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RINGS WITH ZERO INTERSECTION PROPERTY ON ANNIHILATORS: ZIP RINGS

CARL FAITH

Abstract

Zelmanowitz [12] introduced the concept of ring, which we call *right zip rings*, with the defining properties below, which are equivalent:

(ZIP 1) If the right annihilator $X^\perp$ of a subset X of R is zero,then $X_1^\perp = 0$ for a finite subset $X_1 \subseteq X$.

(ZIP 2) If L is a left ideal and if $L^\perp = 0$, then $L_1^\perp = 0$ for a finitely generated left ideal $L_1 \subseteq L$.

In [12], Zelmanowitz noted that any ring R satisfying the d.c.c. on annihilator right ideals (= *dcc* $\perp$) is a right zip ring, and hence, so is any subring of R. He also showed by example that there exist zip rings which do not have *dcc* $\perp$.

In §1 of this paper, we characterize a right zip by the property that every injective right module E is divisible by every left ideal L such that $L^\perp = 0$. Thus, $E = EL$. (It suffices for this to hold for the injective hull of R.)

In §2 we show that a left and right self-injective ring R is zip iff R is pseudo-Frobenius (= PF). We then apply this result to show that a semiprime commutative ring R is zip iff R is Goldie.

In §3 we continue the study of commutative zip rings.

Introduction

Zip rings appear in various guises:

1. Beachy and Blair [4] study rings that satisfy the condition that every faithful right ideal I is co-faithful in the sense that $I_1^\perp = 0$ for a finite subset $I_1 \subseteq I$, equivalently, $R \hookrightarrow I^n$ for $n < \infty$. Right zip rings have this property, and conversely for commutative R.

Moreover,

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A. Theorem. *(Beachy–Blair) If faithful ideals of R are co-faithful then the same is true of $R[x]$, for any commutative ring R, and any set x of variables.*

B. Corollary. *If R is a commutative zip ring then any polynomial ring over R is a zip ring.*

Proof: Obvious from the Beachy–Blair Theorem. ■

2. Vámos [13] characterized a ring R with the property that for any collection $\{I_i\}_{i\in\Lambda}$ of right ideals, there exists a finite subset $\Lambda_1 \subseteq \Lambda$ such that

$$\bigcap_{i\in\Lambda} I_i = 0 \Rightarrow \bigcap_{i\in\Lambda_1} I_i = 0$$

This happens iff R is right semi–Artinian (i.e. has finite essential right socle). Trivially, such rings are right zip. Moreover:

C. Proposition. *Any right essential subring of a right semi–Artinian ring is right zip.*

Proof: This is an application of the following. ■

D. Lemma. *If R is a right essential subring of a right zip ring S, then R is right zip.*

Proof: Let $X \subseteq R$ have zero right annihilator in R. Then X has zero right annihilator in S since $R \underset{ess}{\subset} S$ as a right R–module. Then $X_1^{\perp} = 0$ in S for a finite subset X_1 of X, which is what was to be proved.

A ring R is *left Kasch* if every maximal left ideal has a non zero right annihilator; equivalently, every simple left module embeds in R. Every left Kasch ring is right zip, and conversely if finitely generated left ideals are annihilators (Proposition 1.G). A right $\aleph_0$–injective ring is thereby right zip iff left Kasch (Corollary 1.7).

A right PF ring (see Theorem 1.3) is right semi–Artinian (and right Kasch) hence right zip, and by Lemma C, so is any right essential subring, we characterize these rings via Propositions 1.10 and its Corollary. Furthermore, by a theorem of Kato [6], right PF $\rightarrow$ left Kasch.

We also study Utumi zip rings in §2, and prove *inter alia* that they are Goldie rings.

A commutative ring R is zip iff its classical quotient ring $Q = Q_c(R)$ is zip (Corollary 3.2). When Q is Bezout then R is zip iff Q is Kasch (Corollary 3.6). This holds in particular for any ring R, with Q a chain ring. A similar theorem holds for an FPF zip ring local Q. (Theorem 3.7).

1. PF, Kasch, and Zip Rings

A holomorphism $f : I \to E$ of a right ideal I into a module E is a *Baer homomorphism* if there exists $m \in E$ such that $f(x) = mx \forall \quad x \in I$.

Baer's criterion for injectivity of E states that every homomorphism of any right ideal of R into E is a Baer homomorphism.

A module E is $\aleph_0$*-injective* provided every homomorphism of a finitely generated right ideal into E is Baer. The ring R is *right* $\aleph_0$*-injective* if the canonical right module R is.

