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SPLIT–NULL EXTENSIONS OF STRONGLY RIGHT BOUNDED RINGS

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Abstract

A ring is said to be *strongly right bounded* if every nonzero right ideal contains a nonzero ideal. In this paper strongly right bounded rings are characterized, conditions are determined which ensure that the split–null (or trivial) extension of a ring is strongly right bounded, and we characterize strongly right bounded right quasi–continuous split–null extensions of a left faithful ideal over a semiprime ring. This last result partially generalizes a result of C. Faith concerning split–null extensions of commutative *FPF* rings.

Examples of strongly right bounded rings are: right duo rings (e.g., commutative rings and strongly regular rings) [**8**], [**18**] and [**26**]; right subdirectly irreducible rings [**9**] and [**10**]; right valuation rings which are not subdirectly irreducible [**24**, p. 216]; and bounded principal ideal domains [**20**, p. 41]. In [**13**, p. 364] an example of a strongly left bounded right primitive ring is given. In [**16**, p. 5.3] an example of a strongly right bounded right self–injective ring which is not left selfinjective is presented. Strongly right bounded rings play a fundamental role in the theory of *FPF* rings (e.g., a strongly right bounded right selfinjective ring is right *FPF* and the basic ring of a semiperfect right *FPF* ring is strongly right bounded [**16**]). In fact, according to [**17**, p. 310], C. Faith has conjectured that a right *FPF* ring is Morita equivalent to a strongly right bounded ring.

All rings are associative, R denotes a ring with unity and M will always be a unital (R, R)–bimodule. The *split–null (or trivial extension)* $S(R, M)$ of M by R is the ring formed from the Cartesian product $R \times M$ with component-wise addition and with multiplication given by $(a, m)(b, k) = (ab, ak + mb)$ (cf., [**12**], [**15**], and [**22**]). Annihilators will be symbolized as $\mathfrak{l}_A(X) = \{a \in A | aX = 0\}$ and $\mathfrak{r}_A(X) = \{a \in A | Xa = 0\}$. A *(ring) direct summand* of R will mean a right ideal generated by a (central) idempotent. From [**16**], R is *right FPF* if every finitely generated faithful right R–module generates the category mod–R. From [**3**], R is *right quasi–FPF* if, whenever a faithful right R–module is a direct sum of finitely many cyclic modules, then it is a generator for mod–R. A ring R is *(quasi-) Baer* (cf., [**7**] and [**23**]) if the right annihilator of every (ideal) nonempty subset of R is a direct summand of R. Semiprime right FPF rings are quasi-Baer [**11**, p. 168]. From [**6**] a ring is *right CS* if every right ideal is essential in a direct summand. From [**21**], R is right *quasi-continuous* (also known as π-injective [**19**]) if it is right CS and if P and Q are direct summands of R such that $P \cap Q = 0$, then $P \oplus Q$ is a direct summand of R. Note that if R is right CS and every idempotent is central, then R is right quasi- continuous. Thus in [**14**, p. 83] Faith has shown that every commutative FPF ring is quasi-continuous. R satisfies the *fintersection left annihilator sum property, ILAS*, if whenever X and Y are right ideals such that $X \cap Y = 0$, then $\mathfrak{l}_R(X)R + \mathfrak{l}_R(Y)R = R$ (e.g., right uniform rings, right selfinjective rings [**25**, p. 275], and right quasi-FPF rings [**3**, Lemma 1]).

Proposition 1. *The following conditions are equivalent:*

(i) *R is a strongly right bounded ring.*
(ii) *If xR is a faithful cyclic module, then $\mathfrak{r}_R(x) = 0$.*
(iii) *R is directly finite and every faithful cyclic module is isomorphic to R.*

Proof:

(i) $\to$ (ii). If $\mathfrak{r}_R(x) \neq 0$, then there exists a nonzero ideal $Y \subseteq \mathfrak{r}_R(x)$. Hence $xRY = 0$. Contradiction!
(ii) $\to$ (iii). Assume $R = X \oplus S$ where X and S are right ideals and S is isomorphic to R. Hence R/X is faithful. Therefore, $X = 0$. Consequently, R is directly finite. Clearly every faithful cyclic module is isomorphic to R.
(iii) $\to$ (i). Let X be a right ideal containing no nonzero ideals. Then R/X is isomorphic to R. Hence $R = X \oplus S$ where S is a right ideal. Since R is directly finite, $X = 0$. Consequently, R is strongly right bounded. ∎

