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THE TOPOLOGICAL CENTRALIZERS OF TOEPLITZ FLOWS AND THEIR Z_2–EXTENSIONS

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Abstract

The topological centralizers of Toeplitz flows satisfying a condition (Sh) and their Z_2–extensions are described. Such Toeplitz flows are topologically coalescent. If $\{q_0, q_1, \ldots\}$ is a set of all except at least one prime numbers and $l_0, l_1, \ldots$ are positive integers then the direct sum $\bigoplus_{i=0}^{\infty} Z_{q_i^{l_i}} \oplus Z$ can be the topological centralizer of a Toeplitz flow.

Introduction

In this paper we study the topological centralizers of Toeplitz flows and their Z_2–extensions. Toeplitz flows are obtained as the orbit closure of special points in $\{0,1\}^2$ called Toeplitz sequences. They were introduced by Jacobs and Keane [2]. Many metric and topological properties of Toeplitz flows were investigated by those authors and by Williams [10]. Markley [6], [7] has examined Toeplitz sequences as characteristic sequences over zero-dimensional groups. Lemańczyk [5] used special Toeplitz sequences to produce examples of Z_2-extensions over dynamical system with discrete spectrum that have Lebesgue component of finite multiplicity. A regular Toeplitz sequence ω with a period structure $\{p_t\}$, $t \geq 0$, defines a cocycle $\overline{\psi}_\omega$ on the group G of p_t-adic integers. A cocycle $\overline{\psi}_\omega$ determines a skew product transformation T_ψ on $G \times Z_2$. Each transformation S commuting with T_ψ can be identified with a pair (T_g, f), where T_g is a rotation of G by g and f is a measurable function, $f : G \to Z_2$ [8]. In this case it is natural to say that T_g can be lifted to $S \in C(T_\psi)$ (the metric centralizer of T_ψ). The problem how big is the set of such $g's$ was investigated in [4]. In [3] this set is described for Morse cocycles. But the same problem can be considered from the topological point of view. A dynamical system $(G, m, \widehat{1})$, (m the Haar measure, $\widehat{1}$ the unit element of G) is metrically isomorphic to $\Theta(\omega) = (O(\omega), \mu, \sigma)$, where σ is the shift and μ the unique σ–invariant measure [2]. The cocycle $\overline{\psi}_\omega$ becomes a continuous cocycle ψ_ω, $\psi_\omega : \overline{O(\omega)} \to Z_2$ [5]. The latter enables one to define a Z_2–extension $\widetilde{\Theta(\omega)} = (0(\omega) \times Z_2, \widetilde{\sigma})$ over $\Theta(\omega)$ as a topological dynamical system. Each homeomorphism $\widetilde{S}$ commuting with $\widetilde{\sigma}$ induces a homeomorphism S commuting with σ and S induces a rotation T_g

of G. The problem arises to describe the set of those $g \in G$ which can be lifted to an element of $C(\sigma)$ and those $S \in C(\sigma)$ which can be lifted to an element of $C(\widetilde{\sigma})$. In this paper we answer these questions assuming that ω satisfies a condition (Sh) (separated holes). Next we construct a class of special Toeplitz flows with the topological centralizers as in the abstract.

1. Preliminaries

We summarize some basic definitions and results. We shall use Z, N to denote the integers, the positive integers, respectively. By flow we will mean a pair (X,T), where X is a compact metric space and T is a homeomorphism of X to itself. A flow (X,T) is minimal if X has no proper closed T-invariant subset. A flow (Y,S) is a factor of (X,T) if there is a continuous map Π of X onto Y, with $\Pi \circ T = S \circ \Pi$. If Π is a homeomorphism then (X,T) and (Y,S) are isomorphic as flows. Every minimal flow (X,T) has a maximal equicontinuous factor (G,g), $\Pi : (X,T) \longrightarrow (G,g)$, where G is a compact metric monothetic group with generator g. If $\Pi' : (X,T) \longrightarrow (G',g')$ is any other such factor then there is a factor map $\psi : (G,g) \longrightarrow (G',g')$ such that $\psi \circ \Pi = \Pi'$.

By the topological centralizer of (X,T) we will mean the set of all continuous maps $U : X \longrightarrow X$ which commute with T. We use $C(T)$ to denote the centralizer of T. $C(T)$ is automatically a semigroup and it becomes a group if every $U \in C(T)$ is homeomorphism.

Given a finite abelian group P, let Ω be the space of all bisequences over P with its natural compact metric topology and let σ be the left shift homeomorphism on Ω. If $\omega \in \Omega$ then $\omega[n]$ will denote the value of ω at $n \in Z$, and $O(\omega)$ will denote the orbit of ω. A finite sequence $B = (B[0],\ldots,B[n-1])$, $B[i] \in P$, $n \geq 1$, is called a block. The number n is called the length of B and denoted by $|B|$. If $\omega \in \Omega$ and B is a block then $\omega[i,k]$, $B[i,k]$, $0 \leq i \leq k \leq n-1$, denote the blocks $(\omega[i],\ldots,\omega[k])$ and $(B[i],\ldots,B[k])$ respectively. Let $C = (C[0],\ldots,C[m-1])$ be another block. We say that B appears at the $i-th$ place in ω or C if $\omega[i,i+|B|-1] = B$ or $C[i,i+|B|-1] = B$. If $|C| = |B|$ then the sum of B and C is the block $B + C$ such that

$$B + C = (B[0] + C[0],\ldots,B[n-1] + C[n-1]),$$

where de symbol " + " is the operation of P. Likewise we define a sequence $(\omega + \omega'), \omega' \in \Omega$ as

$$\omega + \omega' = (\ldots\omega[-1] + \omega'[-1], \omega[0] + \omega'[0], \omega[1] + \omega'[1], \ldots).$$

A $\omega \in \Omega$ is called a Toeplitz sequence if there exists a collection of pairwise disjoint arithmetic progressions $\{T_i\}$ whose union is Z and such that $n,m \in T_i$ implies $\omega[n] = \omega[m]$. A Toeplitz sequence ω is regular if the T_i can be choosen so that $\sum_i \frac{1}{q_i} = 1$, where $T_i = \{r_i + k \cdot q_i; k \in Z\}$. Let $\overline{O(\omega)}$ be the orbit closure of ω. The set $\overline{O(\omega)}$ is a closed, σ–invariant subset of Ω. By a Toeplitz flow we will mean a pair $(\overline{O(\omega)}, \sigma) = \Theta(\omega)$, where ω is a Toeplitz sequence.

Assume that ω is a non-periodic, regular Toeplitz sequence. It is know [2] that a Toeplitz flow $\Theta(\omega)$ is minimal and uniquely ergodic. The maximal equicontinuous factor of $\Theta(\omega)$ was constructed in [**10**]. We include a part of this construction to introduce ideas we will use later. For $p \in N$ we set

$$Per_p(\omega) = \{n \in N; \omega[n] = \omega[n'], \text{ whenever } n \equiv n' (\text{mod } p)\}.$$

By the p-skeleton of ω we will mean a sequence ω_p obtained from ω by replacing $\omega[n]$ by a new symbol " _ " for all $n \notin Per_p(\omega)$. Thus p is a period of ω_p. We call p an essential period of ω if p is the smallest period of ω_p. A period structure for ω is an increasing sequence $\{p_t\}$ of natural numbers satisfying

(a) p_t is an essential period of ω for all t,

(b) $p_t | p_{t+1}$

(c) $\bigcup_{t=0}^{\infty} Per_{p_t}(\omega) = Z$.

Every non-periodic Toeplitz sequence has a period structure.

