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# THE PERIODIC SOLUTIONS OF THE SECOND ORDER NONLINEAR DIFFERENCE EQUATION

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*Abstract*

Periodic and asymptotically periodic solutions of the nonlinear equation $\Delta^2 x_n + a_n f(x_n) = 0$, $n \in \mathrm{N}$, are studied.

In several recent papers ([2],[3]) the periodicity of solutions of linear difference equations have been investigated. In this paper we examine the periodic solutions of the nonlinear equation

$$(E) \qquad \Delta^2 x_n + a_n f(x_n) = 0,\ n \in \mathbf{N},$$

where $\mathbf{N} = \{0, 1, 2, ...\}$, $\mathbf{R}$ is the set of real numbers, $f : \mathbf{R} \longrightarrow \mathbf{R}$ and $a, x : \mathbf{N} \longrightarrow \mathbf{R}$ are sequences of real numbers.

Throughout the paper we use the following notations. By $\overline{0,t}$ we denote the set of integers {0,1,2,...,t}. For the function $y$: $\mathbf{N} \longrightarrow \mathbf{R}$ the forward difference operator $\Delta^k$ is defined

$$\Delta y_n = y_{n+1} - y_n,\ \Delta^k y_n = \Delta(\Delta^{k-1} y_n) \text{ for } k > 1.$$

**Definition 1.** *The function $y$ will be called $t$-periodic if $y_{n+t} = y_n$ for all $n \in \mathbf{N}$. (Furthermore we suppose that no $t_1$ exists, $0 < t_1 < t$ such that $y_{n+t_1} = y_n$ for all $n \in \mathbf{N}$ and that $t > 1$).*

**Definition 2.** *The function $y$ will be called asymptotically $t$-periodic $(t > 1)$ if*

$$y = u + v,$$

*where $u$ is a $t$-periodic function and* $\lim_{n\to\infty} v_n = 0$.

**Definition 3.** *We say that the equation $(E)$ has a $p_t$-constant if there exists a constant $p \in \mathbf{R}$, such that the equation*

$$(E_1) \qquad \Delta^2 x_n + a_n f(x_n) = p$$

*has a t-periodic solution.*

*We say that the equation $(E)$ possesses a $p_t^\infty$-constant if there exists a constant $p \in \mathbb{R}$ such that $(E_1)$ has an asymptotically t-periodic solution.*

**Definition 4.** *The equation $(E)$ is said to have a $p_t$-function ($p_t^\infty$-function) if there exists a t-periodic function $p: \mathbb{N} \longrightarrow \mathbb{R}$ such that the equation*

$$(E_2) \qquad \Delta^2 x_n + a_n f(x_n) = p_n$$

*has a t-periodic (asymptotically t-periodic) solution.*

**Remark 1.** Note that if $(E)$ has a $p_t$-constant (function) then $(E)$ has a $p_t^\infty$-constant (function) and if $(E)$ has not a $p_t^\infty$-constant (function) then it has no $p_t$-constant (function).

**Theorem 1.** *Let $f$: $\mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$ and $\lim_{n\to\infty} a_n = 0$. Then the equation $(E)$ has not a $p_t^\infty$-constant for any $t > 1$.*

*Proof:* We show the proof for simplicity in the case $t = 2$. Similar reasoning can be made for $t > 2$.

Suppose that there exists a $p_t^\infty$-constant $q$ such that the equation

$$(E_3) \qquad \Delta^2 x_n + a_n f(x_n) = q$$

has one asymptotically 2-periodic solution $x$.

