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THE CONNECTIVE K-THEORY OF SPINOR GROUPS

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Abstract

We study the properties of the connective K-theory with $\mathbb{Z}_2$ coefficients of the Lie groups $Spin(n)$. This generalises some work of L. Hodgkin.

0. Introduction

In a previous paper [4] we obtained the Hopf algebra structure of $k^*(G;Q(P))$ where G is a compact connected Lie group and $Q(P)$ is the quotient ring of $\mathbb{Z}$ with respect to a multiplicative subset generated by a set of primes, such that $H^*(G;Q(P))$ is torsion free.

Here we study the properties of the connective K-theory of the Lie groups $Spin(n)$. Since $H^*(Spin(n);\mathbb{Z})$ is torsion free if $n \leq 6$, we are only interested in the cases where $n \geq 7$. We will see that $k^*(Spin(n))$ has only two torsion and we will give the algebra structure of $k^*(Spin(n);\mathbb{Z}_2)/\{x \in k^*(Spin(n);\mathbb{Z}_2) : t^{-1}x = 0\}$ for $n \geq 7$ (t^{-1} is the canonical generator of $k^*(pt)$).

Our paper generalises part of earlier work of L. Hodgkin, who computed the usual K-theory of compact Lie groups [3].

1. Preliminaries

Let k and K denote, respectively, the spectrum for connective K-theory and the usual K-theory. There is a natural map of ring spectra $j : k \to K$. We recall that $k^*(pt) = \mathbb{Z}[t, t^{-1}]$, the Laurent polynonial ring generated by the reduced Hopf bundle $t^{-1} \in K^{-2}(pt)$ and its inverse, and that $k^*(pt) = Z[t^{-1}]$. We denote by $m : k \to k$ the morphism of spectra corresponding to multiplication by t^{-1}.

If we introduce $\mathbb{Z}_p$ coefficients in the k-cohomology we still obtain an associative multiplication in $k^*(\quad;\mathbb{Z}_p)$.

Let $H\mathbb{Z}$ be the Eilenberg-MacLane spectrum with integer coefficients. There is a map of ring spectra $n : k \to H\mathbb{Z}$ such that it induces the homomorphism $n^* : k^*(pt) \to H^*(pt;\mathbb{Z})$ given by $n^*(at^{-n}) = \begin{cases} a \text{ if } n=0 \\ 0 \text{ if } n>0 \end{cases}$, $a \in \mathbb{Z}$, $n \in \mathbb{N} \cup \{0\}$ [5].

For each CW complex X and each prime p there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_p \otimes k^*(X;\mathbb{Z}_p) \xrightarrow{\tilde{n}^*} H^*(X;\mathbb{Z}_p) \longrightarrow \mathrm{Tor}_{1,*}^{\mathbb{Z}_p[t^{-1}]}(\mathbb{Z}_p; k^*(X;\mathbb{Z}_p)), \longrightarrow 0$$

where $\tilde{n}^*$ is $1 \otimes n^* : \mathbb{Z} \otimes k^*(X) \to \mathbb{Z} \otimes H^*(X;\mathbb{Z})$ *reduced mod* p [4].

2. Atiyah–Hirzebruch spectral sequence

Let X be a compact CW complex. We are going to consider the following Atiyah– Hirzebruch spectral sequences: $(E_r^{**}, d_r)_{r\geq 2}$ converging to $K^*(X)$ and $('E_r^{**}, d'_r)_{r\geq 2}$ converging to $k^*(X)$. Let $F_p^m = \ker(K^m(X) \to K^m(X^{p-1}))$ and $'F_p^m = \ker(k^m(X) \to k^m(X^{p-1}))$ be the filtrations. The first spectral sequence is compatible with the Bott isomorphism.