Lemma 2. *Let R be a strongly right bounded ring.*

(i) *Every nonzero right ideal is an essential extension of an ideal of R.*
(ii) *R is right nonsingular if and only if R is semiprime if and only if R is reduced (i.e., R has no nonzero nilpotent elements).*

Proof: Part (i) is in [**16**, Note 1.3D]. Part (ii) is in [**4**, Proposition 1]. ∎

Proposition 3. *Let R be a strongly right bounded ring. Then the following conditions are equivalent:*

(i) *R is quasi-Baer.*
(ii) *R is semiprime right quasi-continuous.*
(iii) *R is semiprime right quasi-FPF.*

Proof: This result follows from [**2**, Proposition 1.2], [**3**, Propositions 4 and 6], and Lemma 2. ∎

The following notation will be used: if $V \subseteq S(R, M)$, then V_1 and V_2 are the sets of first and second components of V, respectively.

Lemma 4.

(i) *If V is a right ideal of $S(R,M)$, then V_1 is a right ideal of R, V_2 is a right R–submodule of M, and $\{0\} \times V_1M$ is a right $S(R,M)$–submodule of V.*

(ii) *If W is a right ideal of R and K is a right R–submodule of M such that $WM \subseteq K$, then $W \times K$ is a right ideal of $S(R,M)$.*

(iii) *Let $V \subseteq S(R,M)$. Then $[\mathfrak{l}_R(V_1) \cap \mathfrak{l}_R(V_2)] \times \mathfrak{l}_M(V_1) \subseteq \mathfrak{l}_{S(R,M)}(V)$.*

(iv) *The right ideal $\{0\} \times M$ is right essential in $S(R,M)$ if and only if M is left faithful (i.e., $\mathfrak{l}_R(M) = 0$).*

(v) *If V and W are right ideals of $S(R,M)$ such that $V \cap W = 0$, then $V_1M \cap W_1M = 0$.*

(vi) *Let $S(R,M))$ be strongly right bounded where M is an ideal of R. Then R is strongly right bounded and if $\mathfrak{l}_R(M) \neq 0$, then $\mathfrak{l}_R(M) \cap \mathfrak{r}_R(M) \neq 0$.*

(vii) *Let M be a module such that whenever $A \cap B = 0$, then $AM \cap BM = 0$ where A and B are right ideals of R (e.g., M is an ideal). If $S(R,M)$ satisfies the ILAS condition, then R satisfies the ILAS condition.*

(viii) *Let M be and ideal of R. Then $S(R,M)$ is right uniform if and only if R is right uniform and M is left faithful.*

Proof:

(i) Clearly V_1 is a right ideal of R and V_2 is a right R–submodule of M. Let $w \in V_1$ and $m \in M$. There exists $k \in V_2$ such that $(w,k) \in V$. Then $(w,k)(0,m) = (0,wm) \in V$. Thus $\{0\} \times V_1M$ is a right $S(R,M)$–submodule of V.

(ii) and (iii) are straightforward.

(iv) Suppose $\{0\} \times M$ is right essential in $S(R,M)$ and $0 \neq t \in \mathfrak{l}_R(M)$. There exists $(w,m) \in S(R,M)$ such that $0 \neq (t,0)(w,m) \in \{0\} \times M$. Contradiction! Hence M is left faithful. Conversely, let $(w,m) \in S(R,M)$. If $w = 0$, we are finished. So assume $w \neq 0$. There exists $k \in M$ such that $0 \neq (w,m)(0,k) = (0,wk) \in \{0\} \times M$. Hence $\{0\} \times M$ is right essential in $S(R,M)$.