1.1 Theorem. *(Ikeda–Nakayama [**5**]). Consider the conditions:*

(a) Every homomorphism $f : I \longrightarrow E$ of a right ideal into E is Baer.

(b) $ann_E(I \cap J) = ann_E I + ann_E J$, where I and J are right ideals.

(c) $ann_E K^{\perp} = EK$, where K is a left ideal.

Moreover, let (a^*) denote the restriction of (a) to finitely generated I, and (a^{**}) the restriction to principal I. Similarly for (b^*), (b^{**}), (c^*), and (c^{**}). (Thus, in (c^*), K is a finitely generated left ideal). Then:

i. $(a^{**}) \Rightarrow (c^{**})$.

ii. $(a^*) \Leftrightarrow (b^*), (c^{**})$.

iii. $(a) \Rightarrow (b), (c^*)$.

Thus: E is $\aleph_0$-injective iff (b^*) and (c^{**}) both hold.

Proof: This is proved in [5] for the case $E = R$, and it is easy to prove that this holds for a general module E. (This is made explicit in [**3c**, p. 189, 23.21]).

A module E is FP–*injective* iff for all short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{1}$$

of finitely generated modules A, B and C, it is true that the canonical sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, E) \longrightarrow \mathrm{Hom}_R(B, E) \longrightarrow \mathrm{Hom}_R(A, E) \longrightarrow 0 \tag{2}$$

is exact. $\aleph_0$–injectivity is that statement that (1) exact implies (2) exact for $B = R$ and A finitely, generated.

Any injective module is FP–injective. A coherent $\aleph_0$–injective is FP–injective (Stenstrom [**8b**]–Jain [**14**]) but in general $\aleph_0$–injective does not imply FP–injective.By Jain [**14**] R is right FP–injective iff every finitely presented left R–module is torsionless. ■

1.2 Proposition. *R is a right zip ring iff the injective hull $E = E(R)$ of R in mod-R is divisible by any left ideal I having $I^{\perp} = 0$. In this case I divides any injective right R-module.*

Proof: If R is right zip, and I a left ideal with $I^{\perp} = 0$, then $I_1^{\perp} = 0$ for a finitely generated left ideal $I_1^{\perp} \subseteq I$, then $E = EI_1$ by the Ikeda–Nakayama theorem, so $EI = E$.

Conversely, if $E = E(R)$, then $EI = E$, implies that there exists finitely many $y_i \in E$, $r_i \in I, i = 1, \ldots, n$ such that $1 = \sum_{i=1}^n y_i r_i$. But this implies $\cap_{i=1}^n r_i^\perp = 0$, so R is right zip. ■

Let mod–R be the category of all right R–modules, and let R–mod denote the left–right symmetry.

In general, a module E is a cogenerator of mod–R iff the injective hull $E(V)$ of every simple right R–module V embeds in E (see, e.g. [3], [9)]. A ring R is *right Kasch* if every simple right module $\hookrightarrow R$, or equivalently, ${}^\perp I \neq 0$ for any right ideal $I \neq R$. Thus, a ring R is an injective cogenerator of mod-R iff R is right self–injective and right Kasch. Other characterizations:

1.3 Theorem. *(Azumaya [1], Osofsky [7], Utumi [9]). A ring R is right PF (pseudo–Frobenius) provided the following equivalent conditions hold:*

(PF_1) R is right self-injective and semiperfect with essential right socle. (The socle is the largest semisimple submodule).

(PF_2) R is right self-injective with finite essential right socle.

(PF_3) R is a finite direct sum. $R = \sum_{i=1}^n \oplus e_i R$, where $e_i^2 = e_i \in R$ and $e_I R$ is a projective injective right ideal with simple socle, $i = 1, \ldots, n$.

(PF_4) R is an injective cogenerator in mod–R.

(PF_5) R is right self-injective and right Kasch.

1.4 Theorem. *(Kato [6]). Any right PF ring R is left Kasch.*

A ring R is *left (finitely) annular* if every (finitely generated) left ideal L is an annihilator, that is, $L ={}^\perp (L^\perp)$.

1.5 Theorem. *(1). A right $\aleph_0$–injective ring is left finitely annular.*

(2) A right cogenerator ring R is right annular.

(3) A right PF ring is right annular and left finitely annular.