Let $x_{2n} \longrightarrow C_1, x_{2n+1} \longrightarrow C_2$ as $n \longrightarrow \infty, C_1 \neq C_2$. Hence

$$\Delta^2 x_{2n} \longrightarrow 2C_1 - 2C_2$$

$$\Delta^2 x_{2n+1} \longrightarrow 2C_2 - 2C_1.$$

As result of the assumption we obtain

$$2C_1 - 2C_2 = q$$

$$2C_2 - 2C_1 = q.$$

The above system has a solution if and only if $q = 0$, but in this case we obtain $C_1 = C_2$, which is a contradiction. ∎

**Theorem 2.** *Let $f \neq 0$ on $\mathbb{R}$. If the equation $(E)$ possesses a $p_t$-constant then $a$ is a t-periodic function.*

*Proof:* Let $x$ be a $t$-periodic solution of $(E_3)$. Then $\Delta^2 x$ is $t$-periodic. By virtue of the assumption $f \neq 0$ and we get

$$\frac{\Delta^2 x_n - q}{f(x_n)} = -a_n.$$

The left hand side of the above equality is a $t$–periodic function so the right hand side must also be $t$–periodic. ∎

**Remark 2.** We can prove analogously that if $f \not\equiv 0$ on $\mathbb{R}$, then $t$– periodicity of $a$ is the necessary condition for the existence of a $p_t$– function $q$ for the equation $(E)$. However in this case we do not require for $t$ to be the basic period. Eventually $a$ can be a constant function. It is easy to see that if $f(C_1) = 0$ then the equation $(E)$ has $p_1$–constant $q = 0$. Then a $t$–periodic solution takes the form $x \equiv C_1$.

By $i_R$ we denote the identy function on $\mathbb{R}$.

**Theorem 3.** *Let $a : \mathbb{N} \longrightarrow \mathbb{R}$, let $f$ be a continuous function on $\mathbb{R}$, $f \not\equiv 0$ such that the functions*

$$i_R + a_n f : \mathbb{R} \longrightarrow \mathbb{R} \tag{1}$$

*are surjections for every $n \in \mathbb{N}$. If*

$$\sum_{j=1}^{\infty} j|a_j| < \infty \tag{2}$$

*then the equation $(E)$ has a $p_t^{\infty}$ -function for arbitrary $t \geq 1$.*

*Proof:* Choose $t \geq 1$. By assumption there exist constants $C_r, r = 1, 2, \cdots, t$, $C_i \neq C_j, i \neq j$, such that

$$f(C_r) \neq 0.$$

The case

$$f(C_r) > 0,\ r = 1, 2, \cdots, t \tag{3}$$

will be considered. The proof for the other cases $f(C_i) > 0, f(C_j) < 0$ is similar.

By virtue of the continuity of the function $f$ there exist intervals

$$I_r = [C_{r+1} - \delta, C_{r+1} + \delta],\ r = 0, 1, \cdots, t-1 \tag{4}$$

such that

$$f(u) > 0 \text{ for } u \in I_r,\ r = 0, 1, \cdots, t-1. \tag{5}$$

From (2) it follows that

$$\lim_{n \to \infty} \sum_{j=n}^{\infty} j|a_j| = 0. \tag{6}$$

Let us denote

$$D = \max_{0\le r\le t-1}(\max_{u\in I_r} f(u)) \tag{7}$$

and

$$n_1 = \min\{n \in \mathbb{N} : n = tk + t - 1,\ D\sum_{j=n}^{\infty} j|a_j| \le \delta\}.$$

In the space $l^\infty$ of bounded sequences with the norm

$$\|x\| = sup_{i\ge 0}|x_i|$$

we define the set $T$ in the following way:

$$x = \{x_i\}_{i=0}^{\infty} \in T$$

if

$$x_r = x_{t+r} = x_{2t+r} = \cdots = x_{n_1-t+r+1} = C_{r+1}, x_{tk+r} \in I_{tk+r} :=$$

$$= [C_{r+1} - D\sum_{j=tk+r}^{\infty} j|a_j|; C_{r+1} + D\sum_{j=tk+r}^{\infty} j|a_j|],$$

$$r = 0, 1, \cdots, t-1 : k \in \mathbb{N};\ k > \frac{1}{t}(n_1 + 1 - t).$$

The set $T$ is closed, convex and bounded. Furthermore, by diam $S$ we mean

$$\text{diam } S = sup\{\|x - y\|; x \in S; y \in S\}.$$

So

$$\text{diam } I_{tk+r} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \tag{8}$$