(v) Assume $vm = wk \in V_1M \cap W_1M$ where $v \in V_1$, $w \in W_1$, and $m,k \in M$. There exists $x \in V_2$ and $y \in W_2$ such that $(v,x) \in V$ and $(w,y) \in W$. Consider $(v,x)(0,m) = (0,vm) = (0,wk) = (w,y)(0,k) \in V \cap W = 0$. Therefore, $V_1M \cap W_1M = 0$.

(vi) Let Y be a nonzero right ideal of R. There exists an ideal J of $S(R,M)$ such that J is essential in $Y \times YM$. Since J_1 and J_2 cannot both be zero, Y contains a nonzero ideal. Hence R is strongly right bounded. If $\mathfrak{l}_R(M) \neq 0$, then there exists a nonzero ideal $H \subseteq \mathfrak{l}_R(M) \times \{0\}$. Hence H_1 is a nonzero ideal of R and $(\{0\} \times M)H = \{0\} \times MH_1 \subseteq H$. Therefore, $0 \neq H_1 \subseteq \mathfrak{l}_R(M) \cap \mathfrak{r}_R(M)$.

(vii) Let A and B be right ideals of R such that $A \cap B = 0$. Let $A^* = A \times AM$ and $B^* = B \times BM$. Hence $A^* \cap B^* = 0$. Now $\mathfrak{l}_{S(R,M)}(A^*) = \mathfrak{l}_R(A) \times \mathfrak{l}_M(A)$ and $\mathfrak{l}_{S(R,M)}(B^*) = \mathfrak{l}_R(B) \times \mathfrak{l}_M(B)$. Consequently, $\mathfrak{l}_R(A)R +$

$l_R(B)R = R$.

(viii) Assume $S(R,M)$ is right uniform and let Y be a nonzero right ideal of R. By part (iv) M is left faithful. Let $0 \neq w \in R$. There exists $(t,m) \in S(R,M)$ such that $0 \neq (w,0)(t,m) = (wt, wm) \in Y \times YM$. Therefore, R is right uniform. Conversely, let V be a nonzero right ideal of $S(R,M)$ and $0 \neq (t,m) \in S(R,M)$. By part (iv) $0 \neq V \cap (\{0\} \times M) = \{0\} \times \bar{V}_2$ is essential in V. If $t \neq 0$, there exists $y \in R$ such that $0 \neq ty \in \bar{V}_2$. Since M is left faithful, there exists $k \in M$ such that $0 \neq tyk \in \bar{V}_2$. Thus $0 \neq (t,m)(0,yk) = (0,tyk) \in \{0\} \times \bar{V}_2$. If $t = 0$, then $m \neq 0$ and there exists $q \in R$ such that $0 \neq mq \in \bar{V}_2$. Thus $0 \neq (t,m)(q,0) = (0,mq) \in \{0\} \times \bar{V}_2$. Consequently, in all cases $\{0\} \times \bar{V}_2$ is right essential in $S(R,M)$. Therefore, $S(R,M)$ is right uniform. ■

We note that if R is commutative and M is and ideal of R, then $S(R,M)$ is commutative. However, in Example 9 we shall provide a strongly right bounded ring T_1 and an ideal $(T,0)$ such that $S(T_1,(T,0))$ is not strongly right bounded. Also in [**9**, Example 2.2] the ring R is a strongly right bounded ring; however, from Lemma 4 (vi), $S(R, R(x_1,0)R)$ is not strongly right bounded. Thus it is natural to investigate conditions on R and M which insure that $S(R,M)$ is strongly right bounded. We say M is a *strongly right bounded* module if every nonzero right R –submodule contains a nonzero (R,R)–bisubmodule of M.

Theorem 5. *Let R be a strongly right bounded ring. If either of the following conditions is satisfied, then $S(R,M)$ is a strongly right bounded ring.*

(i) *M is a strongly right bounded module such that $l_R(M)$ contains no nonzero nilpotent ideals of R and $l_R(M) \subseteq r_R(M)$.*
(ii) *M is an ideal of R such that $l_R(M) \cap M = 0$.*