We note that, since $K^q(pt) = 0 = k^q(pt)$ if q is odd and $k^q(pt) = 0$ if $q > 0$, then: $'E_r^{p,q} = 0 = E_r^{p,q}$ for all $p \in \mathbf{Z}$, $r \geq 2$, q an odd integer; $'E_r^{p,q} = 0$ if $q > 0$; and all the differentials of even degree are zero. Moreover, we have for all $i, n \in \mathbf{Z}$: $F_{n-1}^i = F_n^i$ and $'F_{n-1}^i = {'F_n^i}$ if $n - i$ is even; $F_n^i = F_{n+1}^i$ and $'F_n^i = {'F_{n+1}^i}$ if $n - i$ is odd; $m^*(F_n^i) = F_n^{i-2}$ and $'F_n^n = k^n(X)$.

The same notations will be used when we introduce $\mathbf{Z}_p$ coefficients.

2.1 Proposition. [4]:

Let X be a compact CW complex. Then:

i) *$j_s^{**} : {'E_s^{p,q}} \to E_s^{p,q}$ is an isomorphism for $q \leq -\dim X + 1$,*

ii) *$j^*_{|'F_n^m}$ is an isomorphism onto F_n^m for all $m \in \mathbf{Z}$ and $n \geq m + s - 1$, if $d_r = 0$ for $r > s$, where j_s^{**} and j^* are the obvious morphisms induced by $j : k \to K$.*

2.2 Proposition.

Let X be a CW complex such that $K^*(X)$ is torsion free. If the differentials d_r in the spectral sequence converging to $K^*(X)$ are zero for $r > s$ (we suppose s odd since the differentials of even degree are zero), then $\{y \in k^*(X) : t^{(-s+1)/2} y = 0\} = \{y \in k^*(X) : \lambda y = 0$ for some $\lambda \in \mathbf{Z} - \{0\}\}$.

Proof:

We consider the spectral sequences (E_r^{**}, d_r) and $('E_r^{**}, d'_r)$ converging to $K^*(X)$ and $k^*(X)$, respectively.

We have the following commutative diagram for all $m \in Z$, $i \geq 0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 'F^m_{m+2i+2} & \longrightarrow & 'F^m_{m+2i} & \xrightarrow{p'} & 'E_\infty^{m+2i,-2i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^{**}} & & \\
0 & \longrightarrow & F^m_{m+2i+2} & \longrightarrow & F^m_{m+2i} & \longrightarrow & E_\infty^{m+2i,-2i} & \longrightarrow & 0
\end{array}$$

where j^* and j^{**} are the maps induced by $j : k \to K$ and p' is the map of the extension exact sequence.

Let $y \in k^m(X)$ be such that $\lambda y = 0$ for some $\lambda \in \mathbf{Z} - \{0\}$. Since $j^* : {'F_m^{m-s+1}} \to F_m^{m-s+1}$ is an isomorphism (by 2.1) and $K^*(X)$ is torsion free, then $t^{(-s+1)/2} y = 0$.

We suppose now that $y \in k^m(X)$ and $t^{(-s+1)/2}y = 0$. To finish the proof it is enough to show that $\lambda y \in {}'F^m_{m+s-1}$ for some non zero integer λ (by 2.1).

We are going to prove by induction on $i \geq 0$ that there exists $\lambda \in \mathbb{Z} - \{0\}$ such that $\lambda y \in {}'F^m_{m+2i+2}$.

For $i = 0$ we have $j^{**}p'y = 0$ which implies $p'y \in imd_s$. Since all the differentials d_r are torsion–valued [**1**], there exists $\lambda \in \mathbb{Z} - \{0\}$ so that $\lambda p'y = 0$. Therefore, by the exactness of the extension sequence, $\lambda y \in {}'F^m_{m+2}$ as required.

The inductive step is proved in the same way and the result follows. ■

2.3 Proposition. [4]:

Let X be a compact CW complex and let L be equal to $\mathbb{Z}$ or $\mathbb{Z}_p$ (p prime), then $x \in H^p(X; L)$ lies in the image of $n^* : k^*(X; L) \to H^*(X; L)$ if and only if x considered as an element of ${}'E_2^{p,0}$ is an infinite cycle in the spectral sequence $({}'E_r^{**}, d'_r)$ converging to $k^*(X; L)$.