It is easy to find a finite $\epsilon$–net for every $\epsilon > 0$. Therefore by Hausdorff's Theorem the set $T$ is compact. Let us define an operator $A$ for $x \in T$ as follows:

$$Ax = y = \{y_i\}_{i=0}^{\infty}$$

where

$$y_r = y_{t+r} = \cdots = y_{n_1+r+1-t} = C_{r+1}; r = 0, 1, \cdots, t-1,$$

$$y_{tk+1} = C_{r+1} - \sum_{j=tk+r}^{\infty} (j + 1 - tk - r)a_j f(x_j)$$

for $k \in \mathbb{N}$, $k > \frac{1}{t}(n_1 + 1 - t)$, $r = 0, 1, \cdots, t-1$.

Let us observe that

$$I_{tk+r} \subset I_r,\ r = 0, 1, \cdots, t-1,\ k > \frac{1}{t}(n_1 + 1 - t).$$

Hence

$$(9) \qquad \Big|\sum_{j=tk+r}^{\infty}(j+1-tk-r)a_j f(x_j)\Big| \le$$

$$\le \sum_{j=tk+r}^{\infty} j|a_j||f(x_j)| \le D \sum_{j=tk+r}^{\infty} j|a_j|.$$

Therefore $y_{tk+r} \in I_{tk+r}$, $r = 0, 1, \cdots, t-1$, $k \in \mathbb{N}$, $k > (n_1 + 1 - t)/t$ and this means that $A : T \longrightarrow T$. Let us take an arbitrary sequence $\{x^m\}_{m=1}^{\infty}$ of elements of $T$ convergent to some $x^0 \in T$ i.e.

$$\|x^m - x^0\| \longrightarrow 0.$$

Hence we have

$$(10) \qquad \sup_{n\ge 0}|x_n^m - x_n^0| \longrightarrow 0$$

as $m \longrightarrow \infty$. Let $\epsilon_1$ be an arbitrarily taken positive real number. By the uniform continuity of $f$ on the sets $I_r$ we have

$$|u_1 - u_2| < \delta \text{ implies } |f(u_1) - f(u_2)| < \epsilon_1.$$

From (10) it follows that

$$(11) \qquad \sup_{n\ge 0}|x_n^m - x_n^0| < \delta$$

for $m \ge M(\delta)$. Let $y^m = Ax^m$, $m \in \mathbb{N}$; then

$$(12) \qquad \|Ax^m - Ax^0\| =$$

$$= \sup_{n>n_1}\Big|\sum_{j=n}^{\infty}(j+1-n)a_j f(x_j^m) - \sum_{j=n}^{\infty}(j+1-n)a_j f(x_j^0)\Big|.$$

By (9) the series

$$\sum_{j=n}^{\infty}(j+1-n)a_j f(x_j^m),\ m \in \mathbb{N}$$

are absolutely convergent. Hence, by (11) and (12)

$$\|Ax^m - Ax^0\| \le \epsilon_1 \sum_{j=n_1}^{\infty} j|a_j|$$

so that the operator $A$ is continuous on $T$. By Shauder's Theorem there exists $z \in T$ such that $z = Az$. By definition of $A$ this element $z = \{z_i\}_{i=0}^{\infty}$ satifies

$$(13) \qquad z_r = z_{t+r} = \cdots = z_{n_1+r+1-t} = c_{r+1}$$

$$z_{tk+r} = C_{r+1} - \sum_{j=tk+r}^{\infty} (j+1-tk-r)a_j f(z_j)$$

$$k > \frac{1}{t}(n_1 + 1 - t),\ r = 0, 1, \cdots, t-1.$$

Applying the operator $\triangle$ to $z$ we obtain

$$\triangle z_{tk+r} = z_{tk+r+1} - z_{tk+r} =$$

$$= C_{r+2} - C_{r+1} - \sum_{j=tk+r+1}^{\infty} (j+tk-r)a_j f(z_j) + \sum_{j=tk+r}^{\infty} (j+1-tk-r)a_j f(z_j) =$$