Proof: Let V be a nonzero right ideal of $S(R,M)$. If $V_1 = 0$ or $V \cap (\{0\} \times M) \neq 0$, then there exists a nonzero (R,R)–bisubmodule $K \subseteq V_2$ such that $\{0\} \times K \subseteq V$ is an ideal of $S(R,M)$. So assume $V_1 \neq 0$ and $V \cap (\{0\} \times M) = 0$. Let D be a nonzero ideal of R such that $D \subseteq V_1$. Note that with either condition (i) or (ii), $V_1 M = 0 = M V_1$. If condition (i) is satisfied, then $V^2 = V_1^2 \times \{0\} \neq 0$. Hence $D^2 \times \{0\} \subseteq V$ is a nonzero ideal of $S(R,M)$. Now assume condition (ii) is satisfied. If $V_2 = 0$, then $D \times \{0\} \subseteq V$ is a nonzero ideal of $S(R,M)$. If $V_2 \neq 0$, then $V_2 M \neq 0$. But $V(M \times \{0\}) = \{0\} \times V_2 M \subseteq V \cap (\{0\} \times M) = 0$. Contradiction! Therefore, in all cases V contains a nonzero ideal of $S(R,M)$. Consequently, $S(R,M)$ is strongly right bounded. ■

We note that when M is an ideal of R, then $S(R,M)$ is isomorphic to a subring of $T_2(R)$ (i.e., the 2×2 lower triangular matrix ring over R). However, from [**4**, Proposition 10], $T_n(R)$ is never strongly right bounded for $n > 1$.

Corollary 6. *Let M be an ideal of R. Then $S(R,M)$ is strongly right bounded right uniform if and only if R is strongly right bounded right uniform and M is left faithful.*

Proof: This result follows from Theorem 5 and Lemma 4 (viii). ■

Thus, if R is a strongly right bounded domain and M is any ideal of R, then $S(R,M)$ is a strongly right bounded right uniform ring. The ring $H[x]$ where H denotes the real quaternions provides an example of a strongly bounded domain which is neither left nor right duo.

Proposition 7. *Let M be a left faithful ideal of R. Then the following equivalences are true:*

(i) *Every ideal of R is right essential in a (ring) direct summand of R if and only if every ideal of $S(R,M)$ is right essential in a (ring) direct summand of $S(R,M)$.*

(ii) *Every right ideal is right essential in a ring direct summand of R if and only if the same is true for $S(R,M)$.*

Proof:

(i) Let S denote $S(R,M)$ and assume every ideal of R is right essential in a direct summand of R. Let Y be an ideal of S and $V = Y \cap \{0\} \times M$. By Lemma 4 (iv), V is right essential in Y, $V = \{0\} \times V_2$, and V_2 is an ideal of R. Hence there exists a (central) idempotent $e \in R$ such that V_2 is right essential in eR. Consider $(e,0)S$. Let $(x,m) \in S$; then $(e,0)(x,m) = (ex,em)$. Suppose $0 \neq (ex,em)$. If $ex \neq 0$, then there exists $t \in R$ such that $0 \neq ext \in V_2$. Hence $0 \neq (ex,em)(0,t) = (0,ext) \in V$. If $ex = 0$, then there exists $w \in R$ such that $0 \neq emw \in V_2$. Hence $0 \neq (ex,em)(w,0) = (0,emw) \in V$. Therefore, in all cases, V is right essential in $(e,0)S$. Hence Y is right essential in $(e,0)S$. Consequently, every ideal of $S(R,M)$ is right essential in a (ring) direct summand of $S(R,M)$.

Conversely, suppose every ideal of S is right essential in a (ring) direct summand of S. Let K be an ideal of R. Then there exists a (central) idempotent $(e,m) \in S$ such that $\{0\} \times KM$ is right essential in $(e,m)S$. Note that $eme = 0$. Hence (e,m) is central in S if and only if e is central in R and $M = 0$. Now $\{0\} \times KM \subseteq (e,m)(\{0\} \times M) \subseteq (e,m)S$. Hence KM is right essential in eM and eM is right essential in eR because M is left faithful in R. Since K is an ideal and KM is right essential in K, then K is right essential in eR.

(ii) This part is proved in a manner similar to that of part (i). ■

In [**15**] Faith characterizes when $S(R,M)$ is FPF where R is commutative and M is faithful. He poses this characterization as an open problem when R is noncommutative. The following result partially generalizes Faith's result.