3. The Lie group $Spin(n)$

In [**2**] Borel proved that:

i) $H^*(Spin(n); \mathbb{Z}_2)$ is an algebra with a simple system of generators x_i, x where degree $x_i = i \in S_n = \{i \leq n-1 : i \text{ is not a power of } 2\}$ and degree $x = 2^{s(n)-1}$, $s(n)$ is the integer such that $2^{s(n)-1} < n \leq 2^{s(n)}$. Moreover, for all $i \in \mathbb{N}$ we have: $Sq^i x_j = \binom{j}{i} x_{i+j}$ if $j \in S_n$, $i + j \in S_n$ and $Sq^i x_j = 0$ otherwise; $Sq^i x = 0$.

ii) $H^*(Spin(n); \mathbb{Z}_2)$ is torsion free if $n \leq 6$ and is an abelian group isomorphic to the direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$ if $n \geq 7$.

We are interested in the cases where $n \geq 7$.

By the results of [**3; III–2**] and proposition 2.1 we conclude that in the spectral sequence converging to $k^*(Spin(n); \mathbb{Z}_2)$ the only possibily non zero differential is $d'_3 : {}'E_3^{m,q} \to {}'E_3^{m+3,q-2}$ and that it coincides with d_3 when $q < 0$. Therefore, we will use the same symbol d_3 for both.

If we identify $E_3^{m,q}$ with $H^m(Spin(n); \mathbb{Z}_2)$ when q is even and less or equal to zero, we get $d_3 = Sq^1Sq^2 + Sq^2Sq^1$ and we have the following relations:

i) $d_3 x_i = \begin{cases} x_{i+3} & \text{if } i \text{ is odd, } i+3 \in S_n \\ 0 & \text{otherwise} \end{cases}$; $d_3 x = 0$;

ii) if $2i \in S_n$ then $x_{2i} = d_3 x_{2i-3}$;

iii) $x_i^2 = \begin{cases} x_{2i} & \text{if } 2i \in S_n \\ 0 & \text{if } 2i \notin S_n \end{cases}$ and $x^2 = 0$.

We are going to describe $kerd_3/imd_3$. We will use the following notation:

a) $S_n^1 = \{i \text{ odd} \in S_n; i+3 \in S_n\}$, $S_n^2 = \{i \text{ odd} \in S_n : i+3 \in S_n \text{ and } 2i+6 \notin S_n\}$ and $S_n^3 = \{i \text{ odd} \in S_n : i+3 \in S_n \text{ and } 2i+6 \in S_n\}$;

b) $y_j = x_j x_{j+3}$ if $j \in S_n^2$ and $z_k = x_k x_{k+3} + x_{2k+6}$ if $k \in S_n^3$;

c) $\bar{u}$ is the image under the projection $kerd_3 \to kerd_3/imd_3$ of any element $u \in kerd_3$.

3.1 Proposition.

Using the above notation, $kerd_3/imd_3$ is a $\mathbb{Z}_2$ exterior algebra generated by $(x_i)_{i\in S_n^1}$, $\bar{x}$, $(\bar{y}_j)_{j\in S_n^2}$ and $(\bar{z}_k)_{k\in S_n^3}$.

Proof:

We consider the differential algebra $A = (H^*(Spin(n);\mathbb{Z}_2), d_3)$. First we note that it is enough to prove that $H_*(A/(x))$ is an exterior algebra on the generators $(\bar{x}_i)_{i\in S_n^1}$, $(\bar{y}_j)_{j\in S_n^2}$ and $(\bar{z}_k)_{k\in S_n^3}$ ((x) denotes the ideal generated by x).

Let B_n denote $H^*(Spin(n);\mathbb{Z}_2)/(x)$. We are going to prove the result by induction on $n \geq 6$. We note that B_n is isomorphic to a subalgebra of $H^*(Spin(n);\mathbb{Z}_2)$. We shall therefore use the same notation for the corresponding generators of both algebras.

The inductive hypothesis is obviously true for $n = 6$.