$$= C_{r+2} - C_{r+1} + \sum_{j=tk+r}^{\infty} a_j f(z_j),$$

and consequently

$$\triangle^2 z_{tk+r} = \triangle z_{tk+r+1} - \triangle z_{tk+r} =$$

$$= C_{r+3} - 2C_{r+2} + C_{r+1} + \sum_{j=tk+r+1}^{\infty} a_j f(z_j) - \sum_{j=tk+r}^{\infty} a_j f(z_j) =$$

$$= C_{r+3} - 2C_{r+2} + C_{r+1} - a_{tk+r} f(z_{tk+r}),$$

$$r = 0, 1, \cdots, t-1,\ k > \frac{1}{t}(n_1 + 1 - t)$$

where

$$C_{t+1} = C_1,\ C_{t+2} = C_2.$$

Denoting

(14) $$q_{tk+r} = C_{r+3} - 2C_{r+2} + C_{r+1},\ r = 0, 1, \cdots, t-1,$$

we obtain the equation

(15) $$\triangle^2 x_n + a_n f(x_n) = q_n$$

which has an asymptotically $t$–periodic solution defined for $n > n_1$. This follows from (8) and $z_{tk+r} \in I_{tk+r}$, i.e. $z_{tk+r} \longrightarrow C_{r+1}$ as $k \longrightarrow \infty$.

It suffices to show that there exist a solution of (15) which coincides with (13) for $n > n_1$.

For this we observe that the equation (15) can be rewritten in equivalent form

(16) $$x_n + a_n f(x_n) = q_n - x_{n+2} + 2x_{n+1}.$$

Taking $n = n_1$, $x_{n+1} = z_{n_1+1}$, $x_{n+2} = z_{n_1+2}$ we find $x_{n_1}$, which by the assumptions exists (probably more than one). Repeating this reasoning we find $x_i$ for $i = 0, 1, \cdots, n_1 - 1$. This function $x$ is of course a solution of (15) which coincides with $z$ for $n > n_1$ and therefore has the desired asymptotic behaviour. ■

**Remark 3.** If the functions $i_R + a_n f$ are one-to-one mappings of $\mathbb{R}$ onto $\mathbb{R}$ then the solution obtained in the Theorem 3 is unique. The case $t = 1$, i.e. the solutions having the asymptotic property $\lim_{n\to\infty} x_n = C$, was considered in the paper [**1**].

Let us observe that by Theorem 3 if we want to have some solutions which have a given asymptotically $t$-periodic solution, then it suffices to add to equation (E) the periodic perturbation $q$ which can be easily found by (14).

*Example.* As an example we consider the difference equation of the form

$$\Delta^2 x_n + \frac{(-1)^{n+1}}{4[2^n + (-1)^n]} x_n = 0,\ n = 1, 2, \cdots$$

It is evident by d'Alembert criterion that the series

$$\sum_{j=1}^{\infty} \frac{j(-1)^{j+1}}{4[2^j + (-1)^j]}$$

is absolutely convergent. Furthermore the functions

$$x + \frac{(-1)^{n+1}}{4[2^n + (-1)^n]} x$$

are surjections from $\mathbb{R}$ onto $\mathbb{R}$ for all $n$. Therefore the assumptions of the Theorem 3 hold. We show that this equation has a $p_2^\infty$-function and find a 2-periodic solution of the form

$$x_n = (-1)^n + y_n .$$

Applying the proof of the Theorem 3 we see that the $p_2^\infty$-function $q$ takes the form

$$q_n = 4(-1)^n .$$

Considering the equation

$$\Delta^2 x_n + \frac{(-1)^{n+1}}{4[2^n + (-1)^n]} x_n = 4(-1)^n$$

we can observe that this equation has the solution

$$x_n = (-1)^n + \frac{1}{2^n}$$

which is of the desired form.

## References

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