Proof: (1). In (iii) of Theorem 1.1, take $E = R$, and then $EK = K = \mathrm{ann}_R K^\perp ={}^\perp (K^\perp)$ is an annihilator, for any finitely generated K.

(2) A cogenerator E of mod–R has the property

$$I = \mathrm{ann}_R \mathrm{ann}_E I$$

for any right ideal I (see, for instance [4, p.184, 23.13]). Then (2) follows when $E = R$.

(3) R is right annular by 2, and since injective, left finitely annular by 1. ■

1.6 Proposition. *Any left Kasch ring R is right zip. If R is finitely left annular, then conversely.*

Proof: R left Kasch implies $L^\perp \neq 0$ for all left ideals $\neq R$, hence R is right zip.

Conversely, if R is left finitely annular, then right zip implies for any left ideal L with $L^{\perp} = 0$ the existence of a finitely genrated left ideal $L_1 \subseteq L$ with $L_1^{\perp} = 0$. But then $L_1^{\perp} = {}^{\perp}(L_1^{\perp}) = R$, so $L = R$, and R is therefore left Kasch. ■

1.7 Corollary. *A right $\aleph_0$-injective ring R is right zip iff left Kasch.*

1.8 Corollary. *A right PF ring R is right and left zip.*

Proof: R is right annular and left finitely annular by Theorem 1.5, and right and left Kasch by Theorem 1.3 and 1.4, hence right and left zip by Proposition 1.6. ■

1.9 Corollary. *A left and right self-injective ring R is right zip iff right and left PF. (In this case R is left zip).*

Proof: R is left Kasch by Corollary 1.7, hence left PF by Theorem 1.3. However, then R is right Kasch by Theorem 1.4, so R is right PF. (Left zip follows by Corollary 1.8).

Conversely, if R is right and left PF, Corollary 1.8 yields R is zip. ■

1.10 Proposition. *If R is right zip, and if $Q = Q^r_{\max}(R)$ is also a left quotient ring of R (equivalently, $Q \subseteq Q^{\ell}_{\max}(R)$), then Q is right zip. Conversely, if Q is right zip, then so is R.*

Proof: Let L be a left ideal of Q such that $L^{\perp} = 0$ in Q. We shall show that $(L \cap R)^{\perp} = 0$ in R. Suppose not and let $a \in R$ be such that $(L \cap R)a = 0$ and $a \neq 0$. By the assumption $Q \subseteq Q^{\ell}_{\max}(R)$, R is dense in Q as a left R-module, and this implies a contradiction, namely that $La \neq 0$. To prove this, suppose that $x \in L$, and $xa \neq 0$ then there corresponds $r \in R$ with $rx \in R$ and $rxa \neq 0$ ([3b, p.79 Theorem 19.23]). But $rx \in L \cap R$, contradicting $(L \cap R)a = 0$. Therefore by right zip in R, there is a finitely generated left ideal L_1 of R with $L_1 \subseteq L \cap R$ and $L_1^{\perp} = 0$. Then QL_1 is the desired finitely generated left ideal of Q contained in L with $(QL_1)^{\perp} = 0$.

The converse derives from Lemma D. ■

1.11 Corollary. *Let R be right zip. If $Q = Q^r_{\max}(R) = Q^{\ell}_{\max}(R)$ is injective (both sides), then Q is PF (both sides). Conversely.*

Proof: By Proposition 1.10, Q is right zip hence right and left PF by Corollary 1.9. Conversely, if Q is (right) PF, or zip, then R is (right) zip by 1.8 and 1.10. ■

2. Utumi zip rings are Goldie

A ring R is a *right Utumi* ring provided that R is right nonsingular and the following three equivalent conditions hold:

U_1: Every complement (= essentially closed) right ideal is an annihilator.

U_2: Every nonzero left ideal L of $Q = Q^r_{\max}(R)$ meets R, that is, $L \cap R \neq 0$.

U_3: ${}^{\perp}I = 0$. In R for a right ideal I implies that I is essential in R.

U_3 is called *cononsingular*, and implies that R is left nonsingular (proof omitted).

2.12 Theorem. *(Utumi [10]). A nonsingular ring R is right and left Utumi (equivalently right and left cononsingular) iff $Q^r_{\max}(R) = Q^\ell_{\max}(R)$.*

Utumi rings were named by Stenstrom [8].

A ring R is *right Goldie* if R has *acc* $\perp$ and finite right Goldie dimension in the sense that any direct sum $\sum_{a\in A} \oplus X_a$ of right modules embeddable in R has only finitely many $X_a \neq 0$. The latter condition is denoted by *acc*$\oplus$, and is equivalent to the *acc* on complement right ideals.