We have three different cases to consider:

i) $n-1$ is a power of 2. Therefore $B_n = B_{n-1}$.

ii) $n-1$ is odd. Therefore B_n has one more generator, x_{n-1}, than B_{n-1}. Since $d_3x_{n-1} = 0$ and $x_{n-1} \notin imd_3$ the result follows.

iii) $n-1$ is even and it is not a power of 2. Thus B_n has one more generator, x_{n-1} than B_{n-1} and $x_{n-1} = d_3x_{n-4}$.

We consider now the following exact sequence:

$$0 \longrightarrow (x_{n-1}) \xrightarrow{i} B_n \xrightarrow{p} B_n/(x_{n-1}) \longrightarrow 0$$

where i is the inclusion and p is the projection. It induces the exact triangle:

$$\begin{tikzcd} H_*((x_{n-1})) \arrow[rr, "i_*"] & & H_*(B_n) \arrow[dl, "p_*"] \\ & H_*(B_n/(x_{n-1})) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

We can replace $B_n/(x_{n-1})$ by B_{n-1} because they are isomorphic as differential algebras. Clearly, $d_3x_i = 0$ if $i \in S_n^1$, $d_3y_j = x_{j+3}^2 = 0$ if $j \in S_n^2$ and $d_3z_k = x_{k+3}^2 + x_{2k+6} = 0$ if $k \in S_n^3$.

We have to distinguish two subcases:

a) $\frac{n-1}{2}$ is even. Here $S_n^1 = S_{n-1}^1 \setminus \{n-4\}$, $S_n^2 = (S_{n-1}^2 \setminus \{r\} \cup \{n-4\}$ and $S_n^3 = S_{n-1}^3 \cup \{r\}$, where $r = \frac{n-7}{2}$.

By induction, we have $H_*B_{n-1} = \Lambda_{\mathbb{Z}_2}((\bar{x}_i)_{i\in S_{n-1}^1}, (\bar{y}_j)_{j\in S_{n-1}^2}, (\bar{z}_k)_{k\in S_{n-1}^3}$. The generators $(\bar{x}_i)_{i\in S_n^1}$, $(\bar{y}_j)_{j\in S_n^2\setminus\{n-4\}}(\bar{z}_k)_{k\in S_n^3}$ belong to imp_*; $p_*\bar{z}_r = \bar{y}_r$; and $\partial_*\bar{x}_{n-4} = \bar{x}_{n-1}$. As $d_3(a\bar{x}_{n-4}) = a\bar{x}_{n-1}$ in B_n if a is a cycle ,we have that $\partial_*(\bar{a}\bar{x}_{n-4}) = \bar{a}\bar{x}_{n-1}$ if $\bar{a} \in H_*B_{n-1}$. Thus $imp_* = \Lambda_{\mathbb{Z}_2}((\bar{x}_i)_{i\in S_n^1}, (\bar{y}_j)_{j\in S_n^2}, (\bar{z}_k)_{k\in S_n^3-\{r\}})$.

We denote by R the algebra imp_*. The map ∂_* is R–linear and as an R–module $H_*(B_{n-1})$ is free on the two generators 1 and $\bar{x}_{n-4}$. Moreover $\partial_* 1 = 0$, $\partial_* \bar{x}_{n-4} = \bar{x}_{n-1}$ and the map

$$\begin{aligned} B_{n-1} &\to (x_{n-1}) \\ a &\to a x_{n-1} \end{aligned}$$

is an isomorphism of differential modules. Therefore $H_*((x_{n-1}))$ a is free R–module isomorphic to $R\bar{x}_{n-1} \oplus R\bar{x}_{n-4}\bar{x}_{n-1}$ and $H_* B_n$ is isomorphic to $R.1 \oplus R\bar{x}_{n-4}\bar{x}_{n-1}$, as a $\mathbb{Z}_2$–vector space.

