K'_1$, where $K'_0$ is generated by ran $\varepsilon_1$ and $\operatorname{ran}(\delta_1\varepsilon_0)$, and $K'_1$ is generated by ran $\varepsilon_0$ and $\operatorname{ran}(\delta_0\varepsilon_1)$. Since $(\text{ran } \varepsilon)^-$ is finite it follows that $K'$ is finite. Moreover, $K'$ is invariant under $\delta'$ and $\delta'_1$. Invariance under $\delta'$ is obvious, and invariance under $\delta'_1$ follows from the inclusions

$$\begin{aligned}&\delta_1 \text{ ran } (\delta\varepsilon) = (\delta + \varepsilon) \operatorname{ran}(\delta\varepsilon) = \varepsilon \operatorname{ran}(\delta\varepsilon) \subset \text{ran } \varepsilon, \\ &\delta_1 \text{ ran } \varepsilon = (\delta + \varepsilon) \text{ ran } \varepsilon = (\delta + \varepsilon) \text{ ran } \delta \subset \varepsilon \text{ ran } \delta \subset \text{ran } \varepsilon,\end{aligned}$$

where we used the equality $(\delta + \varepsilon)^2 = 0$. Let us denote by $\delta'$ and $\delta'_1$ the restrictions of $\delta$ and $\delta_1$ to $K'$, respectively, and denote by $\delta''$ and $\delta''_1$ the induced complexes on $K'' = K/K'$. Thus we have exact sequences

$$0 \longrightarrow \delta' \xrightarrow{\varphi} \delta \xrightarrow{\psi} \delta'' \longrightarrow 0,$$

$$0 \longrightarrow \delta'_1 \xrightarrow{\varphi} \delta_1 \xrightarrow{\psi} \delta''_1 \longrightarrow 0,$$

where $\varphi$ denotes inclusion, and $\psi$ denotes the canonical projection onto the quotient module. We claim that in fact $\delta'' = \delta''_1$. Indeed, this follows immediately from the fact that $\operatorname{ran}(\delta - \delta_1)$ is contained in $K'$. Moreover, the complexes $\delta'$ and $\delta'_1$ are Fredholm by Proposition 2.1, and

$$\operatorname{ind}(\delta') = \operatorname{ind}(\delta'_1) = [K'_0] - [K'_1]. \tag{4.4}$$

Next we note that $\delta$ has full range, and therefore $\delta_1 = \delta + \varepsilon$ has full range by Lemma 1.8. (ii). We want to argue that $\delta''$ has full range as well. Indeed, consider $k + K' \in (\text{ran } \delta'')^- = (\text{ran } \delta + K')^-$. This means that $k \in ((\text{ran } \delta)^- + K')^-$. Denote by $M$ the cyclic module generated by $k$, and by $p$ the canonical projection onto the quotient $((\text{ran } \delta)^- + K')/(\text{ran } \delta)^-$. Since $pK'$ is dense, this quotient module is finite, and Proposition 1.6 implies the existence of a cyclic vector $k_1$ for $M$ such that $pk_1 \in pK'$ or, equivalently, $k_1 \in (\operatorname{ran}\delta)^- + K'$. Write $k_1 = k_2 + k_3$ with $k_2 \in (\operatorname{ran}\delta)^-$ and $k_3 \in K' = \ker\Psi$, and note that $k_2 + K'$, $k_1 + K'$, and $k + K'$ generate the same module in $K''$. Now, if $M_2$ is the module generated by $k_2$, then $M_2 \subset (\operatorname{ran}\delta)^-$, and hence $M_2 = (\delta N)^-$ for some submodule $N$ because $\delta$ has full range. It is now immediate that $(\delta''\psi N)^- = (\psi\delta N)^-$ is the cyclic module generated by $k + K' = \psi k$. An application of the same argument to $\delta''^*$ shows that $\delta''$ has full range by virtue of Lemma 1.4.

Suppose now that $H_0(\delta)$ is finite. The $C_0$-exact hexagon

$$\begin{array}{ccccc} H_0(\delta') & \longrightarrow & H_0(\delta) & \longrightarrow & H_0(\delta'') \\ \uparrow & & & & \downarrow \\ H_1(\delta'') & \longleftarrow & H_1(\delta) & \longleftarrow & H_1(\delta') \end{array}$$

implies that $H_0(\delta'')$ is also finite. Indeed, both $H_0(\delta)$ and $H_1(\delta')$ are finite (see Proposition 1.5. (ii)). Furthermore, the hexagon

$$\begin{array}{ccccc} H_0(\delta_1') & \longrightarrow & H_0(\delta_1) & \longrightarrow & H_0(\delta_1'') \\ \uparrow & & & & \downarrow \\ H_1(\delta_1'') & \longleftarrow & H_1(\delta_1) & \longleftarrow & H_1(\delta_1') \end{array}$$

implies now that $H_0(\delta_1)$ is finite. Indeed, $H_0(\delta_1')$ and $H_0(\delta_1'') = H_0(\delta'')$ are finite. Thus $\delta_1$ is semi-Fredholm in this case. The case in which $H_1(\delta)$ is finite is treated analogously.