Let G be the group of all p_t-adic integers i.e.

$$G = \{g = \sum_{t \geq 0} g_t \cdot p_{t-1}; \quad 0 \leq g_t \leq \lambda_t - 1\},$$

where $\lambda_t = p_{t-1}/p_t$, $t \geq 0$ and $p_{-1} = 1$. A p_t–adic integer g may be represented also as a class of sequences (n_t), $n_t \in N$, such that $n_{t+1} \equiv n_t (\text{mod } p_t)$, $t \geq 0$. If (n'_t) is another sequence satisfying the above condition then (n_t) and (n'_t) determine the same p_t–adic number g iff $n_t \equiv n'_t (\text{mod } p_t)$, $t \geq 0$. Let T be the translation of G by the unit element $\widetilde{1}$. In [**10**] it is proved that (G, T) is the maximal equicontinuous factor of $\Theta(\omega)$. To define a corresponding homomorphism Π from $(\overline{O(\omega)}, \sigma)$ to (G, T) a special partition $\{\Pi_g\}, g \in G$, of $\overline{O(\omega)}$ was constructed. For fixed t, $t \geq 0$, and n, $0 \leq n \leq p_t - 1$, set

$$\Omega^t_n = \{x \in \overline{O(\omega)}; x \text{ has the same } p_t - \text{skeleton as } \sigma^n(\omega)\}.$$

Then Ω^t_n, $n = 0, 1, \ldots, p_t - 1$, are pairwise disjoint closed and open subsets of $\overline{O(\omega)}$. For $g \in G$, $g = (n_t)$, $0 \leq n_t \leq p_t - 1$, $n_{t+1} \equiv n_t (\text{mod } p_t)$ we set

$$\Omega_g = \bigcap_{t=0}^{\infty} \Omega^t_{n_t}.$$

The family of sets $\{\Omega_g\}$, $g \in G$ is partition of $\overline{O(\omega)}$. Each of them is a closed and non-empty set and

$$\sigma(\Omega_g) = \Omega_{g+\widetilde{1}}.$$

(Here the symbol " + " means the operation in G. We will use this symbol in different meanings and we will not remark if no confusion becomes). The factor map $\Pi : \overline{O(\omega)} \to G$ is defined as

$$\Pi(\Omega_g) = g. \tag{1}$$

The following remark follows easily from the above construction.

Remark 1. If a sequence $\{\sigma^{n_t}(\omega)\}$ is convergent in $\overline{O(\omega)}$ then (n_t) determines a p_t-adic integer, i.e., for any t there exists i_0 such that $n_i \equiv n_j (\text{mod } p_t)$ whenever $i, j \geq i_0$.

Let $A_t = \omega_{p_t}[0, p_t - 1]$. A_t is a block of the length p_t with symbols from P and " _ " (we will call it a "hole"). By a filled place in A_t we will mean each place i such that $A_t[i] \in P$. A sequence of blocks (A_t) satisfies the following conditions:

(A) The block A_{t+1} is obtained as a concatenation of $A_t\, A_t A_t \ldots A_t$, where some "holes" are filled by symbols of P,

(B) $\lim_{t\to\infty} r_t/p_t = 1$, where r_t is the number of the filled places in A_t (regularity),

(C) For every $i \in \mathbf{N}$ there exists an index t such that $A_t[i] \in P$ and $A_t[p_t - i] \in P$.

Conversely, each sequence of blocks $(A_t)_0^\infty$ satisfying (A), (B) and (C) determines a Toeplitz sequence ω (may be periodic).

In the sequel we change a bit a definition of a Toeplitz sequence. Suppose that a sequence $(A_t)_0^\infty$ satisfies the conditions (A) and (B). Then we can define a two-sided sequence ω in such a way that

$$\omega[i \cdot p_t, (i+1)p_t - 1] = A_t, \tag{2}$$

for all $i \in Z$ and $t \geq 0$. We will call it a T°-sequence. The sequence ω can have the symbol " _ " at some places. Let $g = (n_t)$, $0 \leq n_t \leq p_t - 1$, $n_{t+1} \equiv n_t$ (mod p_t), be a p_t-adic integer. We denote by $A_t(g)$ the following block

$$A_t(g) = A_t A_t[n_t, p_t + n_t - 1].$$

The sequence $(A_t(g))_0^\infty$ satisfies the conditions (A) and (B) and hence determines a two-sided sequence $\omega(g)$ given by (2). It is easy to describe the set G_0 of those $g \in G$ for which $\omega(g)$ is a Toeplitz sequence. Let G_2 be the set of all $g = (n_t)_0^\infty$ from G such that $A_t[n_t] =$" _ " for each $t \geq 0$. It follows from (B) that G_2 is of Haar measure zero. Then the set $G_1 = G_2 + Z$ (Z is a subset of G consisting of all elements of the form $k\widetilde{1}$, where k is an integer) is of Haar measure zero again. It is not hard to observe that $G_0 = G - G_1$. Now we can define $\overline{O(\omega)}$ as the orbit closure of ω in the sense that $x = \lim \sigma^{z_t}(\omega)$, $z_t \to \infty$, and $x[i] \in P$ for all $i = 0, \pm 1, \ldots$. For all $g \in G$ we have $\overline{O(\omega)} = \overline{0(\omega(g))}$. If $g \in G_0$ then $\omega(g)$ is a Toeplitz sequence what implies that $\Theta(\omega) = (\overline{O(\omega)}, \sigma)$ is a Toeplitz flow. We define the sets $\Omega_g(\omega)$, $g \in G$, in the same way as above. The construction of the sequences $\omega(h)$ implies

$$\Omega_g(\omega(h)) = \Omega_{g+h}(\omega).$$

Remark 2. If ω is a T°–sequence, then ω satisfies the property from Remark 1. Therefore $x \in \Omega_0(\omega)$ implies that x coincides with ω at each i–th place which $\Omega[i] \in P$. Thus if $g \in G_0$ then Ω_g is an one-point set and $\Omega_g = \{\omega(g)\}$.

Remark 3. For fixed $t \geq 0$ and $0 \leq n \leq p_t - 1$ set

$$C_t(n) = \{g \in G, g = (n_u)_0^\infty; n_t = n\}.$$

The sets $C_t(0), C_t(1), \ldots, C_t(p_t - 1)$ are closed and open subsets of G and

$$\bigcup_{i=0}^{p_t-1} C_t(i) = G.$$

Further we have

$$C_t(0) \xrightarrow{T} C_t(1) \xrightarrow{T} \ldots \xrightarrow{T} C_t(p-1) \xrightarrow{T} C_t(0).$$

Denote by ξ_t a partition of G determined by the family $\{C_t(i)\}$, $0 \leq i \leq p_t - 1$. If ω is a T°–sequence then ω defines a function $\widetilde{\psi}_\omega : G \to P$ such that

$$\widetilde{\psi}_\omega(g) = A_t[i],$$

if $g \in C_t(i)$ and $A_t[i] \in P$. The function $\widetilde{\psi}_\omega$ is defined on G except of the set G_2. If ω is non-periodic then G_2 is just the set of all g for which $\widetilde{\psi}_\omega$ is not continuous. The function $\widetilde{\psi}_\omega$ is $\bigvee_{t=0}^\infty \xi_t$–measurable. Further observe that if $\Pi : (\overline{O(\omega)}, \sigma) \longrightarrow (G, T)$ is the homomorphism defined by (1) then

$$\psi_\omega = \widetilde{\psi}_\omega \circ \Pi \quad \text{on} \quad \Pi^{-1}(G - G_2),$$

where $\psi_\omega(y) = y[0]$, $y \in \overline{O(\omega)}$.