To check that $H_* B_n$ is an exterior algebra on the given elements, it remains to show that they have zero square, which can be easily verified.

b) $\frac{n-1}{2}$ is odd. The proof is similar to the previous one. ∎

3.2 Lemma.

The kernel of $j^* : k^*(Spin(n); \mathbb{Z}_2) \to k^*(Spin(n); \mathbb{Z}_2)$ is equal to the set $A = \{y \in k^*(Spin(n); \mathbb{Z}_2) : t^{-1} = 0\}$.

Proof:

The inclusion $kerj^* \supset A$ follows from the equality $F_r^{r-2} = {}'F_r^{r-2}$ (proposition 2.1). The other inclusion $kerj^* \subset A$ is trivial. ∎

3.3 Proposition.

The kernel of $j^* : k^*(Spin(n); \mathbb{Z}_2) \to k^*(Spin(n); \mathbb{Z}_2)$ is mapped isomorphically onto imd_3 by the map $n^* : k^*(Spin(n); \mathbb{Z}_2) \to H^*(Spin(n); \mathbb{Z}_2)$.

Proof:

We will show first that $n^*(kerj^*) = imd_3$.

Since ${}'E_\infty^{i,0} \approx kerd_3$ and $E_\infty^{i,0} \approx kerd_3/imd_3$ we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}'F_{i+2}^i & \longrightarrow & {}'F_i^i & \xrightarrow{n^*} & kerd_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j^{**}} & & \\
0 & \longrightarrow & F_{i+2}^i & \longrightarrow & F_i^i & \longrightarrow & kerd_3/imd_3 & \longrightarrow & 0
\end{array}$$

where j^{**} is the projection. It follows from the exactness of the rows that $n^*(kerj^*) = imd_3$.

It remains to show that $n^*_{|kerj^*}$ is injective. This is implied by the injectivity of $j^*_{|imm_*}$ (lemma 3.2). ∎

We are now able to describe $k^*(Spin(n); \mathbb{Z}_2)/imj^*$. We note that by propositions 2.3 and 3.3 the map n^* induces an epimorphism $\bar{n}^* : k^*(Spin(n); \mathbb{Z}_2)$

$/imj^* \to kerd_3/imd_3$. Thus we can take elements $(\bar{w}_i)_{i\in S_n^1}$, $(\bar{u}_j)_{j\in S_n^2}$, $(\bar{v}_k)_{k\in S_n^3}$ and $\bar{w}$ in $k^*(Spin(n);\mathbf{Z}_2)/kerj^*$ such that $\bar{n}^*(\bar{w}_i) = \bar{x}_i$, $\bar{n}^*(\bar{u}_j) = \bar{y}_j$, $\bar{n}^*(\bar{v}_k) = \bar{z}_k$ and $\bar{n}^*(\bar{w}) = \bar{x}$. These elements are uniquely determined modulo Imm^*. Furthermore all of them have zero square since the square of the elements of odd degree in $K^*(Spin(n);\mathbf{Z}_2)$ is zero [3].

Therefore there exists an algebra homomorphism

$$g : \Lambda_{\mathbf{Z}_2[t-1]}((\bar{w}_i)_{i\in S_n^1}, (\bar{u}_j)_{j\in S_n^2}, (\bar{v}_k)_{k\in S_n^3}, \bar{w}) \longrightarrow k^*(Spin(n);\mathbf{Z}_2)$$

It is easy to verify that g is an isomorphism.

We have therefore the following theorem.

3.4 Theorem.

The $\mathbf{Z}_2[t^{-1}]$ algebra $k^*(Spin(n);\mathbf{Z}_2)/kerj^*$ is an exterior algebra generated by $(\bar{w}_i)_{i\in S_n^1}$, $(\bar{u}_j)_{j\in S_n^2}$, $(\bar{v}_k)_{k\in S_n^3}$ and $\bar{u}$.

The next proposition is an immediate consequence of proposition 2.2.

3.5 Proposition.

The torsion coefficients of $k^*(Spin(n))$ are two and for all $y \in k^*(Spin(n))$ $2y = 0$ if and only if $t^{-1}y = 0$.

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