If R is right nonsingular then $Q = Q^r_{\max}(R)$ is right self–injective and von Neumann regular. Moreover any annihilator right ideal is a right complement, and the right comlements have the form $eQ \cap R$, where $e = e^2 \in Q$. Moreover, the contraction map $I \longrightarrow I \cap R$ induces bijection between complement right ideals of Q and those of R. Since Q is regular, then the f.a.e.c.'s:

(13.1) R has $(acc)\oplus$.

(13.2) Q has $(acc)\oplus$.

(13.3) Q is semisimple Artin.

Inasmuch contraction induces a surjection between the annihilator right ideals of any ring Q and those of a subring R, then any subring of an Artin (Noether) ring has *dcc* $\perp$ (resp. *acc*l), hence for right nonsingular R, we see that (13.3) implies that

(13.4) R has *acc* $\perp$ and *dcc* $\perp$ (equivalently $\perp$ *acc* and $\perp$ *dcc*) and

(13.5) R is right Goldie.

2.14 A Theorem. *An Utumi right zip ring R is right Goldie, hence satisfies both $\perp$ dcc and dcc $\perp$, so is both right and left zip.*

Proof: Any right nonsingular ring R has injective (and regular) $Q^r_{\max}(R)$, so by the Utumi assumption, $Q = Q^r_{\max}(R) = Q^\ell_{\max}(R)$ is injective on both sides, and the theorem follows from Corollary (1.11), since any PF ring Q is semiperfect. Thus, regularity of Q implies that Q is semisimple Artin, so apply (13.4–5).

If left zip is assumed, we can get the same conclusion assuming Utumi.

2.14 B Theorem. *Any Utumi left zip ring is right Goldie, hence has $\perp$ dcc and dcc $\perp$.*

Proof: Let $I = \sum_{a \in A} \oplus X_a$ be a maximal direct sum of right ideals contained in R. Then I is an essential right ideal, so ${}^{\perp}I = 0$, and left zip implies a finitely generated right ideal $I_1 \subseteq I$ with ${}^{\perp}I_1 = 0$. But then cononsingularity means that I_1 is an essential right ideal. But a direct sum $\oplus_{a \in A} X_a$, has a finitely generated essential submodule iff $X_a \neq 0$ for just finitely many $a \in A$. Thus R has $(acc)\oplus$, so Q has $(acc)\oplus$, i.e. (13.1)–(13.5) hold, proving the theorem. ■

3.Commutative Zip Rings

We now apply earlier results to commutative rings.

3.1 Proposition. *If $R \longrightarrow S$ is an embedding of rings such that*

$$\begin{cases} \textit{ideals } R \longrightarrow \textit{ideals } S \\ \qquad I \longrightarrow IS \end{cases}$$

is surjective, then R zip implies that S is zip.

Proof: Let R be zip, and let I be a faithful ideal of S. Let I_0 be ideal of R such that $I_0 S = I$. Then I_0 is faithful in R, so $I_1^{\perp} = 0$ in R for a finitely generated ideal I_1 of R. Thus,

$$\mathrm{ann}_S(I_1 S) = 0$$

and $I_1 S$ is a finitely generated ideal of S.

This proves that S is zip. ■

3.2 Corollary. *A commutative ring R is zip iff its classical quotient ring $Q = Q_c(R)$ is zip.*

Proof: R zip implies Q zip by the proposition. Conversely, R right essential in Q, Lemma D applies: if Q is right zip so is R.

3.2 also follows from Lemma D and Proposition 1.10. ■

3.3 Corollary. *If R is zip, so is RS^{-1}, for any multiplicative semigroup $S \subseteq R^*$, and conversely.*

Proof: Same as Corollary 3.2 ■

By Proposition 1.6, any commutative Kasch ring is zip this yields:

3.4 Corollary. *If $Q_c(R)$ is Kasch, than R is zip.*

Definition. A commutative ring R is *Bezout* provided that all finitely generated ideals are principal.

Trivially principal ideal rings and chain rings (= rings with linearly ordered ideal lattices) are Bezout, and so is any finite product of Bezout rings. Also any factor ring of a Bezout ring.

R is a (*)–*Bezout ring* if every finitely generated faithful (or dense) ideal is principal. If R is reduced (= semiprime = non–singular) then an ideal I is dense iff I is essential. Since every ideal is a direct summand of an essential ideal (in any ring), then R is (*)–Bezout iff Bezout when R is reduced.