2. Minimality of $\widetilde{\Theta(\omega)}$

Let ω be the regular non-periodic T°–sequence over $Z_2 = \{0, 1\}$ with a period structure $\{p_t\}, t \geq 0$. On the one hand ω determines a Toeplitz flow $\Theta(\omega) = (\overline{O(\omega)}, \sigma)$. On the other hand ω determines a Z_2–extension $\widetilde{\Theta(\omega)}$ of $\Theta(\omega)$ defined by

$$\widetilde{\Theta(\omega)} = (\overline{O(\omega)} \times Z_2, \widetilde{\sigma}),$$

where

$$\widetilde{\sigma}(y,i) = (\sigma(y), i + \psi_\omega(y)),$$

$i \in Z_2$, $y \in \overline{O(\omega)}$. Put $X = \overline{O(\omega)} \times Z_2$ and denote by Π^* the natural projection of X on $\overline{O(\omega)}$ i.e.

$$\Pi^*(y,i) = y.$$

We have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{\sigma}} & X \\ \Pi^* \downarrow & & \downarrow \Pi^* \\ \overline{O(\omega)} & \xrightarrow{\sigma} & \overline{O(\omega)} \\ \Pi \downarrow & & \downarrow \Pi \\ G & \xrightarrow{T} & G \end{array}$$

Let $C(\widetilde{\sigma})$ and $C(\sigma)$ be the topological centralizers of $\widetilde{\sigma}$ and σ respectively. If $S \in C(\sigma)$ then S induces a continuous map S' on G commuting with T because (G,T) is the maximal equicontinuous factor of $(\overline{O(\omega)}, \sigma)$. But S' is a translation by an element $g_0 \in G$. In this case it is natural to say that g_0 can be lifted to S. The question arises which elements $g_0 \in G$ can be lifted to an element of $C(\sigma)$. Notice that if $g_0 \in G$ can be lifted to S, then S is unique because the homomorphism Π is one-to-one on G_0 which is of Haar measure one and the flows (G,T) and $(\overline{O(\omega)}, \sigma)$ are minimal. We will show (proposition 1 below) that if $(X, \widetilde{\sigma})$ is a minimal flow and $\widetilde{S} \in C(\widetilde{\sigma})$ then $\widetilde{S}$ induces an $S \in C(\sigma)$.

The next problem is how to describe those $S \in C(\sigma)$ which can be lifted to elements of $C(\widetilde{\sigma})$. In §3 we answer these questions provided ω satisfies additional conditions.

Suppose that a T^o–sequence ω is determined by a sequence of blocks $\{A_t\}$, $|A_t| = p_t$ and each A_t is partially filled by 0's and 1's. Denote by k_t the smallest distance between neighbouring holes in A_t i.e. if A_t has holes at I_1-th, I_2-th,..., I_s-th places and

$$I_1 < I_2 < \cdots < I_s$$

then

$$k_t = \min\{[I_{j+1} - I_j, \quad j = 1,2,\ldots,s-1], \; p_t - I_s + I_1\}.$$

We say that ω has the property (Sh) (separated holes) if

$$k_t \underset{t\to\infty}{\longrightarrow} \infty.$$

Remark 4. If ω has the property (Sh) then each Ω_g, $g \in G$, contains at most two points.

In fact, let $g = (l'_t) \in G_2$ with $0 \leq l'_t \leq p_t - 1$, $l'_{t+1} \equiv l'_t (\mathrm{mod}\, p_t)$ and suppose $y, y' \in \Omega_g$. Then $y[-l'_t, p_t - l'_t - 1] = y'[-l'_t, p_t - l'_t - 1] = A_t$ what implies that the blocks $y[-k_t, k_t]$ and $y'[-k_t, k_t]$ coincide except at the 0-th place. The condition $k_t \to \infty$ implies that y and y' can differ only at the 0-th place. Simultaneously Ω_g contains precisely two points because $g \in G_2$. If $g \in (G_2 + Z)$ then the same argument shows that Ω_g contains precisely two points. Of course, if $g \in G_0$ then Ω_g consist of the only one point $\omega(g)$.

Proposition 1. *If $\widetilde{\Theta(\omega)}$ is a minimal flow and $\widetilde{S} \in C(\widetilde{\sigma})$ then there exist $S \in C(\sigma)$ and a continuous function $\psi : \overline{O(\omega)} \to Z_2$ such that*

$$\widetilde{S}(y, i) = (S(y), i + \psi(y)).$$

Moreover the function ψ satisfies a condition

$$\psi(y) + (Sy)[0] = y[0] + \psi(\sigma(y)). \tag{4}$$

Proof: For $a \in G$, a set $\Delta_a \subset G \times G$ is defined by

$$\Delta_a = \{(g, g + a); g \in G\}.$$

The sets $\Delta_a, a \in G$, are closed, $T \times T$–invariant and minimal. Consider a family of subsets $\widetilde{\Delta}_a$ of $X \times X$, $a \in G$, where

$$\widetilde{\Delta}_a = (\Pi^*)^{-1}\Pi^{-1}(\Delta_a) \qquad (\text{ see } (3)).$$

The sets $\widetilde{\Delta}_a$ are closed in $X \times X$, $\widetilde{\sigma} \times \widetilde{\sigma}$–invariant, pairwise disjoint and

$$\bigcup_{a \in G} \widetilde{\Delta}_a = X \times X.$$

Take $\widetilde{S} \in C(\widetilde{\sigma})$. The graph Γ of $\widetilde{S}$ is a minimal subset of $(X \times X, \widetilde{\sigma} \times \widetilde{\sigma})$ and hence is contained in one of the $\widetilde{\Delta}'_a s$ i.e.

$$\widetilde{S}\{(\Omega_g, i)\} = \{(\Omega_{g+a}, i)\}, \tag{5}$$

for all $g \in G$ and $i = 0, 1$. Take $g \in G_0 \cap (G_0 - a)$. Then $(g + a) \in G_0$ what means that $\Omega_g = \{\omega(g)\}$ and $\Omega_{g+a} = \{\omega(g + a)\}$. Denote $v = \omega(g)$ and $u = \omega(g + a)$. The condition (5) implies

$$\begin{cases} \widetilde{S}(v, 0) = (u, \psi(v)) \\ \widetilde{S}(v, 1) = (u, 1 + \psi(v)) \end{cases} \tag{6}$$

where $\psi(v) = 0$ or 1.

Now we show that (6) holds for any $y \in \overline{O(\omega)}$. The minimality of the flow $(X, \widetilde{\sigma})$ implies that there exists $r_n \to \infty$ such that

$$(y, 0) = \lim_n \widetilde{\sigma}^{r_n}(v, 0).$$

We have

$$\widetilde{S}(y, 0) = \lim_n \widetilde{S}\widetilde{\sigma}^{r_n}(v, 0) = \lim_n \widetilde{\sigma}^{r_n}(u, \psi(v)) = (u_0, j)$$

and

$$\widetilde{S}(y, 1) = \lim_n \widetilde{S}\widetilde{\sigma}^{r_n}(v, 1) = \lim_n \widetilde{\sigma}^{r_n}(u, 1 + \psi(v)) = (u_0, 1 + j),$$

because $\widetilde{\sigma}^{r_n}(u_n, j_n') = (u_n, j_n')$ implies $\widetilde{\sigma}^{r_n}(u, 1+j') = (u_n, 1+j_n'), j, j', j_n' \in Z_2$, $u_n, u_0 \in \overline{O(\omega)}$.