Corollary 8. *Let R be a semiprime or a right nonsingular ring and M be a left faithful ideal of R. Then the following conditions are equivalent:*

(i) *R is strongly right bounded and right quasi-continuous.*

(ii) *$S(R, M)$ is strongly right bounded and right quasi–continuous.*
(iii) *$S(R, M)$ is strongly right bounded and right quasi–FPF.*

Proof:

(i) $\to$ (ii) By Lemma 2, R is reduced. Hence every idempotent of R is central. Thus every idempotent of $S(R, M)$ is central. By Theorem 5 and Proposition 7, $S(R, M)$ is strongly right bounded and right quasi–continuous.

(ii) $\to$ (iii) By Lemma 4 (vi) and Lemma 2, R is reduced. Hence every idempotent of $S(R, M)$ is central. By [**3**, Proposition 6], $S(R, M)$ is right quasi–FPF.

(iii) $\to$ (i) By Lemma 4 (vi) and Lemma 2, R is reduced strongly right bounded ring. By Lemma 4 (vii), R satisfies the $ILAS$ condition. From [**1**, Lemma 2.2] and Proposition 3, R is right quasi–continuous.

When R is quasi–Baer strongly right bounded and M is a left faithful ideal of R, the sequence of embeddings

$$R \longrightarrow S(R, M) \longrightarrow T_2(R)$$

is interesting in that $S(R, M)$ is strongly right bounded (and right quasi–continuous) but not quasi–Baer (cf., Proposition 3) and $T_2(R)$ is quasi–Baer [**23**] but not strongly right bounded. ■

The following example is a special case of a general procedure indicated in [**5**].

Example 9. Let I denote the ring of integers and T the semigroup ring of A over I_2 (i.e., integers modulo 2) where A is the semigroup on the set $\{a, b\}$ satisfying the relation $xy = y$ for $x, y \in A$. Thus $T = \{0, a, b, a+b\}$. Let T_1 denote the Dorroh extension of T (i.e., the ring with unity formed from $T \times I$ with componentwise addition and with multiplication given by $(x, k)(y, n) = (xy + nx + ky, kn)$). T_1 has the following properties:

(i) The set of nilpotent elements of T_1, $N(T_1) = \{(0,0), (a+b, 0)\}$, is the Jacobson radical and equals the right socle of T_1.

(ii) Every nonzero right ideal of T_1 contains either $N(T_1)$ or a nonzero ideal of the form $(0, 2kI) = \{(0, 2ki) \in T_1 | k$ is a fixed integer and $i \in I\}$. Therefore, T_1 is strongly right bounded.

(iii) T_1 is not right duo since $(a, 1)T_1$ is not an ideal.

(iv) T_1 is not strongly left bounded.

(v) T_1 does not satisfy the $ILAS$ condition since $\mathfrak{l}_{T_1}(N(T_1)) + \mathfrak{l}_{T_1}((a+b, 2)T_1)T_1 \neq T_1$. However if $\{X_i\}$ is a nonempty set of ideals of T_1 such that $\cap X_i = 0$ then $R = \Sigma \mathfrak{l}_{T_1}(X_i)$. Thus T_1 satisfies the $ILAS$ condition defined in [**1**].

(vi) T_1 is not right CS, since $(a+b, 2)T_1$ is not essential in a direct summand. However, every ideal is right essential in a direct summand of T_1.

(vii) $S(I, N(T_1))$ (i.e., split–null extension) is ring isomorphic to the subring $(0, I) + N(T_1)$ of T_1. $S(I, N(T_1))$ provides an example for Theorem 5 (i).

(viii) $S(T_1, (0, k2I))$ provides an example for Theorem 5 (ii).

(ix) $S(T_1, (T, 0))$ is an example of a split–null extension of a strongly right bounded ring which is not strongly right bounded (cf. Theorem 5). To see this observe $((a,1),(0,0))S(T_1,(T,0)) = \{((ka,k),(0,0))|k \in I\}$ contains no nonzero ideals since $((b,0),(0,0))((ka,k),(0,0)) = ((k(a+b),0),(0,0))$.

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