We turn finally to the index. The two exact hexagons above give

$$\begin{aligned} &[H_0(\delta')] + [H_0(\delta'')] + [H_1(\delta)] = [H_0(\delta)] + [H_1(\delta')] + [H_1(\delta'')], \\ &[H_0(\delta_1)] + [H_1(\delta_1')] + [H_1(\delta'')] = [H_0(\delta_1')] + [H_0(\delta'')] + [H_1(\delta_1)], \end{aligned}$$

where we used the fact that $\delta'' = \delta_1''$. If we add these two relations we get

$$\begin{aligned} &[H_1(\delta)] + [H_0(\delta_1)] + [H_0(\delta')] + [H_1(\delta_1')] + [H_0(\delta'')] + [H_1(\delta'')] \\ &\qquad = [H_0(\delta)] + [H_1(\delta_1)] + [H_1(\delta')] + [H_0(\delta_1')] + [H_0(\delta'')] + [H_1(\delta'')]. \end{aligned}$$

Now, Lemma 4.2 shows that

$$\begin{aligned} [H_0(\delta'')] &\le [H_0(\delta)] + [H_1(\delta')], \\ [H_0(\delta'')] &\le [H_0(\delta_1)] + [H_1(\delta_1')], \\ [H_1(\delta'')] &\le [H_0(\delta')] + [H_1(\delta)], \\ [H_1(\delta'')] &\le [H_0(\delta_1')] + [H_1(\delta_1)], \end{aligned}$$

and therefore Lemma 4.1 implies

$$(4.5)\quad [H_1(\delta)] + [H_0(\delta_1)] + [H_0(\delta')] + [H_1(\delta_1')] \\ = [H_0(\delta)] + [H_1(\delta_1)] + [H_1(\delta')] + [H_0(\delta_1')].$$

Using (4.4) we see that

$$\begin{aligned}
[H_0(\delta')] + [H_1(\delta_1')] + [K'] &\\
&= [H_0(\delta')] + [K_1'] + [H_1(\delta_1')] + [K_0'] \\
&= [H_1(\delta')] + [K_0'] + [H_0(\delta_1')] + [K_1'] \\
&= [H_1(\delta')] + [H_0(\delta_1')] + [K'].
\end{aligned}$$

Adding $[K']$ to both sides in (4.5) we get therefore

$$[H_1(\delta)] + [H_0(\delta_1)] + [H_1(\delta')] + [H_0(\delta_1')] + [K'] \\ = [H_0(\delta)] + [H_1(\delta_1)] + [H_1(\delta')] + [H_0(\delta_1')] + [K'],$$

and since $[H_1(\delta')] + [H_0(\delta_1')] \leq [K']$, we get by Lemma 4.1

$$[H_1(\delta)] + [H_0(\delta_1)] + [K'] = [H_0(\delta)] + [H_1(\delta_1)] + [K'].$$

Now clearly $[K'] \leq [(\text{ran }\varepsilon)^-] + [(\text{ran}(\delta\varepsilon))^-] \leq [(\text{ran }\varepsilon)^-] + [(\text{ran }\varepsilon)^-]$, and a final application of Lemma 4.1 yields

$$[H_1(\delta)] + [H_0(\delta_1)] + [(\text{ran }\varepsilon)^-] = [H_0(\delta)] + [H_1(\delta_1)] + [(\text{ran }\varepsilon)^-],$$

which is the desired index relation. ■

## 5. Concluding remarks

Vasilescu [**6**] considered complexes of the form

$$(5.1)\qquad 0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \longrightarrow 0,$$

where $X_0, X_1, \dots, X_n$ are Banach spaces, and $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ are densely defined closed operators. One can always replace (5.1) by a $\mathbb{Z}_2$-graded complex $(Y_0 \oplus Y_1, \delta)$, where $Y_0 = X_0 \oplus X_2 \oplus \dots$, $Y_1 = X_1 \oplus X_3 \oplus \dots$, and

$$\begin{aligned}
\delta_0(x_0 \oplus x_2 \oplus \dots) &= \alpha_0 x_0 \oplus \alpha_2 x_2 \oplus \dots\ , \\
\delta_1(x_1 \oplus x_3 \oplus \dots) &= \alpha_1 x_1 \oplus \alpha_3 x_3 \oplus \dots\ .
\end{aligned}$$

Thus considering $\mathbb{Z}_2$-graded complexes gives a somewhat more general concept of Fredholmness and index. For instance, the requirement that $\delta$ be semi-Fredholm only implies that either the odd-numbered, or the even-numbered homology groups of (5.1) are finite-dimensional.

The theory of Fredholm complexes of $C_0$-modules could also be done with densely defined, closed homomorphisms, but I chose to simplify the exposition by considering only continuous homomorphisms. Our perturbations in Theorem 4.3 correspond in the Banach space case with perturbations of finite rank. Vasilescu allows in [**6**] compact perturbations. I do not know whether there exixts a good correspondent, in the context of $C_0$-modules, of compact operators.

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