The last equalities imply (6) for y. We can rewrite (6) as

$$\widetilde{S}(y, i) = (S(y), i + \psi(y)),$$

$i \in Z_2$ and $y \in \overline{O(\omega)}$. It is a standard argument that $S \in C(\sigma)$ and ψ is a continuous function. The equality (4) follows from the condition $\widetilde{S}\widetilde{\sigma} = \widetilde{\sigma}\widetilde{S}$. In this way the proposition is proved. ■

Proposition 2. *If a T°-sequence ω satisfies (Sh) then $\widetilde{\Theta(\omega)}$ is a minimal flow.*

Proof: Suppose $(X, \widetilde{\sigma})$ is not minimal. It follows from [9] that there exists a continuous function $f : \overline{O(\omega)} \to K$ ($K = \{z; |z| = 1\}$ is the circle group) such that

$$\frac{f(\sigma(y))}{f(y)} = (-1)^{y[0]} \tag{7}$$

for all $y \in \overline{O(\omega)}$. Thus the function f^2 satisfies the condition

$$f^2(\sigma(y)) = f^2(y).$$

This means that f^2 is σ–invariant and hence constant ($f^2 = c$) because $(\overline{O(\omega)}, \sigma)$ is an ergodic system (it is uniquely ergodic). Then the function $\widetilde{f} = \frac{1}{c} \cdot f$ satisfies (7) again and $\widetilde{f}$ admits only 1 and -1 as its value. So we can assume that f has the above property. Define a function $F : \overline{O(\omega)} \to Z_2$ in the following way

$$F(y) = \begin{cases} 0 & \text{if } f(y) = 1 \\ 1 & \text{if } f(y) = -1. \end{cases}$$

Then (7) gives

$$F(\sigma(y)) + F(y) = y[0] \tag{8}$$

for all $y \in \overline{O(\omega)}$.

We will show that (8) implies ω is a periodic sequence. Without loss of generality we can assume that ω is a Toeplitz sequence.

As previously let

$$I_1 < I_2 < \cdots < I_s, \quad s = s(t)$$

be places in A_t such that $A_t[I_j] = " \_ "$, $j = 1, 2, \dots, s$. First we show that (8) implies F is constant on Ω_1^t (see §1) if $I \neq I_1, I_2, \dots, I_s$ and t is large enough. There exist a positive integer L such that

$$F(y) = F(y')$$

whenever $y[-L, L] = y'[-L, L]$. Take I such that $I_1 + L < I < I_2 - L$ (it is possible because $I_2(t) - I_1(t) \geq k_t \to \infty$). Then $y, y' \in \Omega_1^t$ implies

$$y[-L, L] = A_t[I - L, I + L] = y'[-L, L]$$

and hence $F(y) = F(y')$.

Applying (8) we have

$$F(y) = F(y')$$

whenever $y, y' \in \Omega_I^t$, $I_1 < I < I_2$ because $y[0] = A_t[I] = y'[0]$. Now we can repeat the above consideration for all I, $0 \leq I \leq p_t - 1$ such that $I \neq I_1, I_2, \dots, I_s$. As a consequence we obtain that F is constant on Ω_I^t for t large enough and $I \neq I_1, I_2, \dots, I_s$.

Denote, by $\widehat{\omega}$ the one-sided sequence obtained from ω by taking the partial sums of its members in Z_2 i.e.

$$\widehat{\omega} = (0, \omega[0], \omega[0] + \omega[1], \omega[0] + \omega[1] + \omega[2], \dots).$$

Set $F(\Omega_0^t) = 0$ and

$$i_j = F(\Omega_{I_j+1}^t),\ j = 1, 2, \dots, s, \quad i_j \in Z_2.$$

It follows from (8) that

$$F(\Omega_{I_j+1}^t) = i_j + \omega[I_j + 1] + \cdots + \omega[I_j + I - 1] \tag{9}$$

whenever $2 \leq I < I_{j+1} - I_j$, because

$$y[0] = \omega[I_j + I] \ \text{ if } \ y \in \Omega_{I_j+I}^t.$$

Moreover we have

$$F(\Omega_I^t) = \widehat{\omega}[I-1] \tag{10}$$

if $I = 1, 2, \ldots, I_1 - 1$.

Now suppose that $A_{t+1}[I_j] = 0$ or 1 if $j = 1, 2, \ldots, s$ (it is possible because ω is a Toeplitz sequence and $i_1(t) \to \infty$). We obtain from (9) and (10)

$$i_j = F(\Omega_{I_j+1}^t) = F(\Omega_{I_j+1}^{t+1}) = \widehat{\omega}[I_j].$$

Thus we can write (9) as

$$F(\Omega_I^t) = \widehat{\omega}[I-1] \tag{11}$$

whenever $0 \leq I \leq p_t$ and $I \neq I_1, \ldots, I_s$. Using (11) and the condition $i_1(t) \to \infty$ it is not hard to check that $\widehat{\omega}$ is periodic with period p_t if t is large enough. Then ω is a periodic sequence with the same period p_t and moreover

$$\omega[0] + \omega[1] + \cdots + \omega[p_t - 1] = 0. \tag{12}$$

Thus we proved the proposition. ■

Remark 5. If $S \in C(\sigma)$ can be lifted to an $\widetilde{S} \in C(\widetilde{\sigma})$ then it can be lifted to two $\widetilde{S}$, $\widetilde{S}' \in C(\widetilde{\sigma})$.
In fact, if

$$\widetilde{S}(y, i) = (S(y), i + \psi(y)),$$

then the function

$$\psi'(y) = 1 + \psi(y)$$

satisfies (4) what implies that $\widetilde{S}'$ given by

$$\widetilde{S}'(y, i) = (S(y), i + \psi'(y))$$

is an element of $C(\widetilde{\sigma})$. Suppose that $\widetilde{S}_1$ is such that

$$\widetilde{S}_1(y, i) = (S(y), i + \psi_1(y)).$$

Then ψ_1 satisfies

$$\psi_1(y) + (Sy)[0] = y[0] + \psi_1(\sigma(y)).$$

Adding the above equality and (4) in Z_2 we obtain

$$\psi_1(y) + \psi(y) = \psi_1(\sigma(y)) + \psi(\sigma(y)).$$

Thus the function $\psi_1 + \psi$ is σ–invariant and hence constant (=0 or 1) by minimality of $(\overline{O(\omega)}, \sigma)$. So we have

$$\psi_1(y) = \psi(y) \text{ or } \psi_1(y) = 1 + \psi(y).$$

It follows from (4) that $\widetilde{S}$ is an automorphism iff S is an automorphism.

3. The Topological Centralizers of $\Theta(\omega)$ and $\widetilde{\Theta(\omega)}$

Let ω be a T°–sequence satisfying the condition (Sh) and as preceding let G_2 be the set of all $g = (I_t) \in G$, $0 \le I_t \le p_t - 1$, $I_{t+1} \equiv I_t (\mathrm{mod}\ p_t)$ for every $t \ge 0$, and $G_1 = G_2 + Z$.

Proposition 3. *The flow $(\overline{O(\omega)}, \sigma)$ is topologically coalescent i.e. each $S \in C(\sigma)$ is a homeomorphism.*

Proof: Take $S \in C(\sigma)$. By preceding considerations there exists $h = (h_t) \in G$, $0 \le h_t \le p_t - 1$, such that

$$S(\Omega_g) = \Omega_{g+h}$$

for all $g \in G$. If $g \in G_0$ then card $(\Omega_g) = 1$ and then card $(\Omega_{g+h}) = 1$ so that $(G_0 + h) \subset G_0$. We will show that $(G_0 + h) = G_0$.

The map S can be obtained by a code, i.e., there exist integers k, I with $k > 0$ and a function $f : \{0,1\}^k \to \{0,1\}$ such that

$$(Sy)[i] = f(y[i + I, \dots, i + I + k - 1]),$$

for all $i = 0, \pm 1, \dots$ and $y \in \overline{O(\omega)}$. Without loss of generality we can assume that $I = 0$. Choose t large enough so that $k_t > (2k + 1)$ and consider $\omega' \in \Omega_0$. Then $S\omega' \in \Omega_h$.