3.5 Proposition. *Let R be a (*)–Bezout ring.*

(1) R is zip iff every faithful ideal contains a regular element.

(2) In this case, the classical quotient ring $Q = Q_c(R)$ is (*)–Bezout Kasch ring.

Proof: (1) If I is a faithful ideal of a zip ring, and if I, is finitely generated faithful ideal contained in I, then I, is principal, hence generated by a regular element.

(2) Q is also (*)–Bezout, and by (1) every faithful ideal contains a unit, that is, Q is the only faithful ideal. This implies that Q is Kasch. ■

3.6 Corollary. *For a (*)–Bezout ring R the f.a.e.*

(1) $Q = Q_c(R)$ is zip.

(2) Q is Kasch.

Proof: Apply Corollary 3.4 and Proposition 3.5. ■

A ring R is FPF iff every finitely generated faithful module generates mod–R. By [20], a commutative ring R is FPF iff there holds.

(FPF1) Every finitely generated farthful ideal is projective, and (FPF 2) $Q = Q_c(R)$ is self–injective.

In this case Q is FPF.

3.7 Theorem. *If R has a local quotient ring $Q = Q_c(R)$, and if R is an FPF zip ring, then Q is PF.*

Proof: If Q is PF, then Q is Kasch by Theorem 3.1, hence Q and R are zip by Corollary 3.4 (without assuming Q local).

Conversely, if R is zip, then so is Q by Proposition 3.2. Also R FPF implies Q FPF, hence if an ideal I, is finitely generated and faithful in Q, then I is projective by FPF, and free by Kaplansky's theorem on projective modules over local rings. But in a commutative ring, every free ideal is principal, so Q is (*)–Bezout. By Theorem 3.6, Q is Kasch, hence PF by Theorem 1.3. ■

Added in Proof. See **[3c]** for inter alia a study of zip rings, and of an example of a zip ring E with $Q_c(R)$ not Kasch.

The question of when a noncommutative right zip ring R has right zip polynomial ring $R[X]$ is presently open.

In **[3c]**, it is shown that a right zip ring R has the property right ξ min, i.e., every annihilator right ideal $\neq 0$ contains a minimal annihilator right ideal $\neq 0$.

References

1. G., AZUMAYA, Completely faithful modules and self–injective rings, *Nagoya Math. J.* **27** (1966), 697–708.
2. R. BAER,, Abelian groups which are direct summands of every containing group, *Bull. Amer. Math. Soc.* **46** (1940), 800–806.

3a. C. FAITH, Algebra: Rings, modules and categories, *Springer-Verlag* **I** (1973), Berlin and New York.

3b. C. FAITH, Algebra II: Ring theory, *Springer–Verlag* (1976), Berlin and New York.

3c. C. FAITH, Annihilator ideals, associated primes and Kasch–McCoy quotient rings.

4. J. BEACHY AND R. BLAIR, When faithful left ideals are cofaithful, *Pac. J. Math.* (1975).
5. M. IKEDA, T. NAKAYAMA, On some characteristic properties of quasi–Frobenius and regular rings, *Proc. Amer. Math. Soc.* **5** (1954), 15–19.
6. I. KATO, Self–injective rings, *Tokoku Math. J.* **(2) 19** (1967), 485–495.
7. B. OSOFSKY, A generalization of quasi–Frobenius rings, *J. Algebra* **4** (1966), 373–387.

8a. B. STENSTROM, Rings of Quotients.

8b. B. STENSTROM, Coherent and FP–injective modules, *J. London Math. Soc.* **2** (1970), 323–9.

9. Y. UTUMI, Self–injective rings, *J. Algebra* **6** (1967), 56–64.
10. Y. UTUMI, On rings of which every one–sided quotient ring is two–sided, *Proc. Amer. Math. Soc.*.
11. Y. UTUMI, On quotient rings, *Osaka Math* **8** (1958), 1–18.
12. J. ZELMANOWITZ, Finite intersection property on annihilator right ideals, *Proc. Am. Math. Soc.* **57** (1976), 213–216.
13. P. VÁMOS, The dual of the notion of finitely generated, *J: london Math. Soc.* **43** (1968), 643–6.

14. S. JAIN, Flat and FP–injectivity, *Proc. Amer. Math. Soc.* 41 (1973), 437–42.

Department of Mathematics
Rutgers State University
New Brunswick, NJ 08903. USA

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