Suppose that $I_0, I_1, \dots, I_{s-1}$ be places in A_t such that $A_t[I_j] = "\ \_\ "$, $j = 0, 1, \dots, s-1$. Then A_t is of the following form

$$A_t = B_t(0)\,\underline{I_1}\,B_t(1)\,\underline{I_2}\,\dots\,\underline{I_{s-1}}\,B_t(s).$$

The sequences ω'_{p_t} and $(S\omega')_{p_t}$ (see §1) are concatenations of blocks A_t and $(S\omega')_{p_t}$ is the translation of ω'_{p_t} on h_t. We can compare ω'_{p_t} and $(S\omega')_{p_t}$ using the following figure

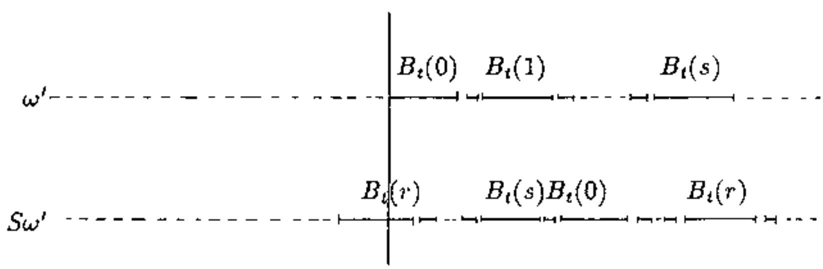

Figure 1.

Using a coding argument it is easy to see that each hole in $(S\omega')_{p_t}$ appears at the place which is distant not more than k places from some hole in ω'_{p_t}. Thus there exists an one-to-one correspondence between the holes in ω'_{p_t} and $(S\omega')_{p_t}$. This means the following: whenever the block $B_t(0)$____$B_t(1)$ appears in ω'_{p_t} then the block $B_t(r)$____$B_t(r+1)$ (see Figure 1) appears in $(S\omega')_{p_t}$ and they are placed as follows

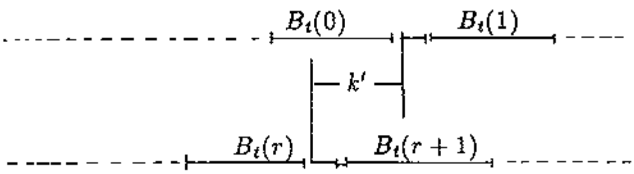

Figure 2.

where $k' \leq k$. Both the blocks $B_t(0)0B_t(1)$ and $B_t(0)1B_t(1)$ appear in ω' and the blocks $B_t(r)0B_t(r+1)$ and $B_t(r)1B_t(r+1)$ appear in $S\omega'$. Using a coding argument again it is clear that whenever the block $B_t(0)0B_t(1)$ appears in ω' then $B_t(r)b_0B_t(r+1)$ appears under it which some $b_0 = 0$ or 1. If $B_t(0)1B_t(1)$ appears in ω' then $B_t(r)\widetilde{b_0}B_t(r+1)$ appears in $S\omega'(\widetilde{b_0} = 1 + b_0)$. We can repeat the same argument for each block $B_t(I)0B_t(I+1)$ and $B_t(I)1B_t(I+1)$, $I = 0, 1, \ldots, s-1$.

Now, take $g \in G_2$. According to Remark 4 there exist precisely two $\omega_1, \omega_2 \in \Omega_g$ such that $\omega_1[i] = \omega_2[i]$ for $i \in Z$, $i \neq 0$, and $\omega_1[0] = \omega_2[0] + 1$. The above reasoning shows that $S\omega_1$ and $S\omega_2$ differ at one place (see Figure 2) and they coincide at the remaining places. This means that card $(\Omega_{g+h}) = 2$ so that $g + h \in G_1$. The last condition implies $(G_1 + h) \subset G_1$ and then $G_1 + h = G_1$ because $(G_0 + h) \subset G_0$. It follows from Remark 4 that S is one-to-one.

Corollary 1. *We conclude from the proof of the above proposition that if $h \in G$ can be lifted to an element of $C(\sigma)$, then*

$$(G_1 + h) = G_1 \tag{13}$$

(equivalently $G_0 + h = G_0$), moreover there exists $k > 0$ such that

$$(G_2 + h) \subset \bigcup_{I=0}^{k} (G_2 + I).$$

To answer the question which $h \in G$ satisfying (13) can be lifted to a $S \in C(\sigma)$ we need two notions. By a t–symbol of ω we mean every block A of length p_t such that

$$A = \omega[Ip_t, Ip_t + p_t - 1], \quad I = 0, \pm 1, \ldots$$

and all the members of $\omega[Ip_t, Ip_t + p_t - 1]$ are 0 or 1. Each t-symbol of ω coincides with A_t except at the I_0-th, $\ldots, I_{s-1}$-th places. The sequence $\omega'(\omega' \in \Omega_0)$ is a concatenation of t–symbols of

$$\omega' = \ldots A_t(j_{-1})A_t(j_0)A_t(j_1)\ldots \tag{14}$$

where $A_t(\cdot)$ denote t–symbols and $A_t(j_0) = \omega'[0, p_t - 1]$. Likewise, if $y \in \Omega_h$, $h = (h_t)$, then y is a concatenation of t–symbols $A_t(h)(\cdot)$ of $\omega(h)$

$$(15) \qquad y = \dots A_t(h)(j_{-1}) \underset{\text{0-th place}}{A_t(h)(j_0)} A_t(h)(j_1) \dots$$

Each of the t-symbols of $\omega(h)$ coincides with $A_t A_t[h_t, h_t + p_t - 1]$ except at the places $I_0 - h_t, \dots, I_{s-1} - h_t$ (taken mod p_t).

Define a two-sided sequence ω^{t+1} as follows

$$\omega^{t+1} = \dots \omega[I_0 - p_t] \dots \omega[I_{s-1} - p_t] \underset{\text{0-th place}}{\omega[I_0]} \dots \omega[I_{s-1}] \omega[I_0 + p_t] \dots \omega[I_{s-1} + p_t] \dots,$$

i.e., ω^{t+1} is the sequence which we should put in the holes of ω_{p_t} to obtain ω. Analogously we define $\omega^{t+1}(h)$.

Theorem 1. *An element $h \in G$ satsifying (13) can be lifted to a $S \in C(\sigma)$ if and only if there exists t such that*

$$\omega^{t+1} + \omega^{t+1}(h)$$

is a periodic sequence with period equal to the number of all holes of A_t.

Proof: It is easy to prove the necesity repeating the same arguments as in the proof of Proposition 3.
Sufficiency. Suppose that the condition

$$(16) \qquad \omega^{t+1} + \omega^{t+1}(h) = \dots B \underset{\text{0-th place}}{B} B \dots$$

holds for $t = t_0$ with $|B| = s$, s–the number of all the holes in A_t, and assume

$$(17) \qquad k_{t_0} > 4k + 1.$$

If (17) is not satisfied then we can take large enough $t > t_0$ because (16) is satisfied for each $t \geq t_0$.

Using this condition we will construct an one-to-one correspondence $f_t(\geq t_0)$ between the sets Z_t and Z'_t of all the t–symbols of ω and $\omega(h)$ respectively. Let $t = t_0$ again, write ω and $\omega(h)$ as in (14) and (15) (except at may be one place). Each t–symbol of ω has the form

$$(18) \qquad A_t(j) = B_t(0)\underline{a_0}B_t(1)\underline{a_1} \dots \underline{a_{s-1}}B_t(s),$$

where $a_0, \dots, a_{s-1}$ are at the I_0-th, $\dots, I_{s-1}$-th places and they depend on j. Thus each t–symbol is completely determined by its values at the places

$I_0, I_1, \ldots, I_{s-1}$. Likewise if $g = (g_t)$, $0 \le g_t \le p_t - 1$, then each t–symbol of $\omega(g)$ is determined by its values at the places $I_0 - g_t, I_1 - g_t, \ldots, I_{s-1} - g_t$ (taken mod p_t). The condition (13) implies that each place $I_0 - h_t, \ldots, I_{s-1} - h_t$ in $A_t(h)$ is distant not more than k places from a place from among $I_0, \ldots, I_{s-1}$ in A_t. That property and (17) define an one-to-one correspondence between the holes in ω_{p_t} and $(\omega(h))_{p_t}$. Moreover (16) implies that if a t–symbol $A_t(j)$ appears in ω then replacing each a_i, $i = 0, \ldots, s-1$, (see (18)) by $a_i + B[i]$, respectively, we obtain a t–symbol $A_t(h)(j)$ of $\omega(h)$. It is easy to see that the above operation determines an one-to-one correspondence f_{t_0} between the sets Z_{t_0} and Z'_{t_0}. Now we can extend that correspondence between Z_t and Z'_t for $t \ge t_0$. Take a t–symbol $A_t(g)$ of ω. Then $A_t(.)$ is a concatenation of t_0–symbols. Then a concatenation of their images by f_{t_0} forms a t–symbol of $\overline{\omega(h)}$ (see (16)). In this manner f_t is defined. To define $S \in C(\sigma)$ take $y \in \overline{O(\omega)}$ with $y \in \Omega_g$, $g = (g_t)$. Then y is a concatenation of t_0–symbols

$$y = \ldots A_{t_0}(j'_{-1})A_{t_0}(j'_0)A_{t_0}(j'_1)\ldots$$

with $y[-g_{t_0}, p_{t_0} - g_{t_0} - 1] = A_{t_0}(j'_0)$. Put

$$S(y) = \ldots A_{t_0}(h)(j'_{-1})A_{t_0}(h)(j'_0)A_{t_0}(h)(j'_1)\ldots,$$

where

$$A_{t_0}(h)(j) = f_t(A_{t_0}(j)),\ j = j'_0, j'_{-1}, j'_1, \ldots.$$

It is evident that S is a continuous map commuting with σ and $S(y) \in \Omega_{g+h}$. This means that S is a lifting of h. Thus the theorem is proved. ■

Theorem 2. *If ω satisfies the condition (Sh) then every $S \in C(\sigma)$ can be lifted to a $\widetilde{S} \in C(\widetilde{\sigma})$.*

Proof: Assume that $h = (h_t)$, $0 \le h_t \le p_t - 1$ satisfies the condition of Theorem 1 with $t = t_0$ and let B be the corresponding block. We will show (Lemma 1) that we can admit

$$B[0] + B[1] + \cdots + B[s-1] = 0 \qquad (\text{in } Z_2). \tag{19}$$

Now suppose that (19) holds. Take a t_0-symbol of the form (18) and construct a t_0-symbol $A_{t_0}(h)(j)$ of $\omega(h)$ as in the proof of Theorem 1. Put $C = A_{t_0}(j) + A_{t_0}(h)(j)$ and denote by $\widehat{C}$ a block obtained from C by taking partial sums of the members of C i.e.

$$\widehat{C}[i] = C[0] + \cdots + C[i] \quad \text{in } Z_2,\ i = 0, 1, \ldots, p_{t_0} - 1.$$

Now we can define a function $\psi : \overline{O(\omega)} \to Z_2$. For $y \in \overline{O(\omega)}$ with $y \in \Omega_g$, $g = (g_t)$, $0 \le g_t \le p_t - 1$, $g_{t+1} \equiv g_t (\mathrm{mod}\ p_t)$ define

$$\Psi(y) = \begin{cases} C[g_{t_0} - 1] & \text{if} \quad g_{t_0} > 0, \\ 0 & \text{if} \quad g_{t_0} = 0. \end{cases} \tag{20}$$

We check that ψ satisfies (4). It follows from considerations of the proof of Theorem 1 that

$$(Sy)[0] = y[0] + C[g_{t_0}]. \tag{21}$$

Now using (19), (20) and (21) we can verify (4) in an easy way. To complete the proof it suffices to show (19). ∎

Lemma 1. *Let $h = (h_t)$ satisfy the conditions (13) and (16). Then we can find t_1 satisfying (16) and the corresponding block B satisfying (19).*

Proof: It suffices to restrict to the case h is not rational integer. Write again the block $A_t(t = t_0)$ in the form

$$A_t = A_t(0)\,\underline{I_0}\,A_t(1)\,\underline{I_1}\,\ldots\,\underline{I_{s-1}}\,A_t(s)$$

and let $I_0, I_1, \ldots, I_{s-1}$ be all places at which A_t has holes. If we draw the block $A_t(h) = A_t A_t[h_t, h_t + p_t - 1]$ under A_t then the condition(16) says that each hole in it is distant not more than k places from a hole of A_t. Suppose that the hole of $A_t(h)$ with the number I_r, $0 \le r \le s-1$ is lying not far from I_0 in A_t. Then the I_{r+1}-th hole is not far from I_1 and so on. Let A^*_{t+1} be the concatenation of λ_{t+1} blocks A_t

$$A^*_{t+1} = \underbrace{A_t(0)\_\_A_t(1)\_\_A_t(s)}_{A_t}\ldots\underbrace{A_t(0)\_\_A_t(1)\_\_\ldots\_\_A_t(s)}_{A_t}.$$

The block A^*_{t+1} has $s\lambda_{t+1}$ holes and to obtain a block A_{t+1} we use a block α^{t+1},

$$\alpha^{t+1} = \alpha^{t+1}(0)\,\underline{x_0}\,\alpha^{t+1}(1)\,\underline{x_1}\,\ldots\,\underline{x'_{s-1}}\,\alpha^{t+1}(s')$$

by putting the successive members of α^{t+1} in holes of A^*_{t+1}. We have $|\alpha^{t+1}| = s\lambda_{t+1}$ and $x_0, x_1, \ldots, x'_{s-1}$ denote the positions in α^{t+1} with holes. If we draw the block $A^*_{t+1}A^*_{t+1}[h_{t+1}, h_{t+1} + p_{t+1} - 1]$ under A^*_{t+1} then there exists exactly one hole in it that appears not far from the first hole in A^*_{t+1}. That hole determines a place r' in α^{t+1}, $0 \le r' \le s\lambda_{t+1} - 1$. Consider the translation to the left of α^{t+1} by r' places. The the block $\alpha^{t+1}\alpha^{t+1}[r', r' + s\lambda_{t+1} - 1]$ is the begining of $\omega^{t+1}(h)$ as well as α^{t+1} is the begining of ω^{t+1}. It is not hard to deduce from (16) that the holes of $\alpha^{t+1}\alpha^{t+1}[r', r' + s\lambda_{t+1} - 1]$ appear precisely under the holes of α^{t+1}. If we denote by H'_0 the subgroup of $Z_{\lambda s}$, $\lambda = \lambda_{t+1}$, generated by r' then the last property means that the set $\{x_0, x_1, \ldots, x'_{s-1}\}$ is a sum of cosets of Z_{λ_s} modulo H'_0. Without loss of generality we can assume that r' is the smallest element of H'_0. Then $r' | s\lambda_{t+1}$. Replacing $t+1$ by $t+v$ eventually we can assume that there exist successive members of α^{t+1} equal 0 or 1 (because $h_t \to \infty$). Suppose that $\alpha^{t+1}[0], \alpha^{t+1}[1], \ldots, \alpha^{t+1}[s-1]$ are 0 or 1.

If we write $r' = qs+u$, $0 \le u \le s-1$, then $v : r' \to u$ defines a homomorphism of H'_0 to Z_s. Let $H_0 = v(H'_0)$ and let c the order of r' in $Z_{\lambda s}$. Then we have

$$\begin{aligned} &\alpha^{t+1}[r'] = \alpha^{t+1}[0] + B[0], \\ &\alpha^{t+1}[2r'] = \alpha^{t+1}[r'] + B[u], \\ &\vdots \\ &\alpha^{t+1}[(c-1)r'] = \alpha^{t+1}[(c-2)r'] + B[(c-2)u], \\ &\alpha^{t+1}[0] = \alpha^{t+1}[cr'] = \alpha^{t+1}[(c-1)r'] + B[(c-1)u]. \end{aligned} \tag{22}$$

The above equalities give

$$B[0] + B[u] + \cdots + B[\underbrace{(c-1)u}_{\text{in } Z_s}] = 0.$$

If $\mu =$ card $\{\ker(v)\}$ then, of course,

$$B[0] + B[u] + \cdots + B[(c-1)u] = \mu \cdot \sum_{i \in H'_0} B[i].$$

Taking in (22) the members $\alpha^{t+1}[r'+j], \alpha^{t+1}[2r'+j], \ldots, \alpha^{t+1}[(c-1)r'+j]$, $j = 1, 2, \ldots, s-1$, we obtain

$$B[j] + B[u+j] + \cdots + B[(c-1)u+j] = 0.$$

Further we have

$$B[j] + B[u+j] + \cdots + B[(c-1)u+j] = \mu \cdot \sum_{i \in A} B[i],$$

where A is a coset of Z_s modulo H'_0. In this way we obtain

$$\mu \cdot \sum_{i=0}^{s-1} B[i] = 0.$$

If μ is odd then we have $\sum_{i=0}^{s-1} B[i] = 0$. If μ is even then we replace $t+1$ by $t+2$. If B' is a block satisfying

$$\omega^{t+2} + \omega^{t+2}(h) = \ldots B' \underset{\text{0-th place}}{\quad} B'B' \ldots$$

and $|B'| = s'$ (the number of all holes in A_{t+1}) then it is not hard to deduce that

$$\sum_{i=0}^{s'-1} B'[i] = \mu \cdot \sum_{A \in C} \Big(\sum_{i \in A} B[i]\Big),$$

where A is a coset of Z_s modulo H'_0 and C a set of cosets (some cosets in C can be repeated). So we have

$$B'[0] + B'[1] + \cdots + B'[s'-1] = 0.$$

In this way the lemma is proved ■

4. k_t–Toeplitz flows

In this section we examine a class of Toeplitz flows determined by special T°–sequences. Given two sequences of positive integers $\mu_0, \mu_1, \ldots; s_0, s_1, \ldots$ such that $\mu_i, s_i \geq 2$, $(\mu_i, \mu_j) = 1$ for $i \neq j$ and $(\mu_i, s_j) = 1$, $i, j = 0, 1, \ldots$, let us denote

$$\lambda_t = \mu_t s_t, \quad k_t = \mu_0 \cdot \ldots \cdot \mu_t, \quad m_t = s_0 \cdot \ldots \cdot s_t, \quad p_t = k_t m_t.$$

We will define a T°–sequence ω determined by a sequence of blocks A_t with $|A_t| = p_t$ in such a way that each A_t has k_t holes with equal distances between them. To make this precise we define

$$A_t = \_\_\_ A_t(0) \_\_\_ \ldots \_\_\_ A_t(k_t - 1),$$

where $A_t(0), \ldots, A_t(k_t - 1)$ are blocks of 0's and 1's with

$$|\_\_\_ A_t(j)| = m_t, \quad j = 0, 1, \ldots, k_t - 1.$$

In order to obtain a block A_{t+1} we use a block α^{t+1} of a form

$$\alpha^{t+1} = \_\_\_ \alpha^{t+1}(0) \_\_\_ \alpha^{t+1}(1) \_\_\_ \ldots \_\_\_ \alpha^{t+1}(k_{t+1} - 1), \tag{23}$$

where $|\_\_\_ \alpha^{t+1}(j)| = s_{t+1}$, $j = 0, 1, \ldots, k_{t+1} - 1$. The block

$$\underbrace{A_t A_t \ldots A_t}_{\lambda_{t+1} \text{ times}}$$

has $k_t \lambda_{t+1}$ holes and we fill them by using the successive members of α^{t+1}. As a consequence we obtain a block A_{t+1} having k_{t+1} holes. If we put

$$\alpha^0 = A_0$$

then we can say that the T°–sequence ω is determined by a sequence $\{\alpha^t\}_0^\infty$ of blocks of the form (23). We will call such a k_t–sequence and corresponding $\Theta(\omega) = (\overline{O(\omega)}, \sigma)$ a k_t–Toeplitz flow. A sequence ω is regular because

$$\frac{k_t}{p_t} = \frac{1}{m_t} \xrightarrow[t \to \infty]{} 0.$$

To assume ω is not periodic we will assume that for all $t \geq 0$ and j; $0 \leq j \leq k_t - 1$, there exist at least two places I, I' in α^{t+1} such that $\alpha^{t+1}[I] = 0$, $\alpha^{t+1}[I'] = 1$ and $I \equiv I' \equiv j (\mathrm{mod}\ k_t)$.

Now we describe the topological centralizer of $\Theta(\omega)$. To do this we need to know the set G_2 of all points in which the corresponding function $\widetilde{\psi}_\omega$ is not continuous (see Remark 3). Let Z_n as usual denote the cyclic group of order n. Define a sequence of group homomorphisms

$$Z_{k_0} \xleftarrow{f_0} Z_{k_1} \xleftarrow{f_1} Z_{k_2} \xleftarrow{f_2} \dots$$

such that

$$f_t(i) = is_{t+1}(\text{mod } k_t), \quad i \in Z_{k_{t+1}}.$$

The condition $(s_{t+1}, k_t) = 1$ implies that f_t is on Z_{k_t}. Let

$$C = \{(j_t m_t)_{t=0}^{\infty} \in G; \quad 0 \le j_t \le k_t - 1; \quad j_t = f_t(j_{t+1}),\ t \ge 0\}.$$

It is easy to see that C is isomorphic to the group Δ of k_t–adic integers numbers and that $G_2 = C$. So $G_1 = G_2 + Z$ is a subgroup of H (not closed) and G_1 is the direct sum of C and Z because $C \cap Z = \{0\}$. Now we want to describe those $h \in C$ that determine $S \in C(\sigma)$. We have $h + G_2 = G_2$ so h can be lifted to a $S \in C(\sigma)$ iff $\omega^{t+1} + \omega^{t+1}(h)$ is k_t–periodic sequence for some $t \ge 0$.

Proposition 4. *If $h \in C$ and h satisfies (16), then the order of h in G is finite.*

Proof: Let $h = (j_t m_t)$; $0 \le j_t \le k_t - 1$; $f_t(j_{t+1}) = j_t$. Then the block $A_t(h)$ has the form

$$A_t(h) = \underline{\quad\ }A_t(j_t)\underline{\quad\ }A_t(j_t+1)\underline{\quad\ }\cdots\underline{\quad\ }A_t(j_t+k_t-1).$$

Therefore the holes in A_t and $A_t(h)$ appear mod k_t precisely at the same places. Assume that the condition (16) holds for $t = t_0$. The sequences ω and $\omega(h)$ are concatenations of the blocks A_t and $A_t(h)$ and the holes in them are filled by the sequences ω^{t+1} and $\omega^{t+1}(h)$ respectively. Thus (16) implies that

$$\omega + \omega(h) = \dots B' \quad \underset{\text{0–th place}}{B'} B' \dots \tag{24}$$

where $|B'| = p_t$. Denote by η_0 the sequence from the right side of (24). Then we have

$$S(y) = y + \sigma^{g_t}\eta_0, \tag{25}$$

where $y \in \overline{O(\omega)}$, $y \in \Omega_g$, $g = (g_t)_{t=0}^{\infty}$, $0 \le g_t \le p_t - 1$. Using (25) several times we obtain

$$S^q(y) = y + \sigma^{g_t}\eta_0 + \sigma^{g_t+h_t}\eta_0 + \cdots + \sigma^{g_t+(q-1)h_t}\eta_0.$$

Let r be the order of h_t in Z_{p_t}. Then we have

$$\begin{aligned} S^{2r}(y) = y + \sigma^{g_t}\eta_0 + \cdots + \sigma^{g_t+(r-1)h_t}\eta_0 + \sigma^{g_t+rh_t}\eta_0 + \cdots + \\ + \sigma^{g_t+(2r-1)h_t}\eta_0 = y + 2(\sigma^{g_t}\eta_0 + \cdots + \sigma^{g_t+(r-1)h_t}\eta_0) = y, \end{aligned}$$

because $rh_t = 0$ (in Z_{p_t}) and $\sigma^q\eta_0 = \eta_0$ whenever $q \equiv 0 \pmod{p_t}$: So we get $S^{2r} = id$ and the order of h in G is finite. This finishes the proof. ■

It is very easy to describe the set of all $h \in C$ with a finite order. Fix $u \geq 0$ and put

$$\mu_u^t = \mu_{u+1} \cdot \ldots \cdot \mu_{u+t} \quad \text{for } t \geq 1.$$

Let

$$H_u^t = \{0, \mu_u^t, \ldots, (k_u - 1)\mu_u^t\}.$$

Then H_u^t is a subgroup of $Z_{k_{u+t}}$ and the order of H_u^t is k_t. Moreover, the homomorphisms f_u, $f_{u+1}, \ldots$ in the following sequence

$$H_u^0 = Z_{k_u} \xleftarrow{f_u} H_u^1 \xleftarrow{f_{u+1}} H_u^2 \xleftarrow{f_{u+2}} \ldots$$

are isomorphism. Define

$$C_u = \{(j_t m_t)_0^\infty;\ 0 \leq j_T \leq k_t - 1,\ j_t = f_t(j_{t+1}),\ j_{u+v} \in H_u^v\}.$$

We have $C_u \subset C_{u+1}$, $u \geq 0$, and each C_u is a subgroup of C with $ord(C_u) = k_u$. It is evident that $C^* = \bigcup_{u \geq 0} C_u$ is a countable subgroup of C and it is the set of all $h \in C$ of a finite order. Thus the topological centralizer of $\Theta(\omega)$ is a subgroup of $C^* \oplus Z$. Now we describe a class of k_t-sequences with topological centralizer equal to $C^* \oplus Z$.

Take $0 \leq j \leq k_t - 1$ and denote by α_j^t the following block

$$\alpha_j^t = \underline{\quad}\alpha^t(j)\underline{\quad}\alpha^t(j+1)\underline{\quad}\ldots\underline{\quad}\alpha^t(k_t - 1 + j).$$

If $h = (j_t m_t) \in C$, then the condition (16) can be formulated as follows:

There exists $t_0 \geq 0$ and a block B with $|B| = k_{t_0}$ such that for all $t \geq t_0$

(26) $$\alpha^{t+1} + \alpha_{j_{t+1}}^{t+1} = \underbrace{B_t B_t \ldots B_t}_{\lambda_{t+1}-\text{times}},$$

where $B_{t_0} = B$ and the next blocks B_{t_0+1}, $B_{t_0+2}, \ldots$ satisfy the recurrent formulas

$$B_{t+1}[j] = B_t[f_t(j)], \quad j = 0, 1, \ldots, k_{t+1} - 1, \quad t > t_0.$$

Now assume additionally that $s_{t+1} > 2k_t + 1$ for every $t \geq 0$. Take any block α^0 of a form (23) such that $p_0 = k_0 m_0$ is the smallest period of the infinite sequence $\alpha^0\alpha^0 \ldots$. Choose blocks

(27) $$\underline{\quad}\alpha^{t+1}(0)\underline{\quad}\ldots\underline{\quad}\alpha^{t+1}(\mu_{t+1} - 1), \quad t \geq 0,$$

in such a way that for each $0 \le j \le k_{t-1}$ there exist $I \neq I'$, $I \equiv I' \equiv j (\mathrm{mod}\ k_t)$ with $\alpha^{t+1}(0)[I] = 0$, ____$\alpha^{t+1}(0)[I'] = 1$, and

$$\mu_{t+1} s_{t+1} \tag{28}$$

is the essential period of (27). Define α^{t+1} of the form (23) by taking the concatenation of k_t copies of the blocks (27). As a consequence we obtain a regular k_t-sequence ω. Moreover the blocks $\{\alpha^t\}$ satisfy (26) if j_{t+1} is the multiplicity of μ_{t+1} and $B_t = 00\ldots0$. By easy considerations we can prove that every $h \in C^*$ can be lifted to a $S \in C(\sigma)$. Therefore $C(\sigma) = C^* \oplus Z$.

It remains to prove that the numbers $\{p_t\}$, $t \ge 0$, form the period structure of ω, i.e., p_t is the essential period of ω_{p_t} for every $t \ge 0$. It is true if $t = 0$ by our choice of α^0. Suppose that p_t is the smallest period of ω_{p_t}. We will show that p_{t+1} is the essential period of $\omega_{p_{t+1}}$. Then the smallest period p' of $\omega_{p_{t+1}}$ is the multiplicity of m_{t+1} because $\omega_{p_{t+1}}$ has the holes every m_{t+1} places starting from 0-th place. On the other hand p' is the multiplicity of p_t. In fact, if

$$p' = I \cdot p_t + r, \quad 0 \le r \le p_t - 1,$$

then p' is a period of ω_{p_t} so r is. The condition $r < p_t$ implies $r = 0$. We have shown that p' is the multiplicity of the smallest common multiplicity of p_t and m_{t+1}. We have

$$[p_t, m_{t+1}] = m_t[k_t, s_{t+1}] = m_t k_t s_{t+1}, \text{ because } (k_t, s_{t+1}) = 1.$$

At the same time it is easy to deduce that the assumption (28) implies p' is the multiplicity of $\mu_{t+1} s_{t+1} m_t$. Now the condition $(k_t, \mu_{t+1}) = 1$ implies $p' = s_{t+1} m_t k_t \mu_{t+1} = m_{t+1} k_{t+1} = p_{t+1}$. In this manner the sequence $\{p_t\}_0^\infty$ is a period structure of ω.

Let q be a fixed prime number and $q_0, q_1, \ldots$ all the remaining prime numbers. We can admit $s_t = q^{t+1}$ and $\mu_t = q_t^{I_t}$ for $t \ge 0$, where I_t are positive integers. Here the group C^* is isomorphic to $\oplus_0^\infty Z_{\mu_t}$ the direct sum of the cyclic groups Z_{μ_t}. Thus the group $\oplus_0 z_{\mu_t} \oplus Z$ can be the topological centralizer of a Toeplitz flow.

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